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On the Existence of Room Squares with Subsquares

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ABSTRACT. We prove several results on the existence of Room squares containing Room subsquares. For example, we prove that if v and w are odd integers, $w \geq 23085$, and $v \geq 3w+40$, then there is a Room square of side v containing a Room subsquare of side w. This result is very close to best possible, since existence of a Room square of side v containing a Room subsquare of side w requires that v and w be odd, and $v \geq 3w+2$. We also prove that if v and w are odd integers, $w \geq 127$, and $v \geq 3w+240$, then there is a Room square of side v containing a Room subsquare of side w. Results are also presented for values of $w < 127$.

1. Introduction. Let S be a set of $v+1$ elements called *symbols*. A *Room square* (on symbol set S) is a v by v array, F, which satisfies the following properties:

(1) every cell of F either is empty or contains an unordered pair of symbols from S,
(2) each symbol of S occurs once in each row and column of F, and
(3) every unordered pair of symbols occurs in precisely one cell of F.

The spectrum for Room squares was determined in 1975 by Mullin and Wallis, who proved the following in **[9]**.

THEOREM 1.1. *There exists a Room square of side v if and only if v is an odd positive integer, $v \neq 3$ or 5.*

Now, suppose F is a Room square of side v on symbol set S. A w by w subarray G of F is said to be a *Room subsquare* of side w if it is itself a Room square of side w on a subset of $w+1$ symbols. In view of Theorem 1.1, no Room square can contain a Room subsquare of side 3 or 5. However, we can construct Room squares *missing* subsquares of these sides. We have the following formal definition.

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Let S be a set of $v+1$ symbols and let T be a subset of S of cardinality $w+1$. A (v, w)-IRS (*incomplete Room square*) is a v by v array F which satisfies the following:

(1) every cell of F either is empty or contains an unordered pair of symbols of S,
(2) there is an empty w by w subarray G of F,
(3) each symbol of $S \setminus T$ occurs once in each row and column of F,
(4) each symbol of T occurs once in each row and column not meeting G, but not in any row or column meeting G, and
(5) the pairs occurring in F are precisely those $\{x, y\}$ where $(x, y) \in (S \times S) \setminus (T \times T)$.

We refer to the subarray T as the *hole*. Observe that the hole can be filled in with any Room square of side w (provided $w \neq 3$ or 5), thereby constructing a Room square of side v containing a subsquare of side w.

In this paper we seek to answer the question: for which ordered pairs (v, w) does there exist a (v, w)-IRS? Of course, v and w must be odd, and it is not difficult to see that $v \geq 3w+2$ (see [**2**]). It has been conjectured that these necessary conditions are sufficient for the existence of a (v, w)-IRS, with the single exception $(v, w) \neq (5, 1)$.

The following theorem summarizes known existence results for (v, w)-IRS.

THEOREM 1.2.

(1) *For all odd* $w \geq 3$, *there is a* $(3w+2, w)$-IRS. ([**12**])
(2) *For all odd* $v \geq 41$, *there is a* $(v, 3)$-IRS. ([**7**])
(3) *For all odd* $v \geq 69$, *there is a* $(v, 5)$-IRS. ([**7**])
(4) *For all odd* $v \geq 55$, *there is a* $(v, 7)$-IRS. ([**7**])
(5) *For all odd* $w \geq 7$ *and all odd* $v \geq 6w+41$, *there is a* (v, w)-IRS. ([**11**])
(6) *For all odd* $w \geq 129$ *and all odd* $v \geq 4w+29$, *there is a* (v, w)-IRS. ([**11**])

In this paper, we obtain significant improvements to (2)–(6), as indicated in the following Theorem.

THEOREM 1.3.

(1) *For* $w = 3, 5, 7$, *and for all odd* $v \geq 3w+2$, *there is a* (v, w)-IRS.
(2) *For all odd* $w \geq 37$ *and all odd* $v \geq (7w-5)/2$, *there is a* (v, w)-IRS.
(3) *For all odd* $w \geq 127$ *and all odd* $v \geq 3w+240$, *there is a* (v, w)-IRS.

2. Room frames. Our main constructions require the use of a generalization of a Room square called a Room frame. Let S be a set, and let $\{S_1, \dots, S_n\}$ be a partition of S. An $\{S_1, \dots, S_n\}$-*Room frame* is an $|S|$ by $|S|$ array, F, indexed by S, which satisfies the following properties:

(1) every cell of F either is empty or contains an unordered pair of symbols of S,
(2) the subarrays $S_i \times S_i$ are empty, for $1 \le i \le n$ (these subarrays are referred to as *holes*),
(3) each symbol of $S \setminus S_i$ occurs once in row (or column) s, for any $s \in S_i$, and
(4) the pairs occurring in F are precisely those $\{s, t\}$, where

$$(s, t) \in (S \times S) \setminus \bigcup_{1 \le i \le n} (S_i \times S_i).$$

The *type* of F is defined to be the multiset $\{|S_i| : 1 \le i \le n\}$. We usually use an "exponential" notation to describe types: a type $t_1^{u_1} \dots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \le i \le k$. In Figure 1, we present a Room frame of type $2^5 4^1$.

