ORTHOGONAL 1-FACTORIZATIONS OF THE COMPLETE MULTIGRAPH

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Abstract. A 1-factorization of the complete multigraph λK_n is called *decomposable* if some proper subset of the factors forms a 1-factorization of $\lambda' K_n$ for some $\lambda' < \lambda$; otherwise it is called *indecomposable*. In this paper the notion of orthogonal 1-factorizations of λK_n is defined. We describe direct starter constructions and recursive frame constructions for orthogonal 1-factorizations. We then determine the spectrum of orthogonal indecomposable 1-factorizations of $2K_n$.

1. Introduction and definitions. The complete multigraph λK_v has v vertices and λ edges joining each pair of vertices. A *1-factor* of λK_v is a set of edges which between them contain each vertex precisely once. A *1-factorization* is a set of 1-factors which partition the edges of λK_v.

Clearly v must be even for a 1-factor to exist; say $v = 2n$. It is well known that K_{2n} has a 1-factorization for every n. Taking λ copies of this factorization yields a 1-factorization of λK_{2n}. Thus every λK_{2n} has a 1-factorization.

Given a 1-factorization F of λK_{2n}, it may be that there exists an integer λ_1 ($\lambda_1 < \lambda$) such that $\lambda_1(2n-1)$ of the 1-factors of F form a 1-factorization of $\lambda_1 K_{2n}$. In such a case F is called *decomposable*; otherwise it is termed *indecomposable*. We abbreviate indecomposable 1-factorization as IOF. A 1-factorization F of λK_{2n} is called *simple* if it has no repeated 1-factors. The following result of Colbourn, Colbourn and Rosa [3] and Archdeacon and Dinitz [1] deals with the existence of simple IOF of λK_n for small λ.

THEOREM 1.1. *A simple indecomposable 1-factorization of λK_{2n} exists as follows:*

$\lambda = 2$: *if and only if* $2n \geq 6$;
$\lambda = 3$ *or* 4: *if and only if* $2n \geq 8$;
$\lambda = 5$: *if* $2n \geq 10$;
$\lambda = 6, 8,$ *or* 9: *if* $2n \geq 12$;
$\lambda = 7$: *if* $2n \geq 16$;
$\lambda = 10$: *if* $2n \geq 14$;
$\lambda = 12$: *if* $2n \geq 32$;

In [1], Archdeacon and Dinitz also proved the following result for all λ.

THEOREM 1.2. *A simple indecomposable 1-factorization of λK_{2n} exists for all $n \geq 2(\lambda + p)$, where p is the smallest prime not dividing λ.*

Two 1-factorizations F_1 and F_2 of λK_n are said to be *orthogonal* if any 1-factor in F_1 and any 1-factor in F_2 have at most 1 edge in common. It is well known that the existence of two orthogonal 1-factorizations of K_{2n} is equivalent to a Room square of side $2n-1$, and that these exist if and only if $2n \geq 8$ [11, 15]. Clearly, if F_1 is orthogonal to F_2, then λ copies of F_1 and λ copies of F_2 will be orthogonal 1-factorizations of λK_n. However, these 1-factorizations will be decomposable and not simple. We will give constructions for orthogonal indecomposable simple 1-factorizations and will discuss the spectrum of these for $\lambda = 2$.

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Given two orthogonal 1-factorizations of λK_{2n}, F and G, we can index the rows of a $\lambda(2n-1)$ by $\lambda(2n-1)$ square L by the 1-factors in F and the columns by the 1-factors in G. Since any edge $\{i,j\} \in \lambda K_{2n}$ occurs in exactly λ 1-factors in F and λ 1-factors of G, we get a λ by λ subarray of L determined by these 1-factors. Place the pair $\{i,j\}$ in the cells of any transversal of this subarray. If this is done for every edge in the graph, the resulting square L satisfies the following properties:

1) every cell of L is either empty or contains an unordered pair of symbols from $\{1,\ldots,2n\}$,
2) each symbol of $\{1,\ldots,2n\}$ occurs once in each row and column of L,
3) every unordered pair of symbols occurs in precisely λ cells of L,
4) if two pairs of symbols occur in two cells of the same row (column), then they do not occur in two cells of any column (row).

We define a *λ-square* $\lambda S(2n,\lambda)$ to be a $\lambda(2n-1)$ by $\lambda(2n-1)$ array satisfying the above four properties. In Figure 1 below we give a λ-square $\lambda S(12,2)$.

It is clear that a Room square of side n is a $\lambda S(n,1)$. λ-squares can also be thought of in terms of other generalizations of Room squares. A λ-square $\lambda S(n,\lambda)$ is a uniform generalized Room square grs $S_2(n,\lambda;n+1)$ (see Rosa [12]) and is also a Room rectangle $RR(n,n;[2,1,1]-[n+1,n+1,n+1],2,[\lambda,1,1])$ [9,12]. It is also a Kirkman square $KS_2(n+1,1,\lambda)$ (see [10,14]). Actually, each of the above squares need only satisfy properties 1, 2 and 3 but not necessarily property 4 of the definition of λ-squares. The relationship between λ-squares and orthogonal 1-factorizations of λK_n is analogous to the relationship between Room squares and orthogonal 1-factorizations of K_n and is given in the following theorem.

THEOREM 1.3. *The existence of two orthogonal 1-factorizations of λK_{2n} is equivalent to the existence of a λ-square $\lambda S(2n-1,\lambda)$.*

Proof. In the discussion preceding the definition of λ-square, it was shown that the existence of two orthogonal 1-factorizations of λK_{2n} implies the existence of a λ-square $\lambda S(2_n-1,\lambda)$. Given a λ-square $\lambda S(2n-1,\lambda)$ it is also clear that the rows and columns give two orthogonal 1-factorizations of λK_{2n}. Note that condition 4 assures the orthogonality of the row and column 1-factorizations. □

Obviously, λ-squares $\lambda S(n,\lambda)$ can be constructed for all odd $n \geq 7$ and all λ by merely putting λ copies of a Room square of side n down the diagonal of a $\lambda \times n$ by $\lambda \times n$ array and leaving the remainder of the array empty. Note that in this

FIGURE 1

A λ-square $\lambda S(12,2)$

0,∞	9,10	6,7										5,8			1,3						2,4
	1,∞	10,0	7,8								3,5		6,9			2,4					
		2,∞		8,9			0,1					4,6		7,10			3,5				
			3,∞	1,2	9,10								5,7		8,0			4,6			
				4,∞	2,3	10,0								6,8		9,1			5,7		
			0,1		5,∞	3,4									7,9		10,2			6,8	
						6,∞	4,5	1,2								8,10		0,3			7,9
							7,∞	5,6	2,3		8;10						9,0		1,4		
									6,7	3,4		9,0						10,1	8,∞	2,5	
4,5									9,∞	7,8			10,1						0,2		3,6
8,9	5,6									10,∞	4,7			0,2						1,3	
				3,7			8,2	10,4			0,∞								6,9		1,5
			2,6		4,8			9,3	0,5			1,∞								7,10	
						5,9			10,4	1,6		3,7	2,∞								8,0
2,7							6,10			0,5	9,1		4,8	3,∞							
1,6	3,8							7,0				10,2		5,9	4,∞						
	2,7	4,9							8,1				0,3		6,10	5,∞					
		3,8	5,10							9,2				1,4		7,0	6,∞				
10,3			4,9	6,0											2,5		8,1	7,∞			
	0,4			5,10	7,1			8,∞								3,6		9,2			
		1,5			6,0	8,2											4,7		10,3	9,∞	
						7,1	9,3				2,6							5,8		0,4	10,∞

construction both the resulting row and column 1-factorizations are decomposable and not simple. Recently, Lamken [10] has constructed indecomposable Kirkman squares $KS_2(n+1, 1, \lambda)$ for all $n \geq 6$ and all $\lambda \geq 1$. In our context, these are squares satisfying properties 1, 2, and 3 above which can not be decomposed into Kirkman squares with smaller λ. In that construction, the row and column 1-factorizations are never simple but are possibly indecomposable.

We define a *simple indecomposable λ-square* $SI\lambda S(n,\lambda)$ to be a $\lambda S(n,\lambda)$ in which the row and column 1-factorizations are both simple and indecomposable.

In this paper we will show the existence of $SI\lambda S(n,2)$ for odd $n \geq 11$. In order to construct λ-squares for these orders we will require a direct construction to get the smaller orders and then a recursive construction for the larger orders. The direct construction is the λ analogue of starters (see [15, 5, 6]) and is discussed in Section 2 while the recursive construction is a frame type construction (see [7]) and is developed in Section 3. In Section 4 we give the spectrum of simple indecomposable λ-squares with $\lambda = 2$.

2. Starter construction. Let G be an additive abelian group of odd order n. A λ-starter in G (of order n) is a set $S = \{S_1, S_2, \ldots S_\lambda\}$ where each S_i is a set of unordered pairs $S_i = \{\{s_{ij}, t_{ij}\}, 1 \leq j \leq \frac{1}{2}(n-1)\}$ and which satisfies the following properties:

1) $\bigcup_j (\{s_{ij}\} \cup \{t_{ij}\}) = G - \{0\}$, for each $1 \leq i \leq \lambda$, and
2) Each $g \in G - \{0\}$ occurs as a difference $\pm(s_{ij} - t_{ij})$, $1 \leq i \leq \lambda$, $1 \leq j \leq \frac{1}{2}(n-1)$, exactly λ times.

We give an example from [3].