A Room frame can be thought of as a Room square from which a spanning set of subsquares has been removed. A Room frame of type 1^v gives rise to a Room square of side v by filling in each diagonal cell (s, s) with the pair $\{\infty, s\}$, where ∞ is a new symbol. More generally, a (v, w)-IRS is equivalent to a Room frame of type $1^{v-w} w^1$.

We remark that Room frames have been used extensively in the literature as a tool in the construction of Room squares (see, for example, **[10]** and **[11]**). In the past, they have usually been referred to simply as "frames". However, frames for other types of combinatorial designs have recently been studied by several researchers. Hence, we will use the term "Room frame" in this paper.

We shall make use of the following Room frames.

THEOREM 2.1 **[3]**. *There exists a Room frame of type* 4^u *if and only if* $u \ge 4$, *and there exists a Room frame of type* 2^u *if and only if* $u \ge 5$.

For our main constructions, we require Room frames of types $2^i 4^{6-i}$, for $0 \le i \le 6$. Theorem 2.1 gives us those of types 4^6 and 2^6. A Room frame of type $2^1 4^5$ was constructed by the intransitive starter-adder method, as

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FIGURE 1. A Room frame of type $2^5 4^1$

defined to be a triple (S, R, C), where

$$S = \bigcup_{1 \le i \le (g-h-2)/2} (\{\{s_i, t_i\} : 1 \le i \le (g-h-2)/2\}) \cup \{\{u\}, \{v\}\},$$

$$C = \{\{p, q\}\}, \text{ and}$$

$$R = \{\{p', q'\}\},$$

satisfying

(1) $\bigcup_{1 \le i \le (g-h-2)/2} (\{s_i\} \cup \{t_i\}) \cup \{u, v, p, q\} = G \backslash H$, and
(2) $\bigcup_{1 \le i \le (g-h-2)/2} (\{\pm(s_i - t_i)\} \cup \{\pm(p-q)\} \cup \{\pm(p'-q')\} = G \backslash H$, and
(3) both $p - q$ and $p' - q'$ have even order in G.

An *adder* for (S, C, R) is an injection $A : S \to G \backslash H$, such that

$$\bigcup (\{s_i + a_i\} \cup \{t_i + a_i\}) \cup \{u + A(u), v + A(v), p', q'\} = G \backslash H$$

THEOREM 2.2 [3, 10]. *If there is an intransitive starter and adder in* $G\backslash H$, *where* $g = |G|$ *and* $h = |H|$, *then there is a Room frame of type* $h^{g/h}2^1$.

In Example 2.1, we present an intransitive starter and adder in $G\backslash H$, where $G = Z_{20}$ and $H = \{0, 10\}$. Hence, there is a Room frame of type $2^1 4^5$.

Example 2.1. A Room frame of type $2^1 4^5$.

	S(starter)		A(adder)	S+A	
	∞_1	1	1	∞_1	2
	∞_2	6	3	∞_2	9
	7	9	9	16	18
	14	18	19	13	17
	11	17	17	8	14
	12	19	12	4	11
	8	16	11	19	7
	4	13	8	12	1
C=	2	3			
R=				3	6

The constructions of the remaining Room frames were accomplished by means of the computer. In order to describe the algorithms used, we require some more definitions. Let G be a graph. A *one-factor* of G is a set of disjoint edges which spans the vertex set (i.e. a perfect matching). A *one-factorization* of G is a set of one-factors which partitions the edge set. Two one-factors of G are *orthogonal* if they contain at most one common edge. A one-factor f is *orthogonal* to a one-factorization $\mathcal{F}$ if f is orthogonal to every one-factor in $\mathcal{F}$. Finally, two one-factorizations $\mathcal{F}_1$ and $\mathcal{F}_2$ are *orthogonal* if every one-factor in $\mathcal{F}_1$ is orthogonal to $\mathcal{F}_2$. It is well-known (see [8]) that a Room square of side n is equivalent to two orthogonal one-factorizations of the complete graph K_{n+1}.

We give an analogous characterization of Room frames. Let $K(t_1^{u_1} \cdots t_k^{u_k})$ denote the complete multipartite graph having u_i parts of size t_i, $1 \le i \le k$. We shall refer to the parts of this graph as *holes*. A *holey one-factor* of $K(t_1^{u_1} \cdots t_k^{u_k})$ is a set of disjoint edges, f, that partitions the vertices not in some hole, which we call the hole *corresponding* to f. A *holey one-factorization* of $K(t_1^{u_1} \cdots t_k^{u_k})$ is a set of holey one-factors that partition the edge set. It is not difficult to prove that if $\mathcal{F}$ is a holey one-factorization of $K(t_1^{u_1} \cdots t_k^{u_k})$, then for every hole H, there are precisely $|H|$ holey one-factors corresponding to H.

Next, we define the idea of orthogonality. Two holey one-factors of $K(t_1^{u_1} \cdots t_k^{u_k})$ are *orthogonal* if they correspond to different holes and conand two holey one-factorizations $\mathcal{F}_1$ and $\mathcal{F}_2$ are *orthogonal* if every holey one-factor in $\mathcal{F}_1$ is orthogonal to $\mathcal{F}_2$. Then we have the following characterization of frames in terms of orthogonal holey one-factorizations. The proof is left to the reader.

THEOREM 2.3. *A Room frame of type* $t_1^{u_1} \cdots t_k^{u_k}$ *is equivalent to the existence of two orthogonal holey one-factorizations of* $K(t_1^{u_1} \cdots t_k^{u_k})$.

In [4], a hill-climbing algorithm is presented for the construction of one-factorizations and orthogonal one-factorizations of K_n. It is not difficult to modify this algorithm to search for (orthogonal) holey one-factorizations of $K(t_1^{u_1} \cdots t_k^{u_k})$. Using the modified algorithm, we found Room frames of types $2^2 4^4$ (presented in Appendix 1 at the end of this paper) and $2^5 4^1$ (presented in Figure 1). However, after more than 30,000 attempts, we were unable to find Room frames of types $2^4 4^2$ and $2^3 4^3$ by this method.

We were able to handle the final two cases by augmenting the hill-climbing algorithm with a backtracking algorithm, as follows.

Algorithm.

(1) Construct a holey one-factorization $\mathcal{F}$ of the graph $K(t_1^{u_1} \cdots t_k^{u_k})$ using the hill-climbing algorithm.

(2) Construct a partial holey one-factorization $\mathcal{F}_1$ of $K(t_1^{u_1} \cdots t_k^{u_k})$ which is orthogonal to $\mathcal{F}$ using the hill-climbing algorithm.

(3) Construct the set $\mathcal{S}$ of all holey one-factors which are orthogonal to $\mathcal{F}$ and disjoint from every holey one-factor in $\mathcal{F}_1$.

(4) Using a backtracking algorithm, attempt to extend $\mathcal{F}_1$ to a holey one-factorization $\mathcal{F}_2$ by using holey one-factors in $\mathcal{S}$. (Note that $\mathcal{F}_2$ will then be orthogonal to $\mathcal{F}$.)

For the graph $K(2^3 4^3)$, we constructed an $\mathcal{F}_1$ (in step 2) consisting of 5 holey one-factors. Then $\mathcal{S}$ contained about 4500 holey one-factors. The exhaustive search in step 4 could then be performed in a reasonable amount of time. On the first four attempts, the algorithm failed. That is, there did not exist any extension of $\mathcal{F}_1$ to a holey one-factorization $\mathcal{F}_2$ orthogonal to $\mathcal{F}$. On the fifth attempt, the algorithm succeeded in finding the desired Room frame.

The case of the graph $K(2^4 4^2)$ was similar. Here, we were able to take $\mathcal{F}_1$ to be empty, and the resulting set $\mathcal{S}$ was still small enough to perform the exhaustive backtrack. We found the desired $\mathcal{F}_2$ on the seventh attempt. This means that there exist holey one-factorizations of $K(2^4 4^2)$ for which it is impossible to find an orthogonal holey one-factorization. The authors

THEOREM 2.4. *There exist Room frames of types* 2^6, $2^5 4^1$, $2^4 4^2$, $2^3 4^3$, $2^2 4^4$, $2^1 4^5$, *and* 4^6.

3. Room squares with holes of size 3, 5, or 7. In the following three theorems, we show that the necessary conditions for the existence of $(v, 3)$-, $(v, 5)$-, and $(v, 7)$-IRS are sufficient. We use these particular IRS for our main constructions in Section 4.

THEOREM 3.1. *A* $(v, 3)$-IRS *exists if and only if* v *is odd and* $v \geq 11$.

PROOF. It was shown in [7] that there exists a $(v, 3)$-IRS if v is odd, $v \geq 11$, and $v \neq 15, 17, 19, 21, 23, 25, 27, 29, 31, 37$, or 39. A $(39, 3)$-IRS follows from the existence of a $(13, 3)$-IRS and [7, Lemma 2.11]. For these 10 remaining values of v, we constructed $(v, 3)$-IRS using the hill-climbing algorithms described in [4]. For $v = 15, 17, 19$, and 21, the squares are presented in Appendix 2. The remaining squares are displayed in the research report [6].

THEOREM 3.2. *A* $(v, 5)$-IRS *exists if and only if* v *is odd and* $v \geq 17$.

PROOF. It was shown in [7] that there exists a $(v, 5)$-IRS if v is odd, $v \geq 17$, and $v \neq$ 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 57, 59, 61, 63, or 67. For 19 of these remaining values of v (excluding $v = 57$ and $v = 63$), we constructed $(v, 5)$-IRS using the hill-climbing algorithms described in [4]. These squares are presented in the research report [5]. The existence of (57,5)- and (63,5)-IRS follow from the existence of (19,5)- and (21,5)-IRS, using [7, Lemma 2.11].

THEOREM 3.3. *A* $(v, 7)$-IRS *exists if and only if* v *is odd and* $v \geq 23$.

PROOF. It was shown in [7] that there exists a $(v, 7)$-IRS if v is odd, $v \geq 23$, and $v \neq 25, 27, 33, 41, 45, 51$, or 53. For these 7 remaining values of v, we constructed $(v, 7)$-IRS using the hill-climbing algorithms described in [4]. These squares are presented in the research report [5].