EXAMPLE 2.1. A 3-starter in the group Z_{11}.

$$S_1 = \{2,3\},\ \{4,5\},\ \{8,9\},\ \{6,10\},\ \{1,7\}$$
$$S_2 = \{1,7\},\ \{2,9\},\ \{4,6\},\ \{3,5\},\ \{8,10\}$$
$$S_3 = \{1,4\},\ \{2,7\},\ \{3,10\},\ \{5,8\},\ \{6,9\}$$

Note that on the vertices labelled by $Z_{11} \cup \{\infty\}$ that $S_i \cup \{0, \infty\}$ is a 1-factor for each i. Each translate of S_i, $S_i + g = \{s_{ij} + g,\ t_{ij} + g\} \cup \{g, \infty\}$ is also a 1-factor for every $g \in G$. In general we get the following relationship between λ-starters and 1-factorizations of λK_n.

THEOREM 2.2. *If there exists a λ-starter of order n then there exists a 1-factorization of λK_{n+1}.*

Proof. Let $S = \{S_1, S_2, \ldots, S_\lambda\}$ be a λ-starter of order n. As noted above, each S_i generates n 1-factors $S_i + g$ of λK_{n+1}. Each pair $\{a, b\} \subset G$, occurs exactly λ times in these 1-factors by property (2) of the definition. The pair $\{\infty, g\}$ also occurs exactly λ times as $\{\infty, g\} \in S_i + g$ for $1 \leq i \leq \lambda$. □

Notice that if the S_i's are distinct then the resulting 1-factorization will be simple. Under certain conditions the 1-factorization generated by the λ-starter will also be indecomposable. One of those conditions is given in the following theorem. We will say that a difference $d \in G$ is *entirely contained* in S_i if all λ occurrences of pairs with difference d are in S_i.

THEOREM 2.3. *Let $S = \{S_1, S_2, ..., S_\lambda\}$ be a λ-starter in the group G, $|G| = n$. If there exists a difference $d \in G$ which is entirely contained in S_i for some i, and if $GCD(\lambda, n) = 1$, then S generates an indecomposable 1-factorization of λK_{n+1}.*

Proof. Assume difference d is entirely contained in S_i. Also assume that the resulting 1-factorization F of λK_{n+1} contains a 1-factorization F' of $\lambda_1 K_{n+1}$ where $\lambda_1 \leq \lambda$. The total number of pairs occurring in F' with difference d is $\lambda_1 \times n$. If t translates of S_i are contained in F', then we get that the total number of pairs occurring in F' with difference d is $t \times \lambda$. Equating, we get $\lambda_1 \times n = t \times \lambda$. Since GCD$(\lambda, n) = 1$ and since $\lambda_1 \leq \lambda$, we conclude that $\lambda_1 = \lambda$ and thus F is indecomposable. □

Notice that in Example 2.1 the difference 1 is entirely contained in S_1. Since GCD(3,11) = 1, we have by Theorem 2.3 that this starter generates a simple indecomposable 1-factorization of K_{12}. If a λ-starter S generates an indecomposable 1-factorization we will say S is an ***indecomposable*** starter. Next we will concentrate on the concept of orthogonal λ-starters.

Let $S = \{S_1, S_2, ..., S_\lambda\}$ and $T = \{T_1, T_2, ..., T_\lambda\}$ be λ-starters of order n (in the group G). We say that S and T are ***orthogonal*** if the following two conditions hold:

1) S_i and T_j have no pairs in common for all $1 \leq i, j \leq \lambda$,
2) If $\{a, b\}, \{e, f\} \in S_i$ with $b - a = d_1$ and $f - e = d_2$, and there exists a j such that $\{x, y\}, \{z, w\} \in T_j$ with $y - x = d_1$ and $w - z = d_2$, then we must have that $z - x \neq e - a$.

Notice again that the above definition is the same as the usual definition for orthogonal starters when $\lambda = 1$ (see [4,5,15]) for results on orthogonal starters).

EXAMPLE 2.4. Two orthogonal 2-starters in Z_{11}.

$$S_1 = \{6,7\},\ \{9,10\},\ \{1,3\},\ \{2,4\},\ \{5,8\}$$
$$S_2 = \{6,9\},\ \{3,7\},\ \{1,5\},\ \{10,4\},\ \{8,2\}$$
$$T_1 = \{4,5\},\ \{8,9\},\ \{10,3\},\ \{2,7\},\ \{1,6\}$$
$$T_2 = \{3,5\},\ \{8,10\},\ \{4,7\},\ \{9,1\},\ \{2,6\}$$

Just as orthogonal starters generate orthogonal 1-factorizations of K_n, orthogonal λ-starters generate orthogonal 1-factorizations of λK_n and thus also λ-squares $\lambda S(n-1, \lambda)$. We state the following theorem and leave the proof to the interested reader.

THEOREM 2.5. *If there exist t orthogonal λ-starters of order n, then there exist t pairwise orthogonal 1-factorizations of λK_{n+1}.*

The λ-square in Figure 1 was constructed by use of the two orthogonal 2-starters in Z_{11} given in Example 2.4. The rows of the square are the 1-factors obtained from translates of S_1 and S_2 and the columns are generated from the translates of T_1 and T_2. Combining Theorems 2.5 and 1.2 gives the following result concerning λ-squares.