4. Main results. In this section, we prove our main results using recursive constructions for Room frames. We first need to define some design-theoretic terminology. A *pairwise balanced design* (or, PBD) of index 1 is a pair $(X, \mathcal{A})$, such that X is a set of elements (called *points*) and $\mathcal{A}$ is a set of subsets of X (called *blocks*), each of cardinality at least two, such that every unordered pair of points is contained in a unique block of $\mathcal{A}$. If v is a positive integer and K is a set of positive integers, each of which is greater than or equal to 2, then we say that $(X, \mathcal{A})$ is a (v, K)-PBD if $|X| = v$, and $|A| \in K$ for every $A \in \mathcal{A}$. The integer v is called the *order* of the PBD.

Using this notation, a $(v, k, 1)$-BIBD (*balanced incomplete block design*)

A *group-divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$, which satisfies the following properties:

(1) $\mathcal{G}$ is a partition of X into subsets called *groups*,
(2) $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point, and
(3) every pair of points from distinct groups occurs in a unique block.

The *group-type* of a GDD $(X, \mathcal{G}, \mathcal{A})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. As with Room frames, we shall use an exponential notation for group-types, for convenience. We will say that a GDD is a K-GDD if $|A| \in K$ for every $A \in \mathcal{A}$

A *transversal design* TD(k, m) can be defined to be a $\{k\}$-GDD of type m^k. It is well-known that a TD(k, m) is equivalent to $k-2$ mutually orthogonal Latin squares of order m. For results on the existence of transversal designs we refer to [**1**].

The following is our main recursive construction for Room frames.

CONSTRUCTION 4.1. [**10**, Construction 2.2] *Let $(X, \mathcal{G}, \mathcal{A})$ be a* GDD, *and let $w : X \to Z^+ \cup \{0\}$ (we say that w is a weighting). For every $A \in \mathcal{A}$, suppose there is a Room frame of type $\{w(x) : x \in A\}$. Then there is a Room frame of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

Once we have constructed Room frames, we fill in the holes (see [**10**, Corollary 4.9]).

LEMMA 4.2. (i) *Suppose there is a Room frame of type $t_1^{u_1} \dots t_k^{u_k}$. Let $a \geq 0$. For $1 \leq i \leq k$, suppose there is a $(t_i + a, a)$*-IRS. *Then, for $1 \leq i \leq k$, there is a $(\sum_{1 \leq i \leq k} t_i u_i + a, t_i + a)$*-IRS, *and a $(\sum_{1 \leq i \leq k} t_i u_i + a, a)$*-IRS.
(ii) *Suppose there is a Room frame of type $t_1^{u_i} \dots t_k^{u_k}$, where $u_1 = 1$. Let $a \geq 0$. For $2 \leq i \leq k$, suppose there is a $(t_i + a, a)$*-IRS. *Then there is a $(\sum_{1 \leq i \leq k} t_i u_i + a, t_1 + a)$*-IRS.

We apply these constructions as follows.

THEOREM 4.3. *Suppose there is a resolvable $(20m + 5, 5, 1)$*-BIBD. *Let $a = 1$ or 3; let $5m \leq t \leq 10m$; and let $20m + 5 \leq s \leq 40m + 10$. Then there exists a $(2s + 2t + a, 2t + a)$*-IRS.

PROOF. Adjoin infinite points to all but one parallel class of the BIBD, giving rise to a $\{6\}$-GDD of group-type $5^{4m+1}(5m)^1$. Give the points in the group G of size $5m$ weights 2 and 4 in such a way that

$$\sum_{x \in G} w(x) = 2t,$$

and give the other points in the GDD weights 2 and 4 in such a way that

Apply Construction 4.1. Now adjoin a infinite points to the Room frame, and apply Lemma 4.2 (ii). We fill in the appropriate $(v, 1)$-IRS $(v = 11, 13, 15, 17, 19, 21)$, if $a = 1$; or the appropriate $(v, 3)$-IRS $(v = 13, 15, 17, 19, 21, 23)$, if $a = 3$.

THEOREM 4.4. *For any odd $w \geq 127$, and for any odd $v \geq 3w + 240$, there is a (v, w)-IRS.*

PROOF. Write $w = 20m_0 + 3 - 2\alpha$, where $0 \leq \alpha \leq 9$. Now, there is an m, $m_0 \leq m \leq m_0 + 5$, such that a resolvable $(20m + 5, 5, 1)$-BIBD exists (see Table 1 of [**13**]). Apply Theorem 4.3 with $a = 3$. In order to construct a (v, w)-IRS, we need to have

$$10m + 3 \leq w \leq 20m + 3.$$

The second inequality is clearly true, and the first inequality will be true provided

$$10(m_0 + 5) + 3 \leq 20m_0 + 3 - 18,$$

or

$$10m_0 \geq 68.$$

But $w \geq 127$, so $m_0 \geq 7$. By Theorem 4.3, we can handle all odd v where

$$40m + 10 + w \leq v \leq 80m + 20 + w.$$

Now, we have

$$40m + 10 \leq 40m_0 + 210 \leq 2w + 240;$$

and

$$80m + 20 \geq 80m_0 + 20 \geq 4w + 8.$$

So, we can handle all odd v such that

$$3w + 240 \leq v \leq 5w + 8.$$

For odd $w \geq 127$ and odd $v \geq 5w + 2$, there is a (v, w)-IRS by [**11**, Theorem 3.13]. This completes the proof.

By similar methods, we can prove

THEOREM 4.5. *For any odd $w \geq 23085$, and for any odd $v \geq 3w + 40$, there is a (v, w)-IRS.*

PROOF. As before, write $w = 20m_0 + 3 - 2\alpha$, where $0 \leq \alpha \leq 9$. Since $w \geq 23085$, we have $m_0 \geq 1155$, and hence a resolvable $(20m_0 + 5, 5, 1)$-BIBD exists, by [**13**, Theorem 1.1]. Proceed as in Theorem 4.4

For some values of w, the existence of (v, w)-IRS is completely determined. For example, we have the following.

THEOREM 4.6. *For any $w \geq 23103$ $w \equiv 3$ modulo 20, and for any odd $v \geq 3w + 2$, there is a (v, w)-IRS.*

THEOREM 4.7. *Suppose there is a* TD$(6, m)$, *and suppose there exists a* $(2r+a, a)$-IRS *for all* r *such that* $m \le r \le 2m$. *Let* $m \le t \le 2m$ *and let* $5m \le s \le 10m$. *Then there exists a* $(2s+2t+a, 2t+a)$-IRS.

PROOF. Give the points in one group G of the TD weights 2 and 4 in such a way that

$$\sum_{x \in G} w(x) = 2t,$$

and give the other points in the TD weights 2 and 4 in such a way that

$$\sum_{x \notin G} w(x) = 2s.$$

Apply Construction 4.1. Now adjoin a infinite points to the Room frame, filling in $(2r+a, a)$-IRS for the required values of r (Lemma 4.2 ii)). This constructs the desired Room square.

COROLLARY 4.8. *Suppose there is a* TD$(6, m)$, *where* $m \ge 8$. *Let* $a = 1, 3, 5$ *or* 7; *let* $m \le t \le 2m$; *and let* $5m \le s \le 10m$. *Then there is a* $(2s+2t+a, 2t+a)$-IRS.

PROOF. Apply Theorem 4.7 and Theorems 3.1–3.3.

THEOREM 4.9. *For any odd* $w \ge 37$, *and for any odd* $v \ge (7w-5)/2$, *there is a* (v, w)-IRS.

PROOF. First, write $w = 4m_0 + a$, where m is odd and $a = 1, 3, 5$, or 7. Second, write $w = 2m_1 + a$, where m is odd and $a = 1$ or 3. We apply Corollary 4.8 for any odd value of m such that $m_0 \le m \le m_1$. Note that a TD$(6, m)$ exists for all odd $m > 5$; see [**1**]. This enables us to construct a (v, w)-IRS for all odd v where

$$10m_0 + w \le v \le 20m_1 + w.$$

Now, we have $m_0 \le (w-1)/4$ and $m_1 \ge (w-3)/2$. So, we can handle all odd v such that

$$(7w-5)/2 \le v \le 11w - 30.$$

By [**11**,Corollary 3.7], there exists a (v, w)-IRS for all odd $v \ge 6w+41$, and $11w - 30 \ge 6w + 41$ when $w \ge 15$. But $w \ge 37$, so we are done. This completes the proof.

5. Room squares with holes of size ≤ 35. We have already shown in Section 3 that for $w = 3, 5$, or 7, there is a (v, w)-IRS if and only if v is odd and $v \ge 3w+2$. For $9 \le w \le 31$, we now updat; bounds given in

TABLE 1
Bounds on the existence of Room squares with subsquares

	old bound	new bound		old bound	new bound
w	v_0	v_0	w	v_0	v_0
1	7	7	3	41	11
5	69	17	7	55	23
9	85	33	11	87	43
13	89	63	15	77	65
17	71	67	19	97	69
21	137	71	23	117	73
25	127	95	27	137	97
29	195	99	31	197	101
33	135	103	35	177	115

THEOREM 5.1. *A* $(v, 9)$-IRS *exists if* $v = 29$, *or if* v *is odd and* $v \geq 33$.

PROOF. It was shown in [11] that there exists a $(v, 9)$-IRS if v is odd, $v \geq 85$. For $v = 37, 39, 45, 49, 57, 59, 63, 69, 73, 79$, and 81, the existence of $(v, 9)$-IRS follow easily from [7, Lemma 2.4 and 2.7–2.10]. For $v = 33, 35, 41, 43, 47, 51, 53, 55, 61, 65, 67, 71, 75, 77$, and 83, we constructed $(v, 9)$-IRS using the hill-climbing algorithms described in [4]. These squares are presented in the research report [5].

THEOREM 5.2. *A* $(v, 11)$-IRS *exists if* $v \geq 43$.