PROPOSITION 2.6. *If there exists two orthogonal λ-starters of order n, then there exists a λ-square $\lambda S(n-1, \lambda)$. Furthermore, if the starters are both simple and indecomposable then there exists a $SI\lambda S(n-1, \lambda)$.*

Orthogonal indecomposable λ-starters can be constructed on the computer by means of a modified version of the hill-climbing algorithm for strong starters described in [6]. In the appendix we give two orthogonal 2-starters in Z_{2n-1} for $6 \leq n \leq 30$ (orthogonal 2-starters for $K_{2n}, 6 \leq n \leq 30$). Since GCD$(2, 2n-1) = 1$, by Theorem 2.3 we have that each of these starters is indecomposable. They are also clearly simple. Thus, by Proposition 2.6 we have

THEOREM 2.7. *There exists a $SI\lambda S(2n-1, 2)$ for all $6 \leq n \leq 30$.*

3. Subsquares and frame construction. Before beginning our discussion of frames, we first must make some remarks concerning subsquares. Let L be a λ-square $\lambda S(n, \lambda)$. A square $w\lambda$ by $w\lambda$ subarray M of L is said to be a ***sub-λ-square*** if it is itself a λ-square $\lambda S(w, \lambda)$ on a subset of $w + 1$ symbols. Of importance to our constructions will be the existence of sub-λ-squares of order 1 (i.e. $\lambda S(1, \lambda)$). In the case of Room squares it is obvious that there always exists a subsquare of side 1. It is not quite so obvious to see this when $\lambda \neq 1$, so we will give the short proof of this fact.

LEMMA 3.1. *Every $\lambda S(n, \lambda)$ contains a sub-$\lambda S(1, \lambda)$.*

Proof. Let L be a $\lambda S(n, \lambda)$ and let $\{a, b\}$ be any pair in L. Rearrange the rows and columns of L so that the first λ of the rows and of the columns are the ones which contain $\{a, b\}$. Call this λ by λ square M. We will be done if we can show that all the cells of M which do not contain the pair $\{a, b\}$ are empty. Assume the pair $\{c, d\} \neq \{a, b\}$ occurs in M. But since $\{a, b\}$ and $\{c, d\}$ both occur in a row *and* a column of L this violates condition 4 of the definition of λ-square and the result follows. □

The following result will be used to show that larger squares are indecomposable.

THEOREM 3.2. *If L is a $\lambda S(n, \lambda)$ which contains an indecomposable sub-$\lambda S(m, \lambda)$, then L is indecomposable.*

Proof. Let $M \subseteq L$ be an indecomposable sub-$\lambda S(m, \lambda)$. Since any decomposition of L induces a decomposition of M, the result follows. □

Our main recursive construction requires the use of a generalization of a Room square called a Room frame. Room frames have been extremely useful in problems related to Room squares [4, 7, 8] and are examples of the usefulness of "holes" in combinatorial designs. Let S be a set, and let $\{S_1, ..., S_n\}$ be a partition of S. An $\{S_1, ..., S_n\}$-*Room frame* is an $|S|$ by $|S|$ array, F, indexed by S which satisfies the following properties:

1) every cell of F either is empty or contains an unordered pair of symbols of S,
2) the subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are referred to as holes),
3) each symbol of $S - S_i$ occurs once in row (column) s, for any $s \in S_i$, and
4) the pairs occurring in F are precisely those $\{s, t\}$, where

$$(s,t) \in (S \times S) - \bigcup_{1 \leq i \leq n} (S_i \times S_i).$$

The type of F is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. We usually use an "exponential" notation to describe types: a type $t_i^{u_i} \ldots t_k^{u_k}$ denotes u_i occurences of t_i, $1 \leq i \leq k$. A Room frame can be thought of as a Room square from which a spanning set of subsquares has been removed. A Room frame of type 1^v gives rise to a Room square of side v by filling in each diagonal cell (s, s) with the pair $\{\infty, s\}$, where ∞ is a new symbol. In Figure 2 we give a Room frame of type $2^5 4^1$.

We will make use of the following frames which were recently constructed by Dinitz and Stinson.

THEOREM 3.3. [8, THEOREM 2.3]. *There exist Room frames of types* 2^6, $2^5 4^1$, $2^4 4^2$, $2^3 4^3$, $2^2 4^4$, $2^1 4^5$, *and* 4^6.

In order to make use of these frames we need to define some design-theoretic terminology. Let X be a set of points. A group-divisible design (or GDD) is a triple (X, G, A), which satisfies the following properties:

1) G is a partition of X into subsets called *groups*,
2) A is a set of subsets of X (called *blocks*) such that a group and a block contain at most one point in common, and
3) every pair of points from distinct groups occurs in a unique block.

The *group-type*, of a GDD(X, G, A) is the multiset $\{|G| : G \in G\}$. As with Room frames, we shall use an exponential notation for group types, for convenience. We will say that a GDD is a k-GDD if $|A| = k$ for every $A \in A$.