PROOF. It was shown in [11] that there exists a $(v, 11)$-IRS if v is odd, $v \geq 87$. For $61 \leq v \leq 85$, the existence of $(v, 11)$-IRS follows from Theorem 4.7 with $m = 5$, $t = 5$, $a = 1$. For $v = 43, 45, 51$, and 55, the existence of $(v, 11)$-IRS follow easily from [7, Lemmata 2.4,2.5,2.7 and 2.8]. For $v = 47, 49$, and 53, we constructed $(v, 11)$-IRS using the hill-climbing algorithms described in [4]. These squares are presented in the research report [5]. It remains to construct $(57, 11)$- and $(59, 11)$-IRS. These are done by means of a frame construction. To construct a $(57, 11)$-IRS, start with a TD$(6, 5)$ and give every point weight 2, except for 3 points in one group, which get weight 0. Apply Construction 4.1, using Room frames of types 2^5 and 2^6, which exist from Theorem 2.1. A Room frame of type $10^5 6^1$ results. Now Apply Lemma 4.2 i) with $a = 1$. To construct a $(59, 11)$-IRS, proceed similarly, but give only two points weight 0. We get a Room frame

TABLE 2
Applications of Theorem 4.7

w	m	t	a	interval covered
13	5	6	1	$63 \le v \le 113$
15	5	7	1	$65 \le v \le 115$
17	5	8	1	$67 \le v \le 117$
19	5	9	1	$69 \le v \le 119$
21	5	10	1	$71 \le v \le 121$
21	7	10	1	$91 \le v \le 161$
23	5	10	3	$73 \le v \le 123$
25	7	12	1	$95 \le v \le 165$
27	7	13	1	$97 \le v \le 167$
29	7	14	1	$99 \le v \le 169$
29	9	14	1	$119 \le v \le 209$
31	7	14	3	$101 \le v \le 171$
31	9	15	1	$121 \le v \le 211$
33	7	14	5	$103 \le v \le 173$
35	8	16	3	$115 \le v \le 195$

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Appendix 1

1 – 4								12, 13	7, 11	8, 9	14, 15		5, 16		6, 10
					10, 16				5, 15		8, 13		6, 12	7, 14	9, 11
					11, 14				8, 16	10, 15		6, 9		5, 13	7, 12
				9, 12		10, 13		7, 15		5, 14	6, 16			8, 11	
12, 16			10, 14	5 – 8							4, 9	3, 15	1, 11		2, 13
		12, 15							2, 14	3, 13		11, 16	4, 10	1, 9	
11, 15	9, 14								1, 13	2, 16	3, 10			4, 12	
			13, 16					4, 11	3, 12				9, 15	2, 10	1, 14
7, 13		5, 11		3, 16		12, 14	4, 15	9 – 10		1, 6			2, 8		
	7, 16	6, 13	8, 15	4, 14	1, 12	3, 11					2, 5				
5, 9		7, 10		13, 15		4, 16	3, 14	2, 6		11 – 12		1, 8			
	5, 10					1, 15	9, 13	14, 16				2, 7		3, 6	4, 8
	8, 12	9, 16			2, 15		10, 11		4, 6		1, 7	13 – 14			3, 5
8, 10	6, 15		5, 12	2, 11	3, 9		1, 16			4, 7					
6, 14	11, 13		7, 9	1, 10			2, 12	3, 8				4, 5		15 – 16	
		8, 14	6, 11		4, 13	2, 9		1, 5				10, 12	3, 7		

A Room frame of type $2^4 4^2$

1 – 4					11, 17	9, 15	13, 16	8, 18		5, 14			6, 12		7, 10		
									6, 13	16, 18		7, 15	9, 17	5, 12	11, 14		8, 10
				9, 18			12, 17		14, 16		7, 13	8, 11	5, 15	6, 10			
				12, 16	10, 13			7, 14			15, 18				6, 17	8, 9	5, 11
			11, 16	5 – 8				15, 17				3, 10		1, 9	13, 18	2, 14	4, 12
	14, 17							3, 16				2, 12	1, 10	11, 18	4, 9		13, 15
14, 18	9, 13	10, 15									4, 17		2, 11			3, 12	1, 16
	11, 15	12, 13							2, 17		1, 14	9, 16	3, 18			4, 10	
5, 13		8, 14		1, 17		2, 16		9 – 12				4, 18				6, 15	3, 7
	6, 18	16, 17	8, 15		3, 14	4, 13									2, 5	1, 7	
			6, 14	3, 13		1, 18	4, 15					5, 17	8, 16	2, 7			
	7, 16	5, 18	13, 17		1, 15									4, 14	3, 8		2, 6
8, 17		7, 9	10, 18			3, 11		4, 6		2, 15		13 – 14			1, 12	5, 16	
12, 15	5, 10				2, 9		1, 11		7, 18	4, 8	6, 16			3, 17			
6, 9				4, 11	12, 18		10, 14	2, 13	1, 8	7, 17	3, 5			15 – 16			
	8, 12					10, 17	2, 18	1, 5		3, 6			4, 7			11, 13	9, 14
7, 11			5, 9	2, 10	4, 16	12, 14			3, 15			1, 6		8, 13		17 – 18	
10, 16		6, 11	7, 12	14, 15			3, 9		4, 5	1, 13	2, 8						