A *transversal design* TD(k, m) can be defined to be a k-GDD of type m^k. It is well-known that a TD(k, m) is equivalent to $k-2$ mutually orthogonal Latin squares of order m. For results on Latin squares and transversal designs we refer to [2]. The following is the main recursive construction for Room frames.

FIGURE 2

A Room Frame of Type $2^5 4^1$

1 - 2			10, 12		3, 13			8, 11	7, 14		4, 6	5, 9	
		7, 11		9, 13		6, 14	4, 12			8, 10	3, 5		
9, 12		3 - 4		8, 14		5, 11		7, 13		1, 6			2, 10
	9, 14				10, 11			6, 12	1, 13	5, 7		2, 8	
		8, 12	1, 14	5 - 6		10, 13	9, 11			2, 3		4, 7	
4, 13	3, 12	10, 14							2, 11		8, 9		1, 7
3, 11		5, 13		2, 12	4, 14	7 - 8					1, 10		6, 9
			2, 13	1, 11				3, 14	5, 12	4, 9		6, 10	
5, 14	4, 11					1, 12	6, 13	9 - 10			2, 7		3, 8
	8, 13		6, 11		7, 12		2, 14					1, 3	4, 5
6, 8	5, 10			3, 7	2, 9			1, 4		11 - 14			
	6, 7	1, 9	5, 8			2, 4	3, 10						
			7, 9	4, 10	1, 8			2, 5	3, 6				
7, 10		2, 6				3, 9	1, 5		4, 8				

Construction 3.4. [13, Construction 2.2] Let (X, G, A) be a GDD, and let w: $X \to Z^+ \cup \{0\}$ (we say that w is a *weighting*). For every A$\in$ A, suppose that there is a Room frame of type $\{w(x) : x \in A\}$. Then there is a Room frame of type $\{\sum_{x \in G} w(x) : G \in G\}$.

Once we have constructed the Room frames, we can fill in the holes. The next theorem is essentially [13, Corollary 4.9], however, we will take multiple copies of the frame in order to obtain a λ-square.

LEMMA 3.5. *Suppose there is a Room frame of type* $t_i^{u_i} \ldots t_k^{u_k}$. *For* $1 \leq i \leq k$, *suppose there is a simple* λ-*square* $\lambda S(t_i + 1, \lambda)$. *Then there is a simple* $\lambda S(\sum_{1 \leq i \leq k} t_i u_i + 1, \lambda)$. *Furthermore, if any one of the* λ-*squares* $\lambda S(t_i + 1, \lambda)$ *is indecomposable, then so is the resulting* λ-*square.*

Proof. Let F be an $\{S_i, \ldots S_n\}$ frame of type $t_i^{u_i} \cdots t_k^{u_k}$ where $\bigcup_{1 \le i \le n} S_i = S$, $\sum_{1 \le i \le n} u_i = n$ and $|S| = s$. Let $T_1 = T_2 = \cdots = T_\lambda = S$ be λ identical copies of S and let L be a $\lambda s + \lambda$ by $\lambda s + \lambda$ array indexed by $(\bigcup_{1 \le i \le n} T_i) \cup \{\infty_1, \cdots, \infty_\lambda\}$. We begin filling in L by defining $L(x, y) = F(x, y)$ if $(x, y) \in \bigcup_{1 \le i \le \lambda} T_i^2$.

Next we must fill in the "holes" in the frames. By hypothesis there is a simple λ-square $\lambda S(|S_i| + 1, \lambda)$ for each i, $1 \le i \le n$. Thus by Lemma 3.1 there exists a simple $\lambda S(|S_i| + 1, \lambda)$ containing a subsquare of order 1 for each i, $1 \le i \le n$. Call these λ-squares $\Lambda_1, \ldots, \Lambda_n$ where Λ_i is indexed by $\{\infty_1, \ldots \infty_\lambda\} \cup S_i$ and each Λ_i has its sub-λ-square indexed by the ∞'s. Λ_i uses the symbol set. $S_i \cup \{\infty\} \cup \{\omega\}$ where ω is a new symbol and has $\{\infty, \omega\} = \Lambda_i(\infty_j, \infty_j)$ for $1 \le j \le \lambda$. (Note that there are λ copies of the indices S_i, one in each T_i). Define $L(x, y) = \Lambda_i(x, y)$ if $(x, y) \in (\{\infty_1 \ldots, \infty_\lambda\} \cup S_i) \times (\{\infty_1, \ldots, \infty_\lambda\} \cup S_i)$.

It may be checked that L is the desired λ-square. Furthermore, since each Λ_i is a sub-λ-square, if any of these squares are indecomposable, then by Theorem 3.2, L is indecomposable. Note that this construction basically just puts λ copies of the frame down the diagonal of L and then fills in the holes with the small λ squares.

We apply Construction 3.4 and Lemma 3.5 to the frames given by Theorem 3.3 to obtain the following theorem, which is our main recursive construction.