A Room frame of type $2^3 4^3$

1 – 2			12, 13		10, 16						8, 17	9, 18	5, 19	11, 20		6, 14	4, 7		3, 15
						13, 20	10, 17		14, 19				4, 18	6, 9	3, 8	7, 16	5, 12	11, 15	
		3 – 4		11, 17			9, 19		6, 13	7, 18	16, 20		8, 12			2, 10	1, 15		5, 14
					11, 18	10, 15	1, 13	8, 14		16, 19	2, 5	6, 20			12, 17			7, 9	
13, 18		2, 16		5 – 8				15, 17	3, 20			4, 12	1, 9		11, 19			10, 14	
	12, 18		11, 14					3, 19		2, 15			10, 20	1, 17			9, 16		4, 13
14, 17	3, 16									4, 20				12, 19	2, 18	9, 15	11, 13		1, 10
12, 20		9, 13	10, 18						2, 17		15, 19	1, 11					3, 14	4, 16	
8, 15	13, 19	1, 6		2, 14			16, 18	9 – 12				3, 17		4, 5	7, 20				
6, 19		8, 20	1, 7	15, 18		16, 17	4, 14						2, 3			5, 13			
	14, 20		5, 16	1, 3	4, 15							7, 19	6, 17	8, 18				2, 13	
		14, 18			13, 17	1, 19	15, 20							2, 7		4, 8		3, 5	6, 16
4, 10	6, 11		9, 17		2, 19	3, 18		5, 20				13 – 16						1, 8	7, 12
	7, 10		8, 19		1, 20	4, 11	3, 12	6, 18		5, 17									2, 9
3, 7	4, 17	10, 19		9, 20					5, 18							1, 12	2, 6		8, 11
5, 9		7, 17	2, 20	4, 19							1, 18					3, 11	8, 10	6, 12	
11, 16			6, 15		3, 9	2, 12			1, 4	8, 13	7, 14	5, 10				17 – 20			
	5, 15			10, 13	12, 14			1, 16		3, 6		2, 8	7, 11		4, 9				
	8, 9	5, 11		12, 16				2, 4	7, 15	1, 14	3, 13				6, 10				
		12, 15				9, 14	2, 11	7, 13	8, 16	4, 6				3, 10	1, 5				

A Room frame of type $2^2 4^4$

Appendix 2

							14, 12	5, 9		6, 10	16, 15	8, 11		7, 13
				9, 10	12, 7	14, 16					5, 11	6, 13	8, 15	
				11, 7			8, 13		5, 6		14, 10	12, 15	9, 16	
15, 9	10, 11	5, 14		16, 12			7, 1			8, 2	6, 4		3, 13	
6, 16						12, 9	10, 15	13, 2	14, 11	5, 4	1, 8	7, 3		
	12, 5	9, 6		13, 14	15, 3		16, 4		10, 8		2, 7			11, 1
	7, 15		6, 12		2, 10	5, 8	9, 11	1, 16	13, 4	14, 3				
			16, 3	15, 2				10, 4	9, 7		13, 12	5, 1	6, 11	8, 14
				8, 4	14, 6	1, 13				15, 11	9, 3	16, 10	7, 5	12, 2
11, 13		12, 10	15, 5	6, 1				8, 3		16, 7			14, 2	9, 4
5, 10		7, 8	14, 1			15, 4	6, 3	11, 12	16, 2	9, 13				
14, 7		13, 15	8, 9		4, 11	6, 2			12, 3				10, 1	16, 5
	9, 14		13, 10		16, 8	11, 3	5, 2	6, 7	1, 15				12, 4	
8, 12	16, 13		11, 2	5, 3	1, 9	7, 10						14, 4		6, 15
	6, 8	16, 11	7, 4		13, 5			15, 14		12, 1		9, 2		10, 3

A (15, 3)-incomplete Room square

			5, 15					18, 17	7, 11	8, 10	12, 9	14, 13			6, 16	
			10, 14		11, 15		12, 6			13, 16			9, 17	18, 5	7, 8	
			17, 7				11, 13		12, 16	5, 14	8, 18	6, 9	10, 15			
	6, 14	15, 7			16, 4	8, 1	9, 5	12, 10			13, 17				11, 2	3, 18
12, 14	8, 17			13, 15	18, 1	6, 10		5, 2		11, 4		3, 7		16, 9		
			16, 2			13, 18	10, 3	15, 9	8, 5			11, 1	4, 12	7, 6		17, 14
6, 11	13, 9		12, 3	16, 18	10, 7	17, 2							1, 5		15, 14	8, 4
	15, 18	13, 12	6, 8		9, 3	5, 4		1, 14					11, 16	10, 17		2, 7
5, 17		8, 9		1, 6		15, 12	14, 18	16, 7				10, 2		3, 11	13, 4	
	10, 16		9, 4	14, 11	6, 17			13, 3	18, 2		5, 7	8, 15		12, 1		
		16, 14		7, 9			1, 17	8, 11		18, 12	6, 2			4, 15	3, 5	13, 10
9, 10			18, 11	4, 17	12, 8		2, 15		13, 6	1, 7			3, 14			5, 16
		17, 11		8, 3	2, 14				4, 10		16, 1	12, 5	7, 13		9, 18	15, 6
18, 7				5, 10		9, 11	16, 8		15, 1	6, 3	14, 4			2, 13	17, 12	
	7, 12	5, 6	13, 1			3, 16			9, 14	17, 15	10, 11	18, 4	2, 8			
16, 15					5, 13		7, 4		17, 3	2, 9			18, 6	8, 14	1, 10	11, 12
8, 13	11, 5	10, 18		2, 12		7, 14		6, 4			3, 15	17, 16				9, 1