THEOREM 3.6. *Suppose there exists a $TD(6, m)$, and suppose there exists a $SI\lambda S(2r + 1, \lambda)$ for all r such that $m \le r \le 2m$. Then there exists a $SI\lambda S(2s+1, \lambda)$ for all $6m \le s \le 12m$.*

Proof. Let (X, G, A) be a $TD(6, m)$. Give the points in the TD weights 2 and 4 in such a way that $\sum_{x \in X} w(x) = 2s$. Apply construction 3.4 with the Room frames supplied by Theorem 3.3. Note that the "holes" in this frame are of size $2r$ for $2m \le 2r \le 4m$. Now use Lemma 3.5 and the hypothesized $SI\lambda S$'s to complete the construction. □

4. The Spectrum. In this section we will discuss the spectrum of $SI\lambda S(n, 2)$

LEMMA 4.1. *There exists a $SI\lambda S(2s + 1, 2)$ for $30 \le s \le 732$.*

Proof. We will use Theorem 3.6 to construct $SI\lambda S(2s + 1, 2)$. The ingredients are a $TD(6, m)$ and a $SI\lambda S(2r + 1, \lambda)$ for all r such that $2m + 1 \le 2r + 1 \le 4m + 1$.

Consider the following table

m	$2r + 1$	$2s + 1$	Comments
5	11–21	61–121	11-21 exist by Theorem 2.7
9	19–37	108–217	19-37 exist by Theorem 2.7
17	35–69	205–408	Theorem 2.7 and Line 1 in this table
31	63–125	372–745	Lines 1 and 2 in this table
61	123–245	732–1465	Lines 2 and 3 in this table

THEOREM 4.2. *There exists a $SI\lambda S(2s + 1, 2)$ for all $s \ge 5$.*

Proof. If $5 \le s \le 372$ the result follows from Theorem 2.7 and Lemma 4.1. We assume $s \ge 732$ and proceed by induction. Let $m = [s/10]$, where [ ] denotes the greatest integer function. Then, since $m \ge 73$, there exists a TD(6, m) [2]. By induction, there exist $SI\lambda S(2r + 1, 2)$ for all $m \le r \le 2m$. Thus, by Theorem 3.6 there exists a $SI\lambda S(2s + 1, 2)$. □

COROLLARY 4.3. *There exist two orthogonal indecomposable 1-factorizations of $2K_{2n}$ for all $n \ge 6$.*

Conclusion. In this paper we have constructed orthogonal indecomposable 1-factorizations of $2K_{2n}$ for all $n \ge 6$. We used starter techniques to construct the small orders of n and then used recursive frame techniques to complete the spectrum. We would have liked to solve this problem for all λ, however, finding pairs of orthogonal starters for $\lambda > 2$ proved to be difficult (on the computer). We would like to note that all the recursive constructions in Section 3 work for *all* λ. Thus, if methods can be found to find small orders for a given λ, then the recursive techniques of this paper can be used to construct the large orders.

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APPENDIX

orthogonal λ-starters of $2K_{12}$

S_1: 6,7 9,10 2,4 1,3 5,8
S_2: 6,9 3,7 1,5 10,4 8,2
T_1: 4,5 8,9 10,3 2,7 1,6
T_2: 3,5 8,10 4,7 9,1 2,6

orthogonal λ-starters of $2K_{14}$

S_1: 1,2 11,12 3,5 8,10 4,7 6,9
S_2: 5,9 10,1 3,8 2,7 6,12 11,4
T_1: 9,10 4,5 3,6 11,2 7,12 8,1
T_2: 4,6 5,7 11,1 8,12 10,2 3,9

orthogonal λ-starters of $2K_{16}$

S_1: 7,8 4,5 10,12 1,3 6,9 11,14 13,2
S_2: 6,11 9,14 1,7 12,3 10,2 13,5 4,8
T_1: 2,3 12,13 5,8 7,10 11,1 4,9 14,6
T_2: 9,11 12,14 3,7 1,5 2,8 4,10 6,13

orthogonal λ-starters of $2K_{18}$

S_1: 8,9 14,15 11,13 3,5 1,4 7,10 12,16 2,6
S_2: 8,13 7,12 14,3 4,10 15,5 16,6 1,9 11,2
T_1: 4,5 1,2 10,13 12,15 3,8 6,11 7,14 9,16
T_2: 8,10 9,11 3,7 14,1 16,5 13,2 4,12 15,6

orthogonal λ-starters of $2K_{20}$

S_1: 4,5 8,9 1,3 11,13 12,15 7,10 17,2 14,18 16,6
S_2: 2,7 9,14 6,12 11,17 3,10 13,1 16,5 15,4 18,8
T_1: 17,18 9,10 4,7 5,8 11,16 15,1 14,2 6,13 3,12
T_2: 3,5 7,9 12,16 11,15 14,1 2,8 10,18 17,6 4,13

orthogonal λ-starters of $2K_{22}$

S_1: 3,4 12,13 17,19 6,8 7,10 15,18 16,20 5,9 14,2 1,11
S_2: 8,13 19,3 16,1 6,12 2,9 11,18 7,15 17,4 5,14 10,20
T_1: 11,12 14,15 19,1 13,16 2,7 4,9 3,10 20,6 17,5 8,18
T_2: 18,20 14,16 1,5 4,8 9,15 7,13 11,19 2,10 3,12 17,6

orthogonal λ-starters of $2K_{24}$

S_1: 21,22 4,5 8,10 13,15 7,12 18,1 19,3 9,17 11,20 6,16 14,2
S_2: 6,9 22,2 7,11 14,18 8,13 15,21 17,1 19,4 3,12 10,20 5,16
T_1: 14,15 18,19 10,13 5,9 16,21 1,7 20,4 3,11 22,8 2,12 6,17

T_2: 6,8 9,11 17,20 10,14 21,3 7,13 18,2 16,1 19,5 12,22 4,15

orthogonal λ-starters of $2K_{26}$

S_1: 12,13 23,24 4,6 16,18 15,20 21,2 3,10 9,17 5,14 1,11 22,8 7,19

S_2: 7,10 19,22 17,21 24,3 1,6 9,15 11,18 12,20 5,14 23,8 2,13 4,16

T_1: 6,7 19,20 21,24 11,15 12,17 10,16 1,8 22,5 4,13 18,3 23,9 2,14

T_2: 12,14 2,4 15,18 3,7 17,22 10,16 13,20 1,9 24,8 21,6 19,5 11,23

orthogonal λ-starters of $2K_{28}$

S_1: 1,2 9,10 18,20 22,24 16,21 8,14 4,11 26,7 6,15 13,23 19,3 5,17 12,25

S_2: 9,12 15,18 2,6 16,20 21,26 5,11 24,4 14,22 10,19 25,8 23,7 1,13 17,3

T_1: 5,6 16,17 7,10 19,23 3,8 20,26 15,22 21,2 9,18 4,14 1,12 13,25 11,24

T_2: 13,15 12,14 4,7 26,3 23,1 19,25 17,24 8,16 2,11 10,20 22,6 9,21 5,18

orthogonal λ-starters of $2K_{30}$

S_1: 16,17 9,10 12,14 1,3 22,27 5,11 18,25 20,28 26,6 13,23 4,15 19,2 24,8 7,21

S_2: 10,13 23,26 3,7 20,24 6,11 15,21 9,16 17,25 28,8 4,14 19,1 22,5 18,2 27,12

T_1: 20,21 11,12 1,4 28,3 9,14 19,25 8,15 23,2 17,26 6,16 13,24 27,10 5,18 22,7

T_2: 8,10 7,9 17,20 21,25 14,19 28,5 4,11 23,2 15,24 3,13 16,27 18,1 22,6 12,26

orthogonal λ-starters of $2K_{32}$

S_1: 30,28 20,22 8,9 17,18 21,23 12,14 5,10 29,4 25,1 11,19 15,24 27,6 2,13 26,7 3,16

S_2: 22,25 3,6 7,11 5,9 21,26 2,8 13,20 15,23 18,27 19,29 1,12 16,28 4,17 10,24 30,14

T_1: 26,27 12,13 19,22 30,3 4,9 17,23 8,15 21,29 11,20 28,7 5,16 2,14 24,6 18,1 10,25

T_2: 24,26 11,13 18,21 10,14 7,12 23,29 1,8 9,17 28,6 20,30 25,5 15,27 22,4 2,16 19,3

orthogonal λ-starters of $2K_{34}$

S_1: 14,15 21,22 3,6 19,23 2,7 24,30 5,12 17,25 28,4 32,9 18,29 8,20 31,11 13,27 1,16 10,26

S_2: 9,11 29,31 24,27 3,7 30,2 15,21 12,19 14,22 8,17 16,26 28,6 25,4 10,23 32,13 5,20 18,1

T_1: 27,28 5,6 17,20 18,22 9,14 25,31 30,4 11,19 7,16 2,12 23,1 29,8 13,26 10,24 21,3 32,15

T_2: 13,15 28,30 16,19 6,10 7,12 8,14 20,27 21,29 22,31 1,11 24,2 26,5 23,3 4,18 17,32 9,25

orthogonal λ-starters of $2K_{36}$

S_1: 33,34 22,23 13,15 18,20 5,10 31,2 7,14 9,17 19,28 29,4 21,32 24,1 3,16 11,25 26,6 [illegible] 30,12

S_2: 2,5 23,26 28,32 12,16 20,25 18,24 3,10 21,29 33,7 4,14 11,22 31,8 6,19 34,13 15,30 [illegible] 27,9

T_1: 4,5 20,21 24,27 22,26 25,30 32,3 10,17 6,14 9,18 1,11 2,13 19,31 34,12 15,29 28,8 [illegible] 16,33