A (17, 3)-incomplete Room square

				20, 15	13, 5	8, 12	18, 16			7, 6			11, 10	17, 9			14, 19	
						14, 20	19, 10			8, 13		12, 7			18, 11	17, 5	9, 15	6, 16
			5, 16	9, 7				14, 8	12, 6		11, 15			20,13			10, 17	18, 19
					15, 8	17, 16		18, 5		20, 3		13, 19	9, 1		2, 14	4, 6	11, 12	7, 10
		17, 7		14, 10		3, 6	2, 8	1, 20		15, 19		11, 5	4, 16			16, 12		13, 9
13, 15			3, 9	12, 16	11, 6					17, 2	14, 7	10, 18		4, 19			8, 20	5, 1
		5, 15	20, 4		1, 14	2, 19		9, 11	18, 17		10, 3		8, 6		12, 13		16, 7	
6, 20	12, 14	16, 8		19, 11	2, 10		1, 17					4, 9	13, 18		5, 3	15, 7		
			12, 10	3, 8	18, 4				16, 9		17, 13	6, 15	19, 5		1, 7	11, 14		20, 2
5, 14	9, 10	13, 11			3, 16	15, 18		7, 19	8, 4	12, 1		20, 17					2, 6	
12, 9			19, 8	18, 6		11, 1	7, 3	10, 4	14, 13		20, 5	16, 2						15, 17
8, 10			7, 11	13, 1		5, 9		17, 6	15, 2		19, 16		12, 20	3, 18				4, 14
16, 11	20, 18		17, 14		19, 9		6, 5	12, 2						7, 8	15, 4	10, 1	13, 3	
7, 18	8, 5		13, 6		17, 12				20, 10	4, 11	2, 9		15, 14	16, 1		19, 3		
	6, 19		15, 1	17, 4		13, 10	14, 9		7, 5		8, 18			2, 11		16, 20		12, 3
		19, 12		2, 5		7, 4	11, 20	13, 16		9, 18	6, 1	3, 14		10, 15	8, 17			
	7, 13	9, 20	18, 2				12, 15		1, 19				17, 3	14, 6	10, 16		5, 4	11, 8
19, 17	16, 15	14, 18			20, 7				11, 3	10, 5	4, 12	8, 1			6, 9	13, 2		
	17, 11	10, 6					13, 4	15, 3		14, 16			7, 2	12, 5	20, 19	8, 9	1, 18	

A (19, 3)-incomplete Room square

			18, 15	21, 5		9, 12	14, 6			17, 10				22, 7		16, 13	8, 20	11, 19		
					18, 14					19, 16	21, 10		22, 11	15, 20	5, 6		13, 7		12, 8	17, 9
					6, 21	7, 15	19, 10		17, 22		5, 14	12, 13	20, 16		8, 9		11, 18			
		6, 19					1, 16	4, 9	20, 13		11, 15	17, 18				22, 8	10, 2	21, 12	7, 5	3, 14
	11, 14	13, 18	8, 1		7, 12	17, 19		22, 2		21, 3					16, 10			5, 9	6, 15	4, 20
10, 22			9, 19			5, 4	12, 20	13, 17	1, 15			16, 13	7, 6	21, 14	11, 2					8, 18
	17, 21	22, 12		20, 18	2, 13			11, 1	14, 4							10, 6	15, 5	16, 8	3, 9	7, 19
	13, 6	14, 15	7, 21		3, 22			19, 5	18, 16		12, 2	10, 9		4, 8		17, 1			11, 20	
	5, 20			10, 3	15, 16		11, 8	18, 12			1, 9	2, 7			19, 4		14, 17	22, 6	21, 13	
17, 7	12, 19		4, 6		5, 1			14, 16	9, 2	8, 13	22, 20	21, 11						18, 3		10, 15
						20, 1	18, 9	21, 15	3, 7		6, 17	19, 22	8, 2	5, 10	13, 14		12, 4			11, 16
		5, 16	17, 11	4, 22		21, 8			10, 12	9, 6	13, 3	14, 20			7, 1	19, 15			2, 18	
		11, 10	5, 3	2, 19			15, 13			14, 1	7, 8		18, 4		17, 12	20, 9	16, 6			22, 21
12, 11	8, 10			9, 13		3, 6	5, 22			20, 7		15, 4	19, 1		21, 18	14, 2			16, 17	
12, 16	7, 18		20, 2	8, 14	9, 11								17, 5	12, 6	15, 22		19, 3		10, 4	13, 1
14, 9	22, 16			12, 1		13, 11		7, 10	6, 8	5, 2	19, 18		15, 3			4, 21		17, 20		
	15, 9	8, 17	13, 22			16, 2	4, 7	20, 6	21, 19				12, 14	3, 11		18, 5		10, 1		
19, 20			10, 14	17, 15			21, 2	3, 8		18, 22		1, 6		16, 9		7, 11		4, 13		15, 12
5, 13				11, 6	19, 8	10, 18				12, 15	16, 4		9, 21	17, 2	20, 3			7, 14	22, 1	
8, 15		21, 20		16, 7	17, 4				5, 11				13, 10	1, 18		12, 3	9, 22		19, 14	6, 2
18, 6		9, 7	12, 16		20, 10	22, 14	17, 3			11, 4		8, 5		19, 13			21, 1	15, 2		

A (21, 3)-incomplete Room square

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