T_2: 10,12 11,13 24,27 17,21 29,34 14,20 23,30 28,1 32,6 33,8 26,2 4,16 9,22 5,19 3,18 [illegible] 25,7

orthogonal λ-starters of $2K_{38}$

S_1: 13,14 17,18 9,12 3,7 27,32 36,5 16,23 35,6 24,33 20,30 8,19 22,34 25,1 15,29 26,4 31,10 11,28 21,2

S_2: 33,35 11,13 27,30 5,9 19,24 22,28 18,25 8,16 23,32 7,17 1,12 29,4 34,10 26,3 6,21 36,15 14,31 2,20

T_1: 31,32 12,13 35,1 20,24 21,26 10,16 8,15 25,33 9,18 4,14 6,17 27,2 29,5 22,36 19,34 28,7 23,3 30,11

T_2: 1,3 26,28 7,10 14,18 29,34 35,4 25,32 12,20 15,24 21,31 2,13 11,23 9,22 5,19 30,8 27,6 16,33 36,17

orthogonal λ-starters of $2K_{40}$

S_1: 34,35 26,27 33,36 21,25 2,7 16,22 12,19 30,38 6,15 3,13 37,9 8,20 11,24 18,32 28,4 1,17 14,31 5,23 10,29

S_2: 6,8 26,28 24,27, 14,18 11,16 31,37 22,29 33,2 25,34 7,17 4,15 1,13 23,36 35,10 5,20 3,19 21,38 30,9 32,12

T_1: 18,19 7,8 9,12 27,31 20,25 32,38 6,13 26,34 1,10 5,15 22,33 17,29 28,2 23,37 35,11 14,30 4,21 24,3 36,16

T_2: 13,15 34,36 25,28 26,30 4,9 23,29 3,10 16,24 5,14 1,11 7,18 19,31 38,12 21,35 32,8 17,33 20,37 27,6 22,2

orthogonal λ-starters of $2K_{42}$

S_1: 10,11 29,30 9,12 27,31 39,3 40,5 18,25 26,34 23,32 7,17 4,15 35,6 20,33 14,28 21,36 8,24 37,13 1,19 38,16 2,22

S_2: 22,24 23,25 28,31 36,40 4,9 8,14 35,1 5,13 7,16 17,27 19,30 26,38 21,34 33,6 3,18 37,12 39,15 2,20 10,29 32,11

T_1: 34,35 8,9 3,6 38,1 17,22 25,31 26,33 2,10 15,24 11,21 29,40 4,16 7,20 23,37 12,27 14,30 19,36 28,5 13,32 39,18

T_2: 2,4 18,20 36,39 34,38 3,8 11,17 16,23 29,37 19,28 40,9 24,35 13,25 14,27 7,21 32,6 26,1 5,22 15,33 12,31 10,30

orthogonal λ-starters of $2K_{44}$

S_1: 21,22 7,8 3,6 27,31 13,18 34,40 4,11 29,37 26,35 15,25 5,16 41,10 39,9 24,38 17,32 20,36 28,2 12,30 14,33 42,19 23,1

S_2: 28,30 9,11 4,7 12,16 26,31 32,38 18,25 37,2 24,33 13,23 40,8 10,22 1,14 34,5 27,42 19,35 3,20 21,39 41,17 29,6 15,36

T_1: 5,6 17,18 33,36 24,28 30,35 23,29 13,20 42,7 37,3 21,31 8,19 40,9 34,4 1,15 11,26 39,12 10,27 14,32 22,41 25,2 38,16

T_2: 23,25 40,42 26,29 30,34 12,17 4,10 8,15 19,27 28,37 39,6 41,9 20,32 35,5 7,21 1,16 38,11 14,31 18,36 3,22 13,33 24,2

orthogonal λ-starters of $2K_{46}$

S_1: 40,41 30,31 19,22 43,2 12,17 38,44 7,14 28,36 9,18 25,35 21,32 3,15 16,29 42,11 24,39 4,20 33,5 37,10 8,27 6,26 13,34 1,23

S_2: 14,16 27,29 34,37 20,24 7,12 11,17 21,28 2,10 39,3 41,6 35,1 26,38 40,8 22,36 18,33 44,15 13,30 31,4 23,42 5,25 43,19 32,9

T_1: 43,44 24,25 12,15 33,37 26,31 16,22 35,42 32,40 10,19 7,17 30,41 8,20 36,4 14,28 39,9 2,18 6,23 3,21 27,1 38,13 29,5 34,11

T_2: 32,34 8,10 35,38 44,3 4,9 15,21 20,27 16,24 17,26 23,33 1,12 29,41 30,43 37,6 13,28 31,2 5,21
7,25 40,14 36,11 18,39 42,19

orthogonal λ-starters of $2K_{48}$

S_1: 32,33 23,24 40,43 14,18 37,42 39,45 31,38 21,29 10,19 7,17 44,8 3,15 12,25 35,2 5,20 6,22 41,11
16,34 27,46 28,1 9,30 4,26 13,36

S_2: 7,9 42,44 37,40 10,14 1,6 13,19 28,35 4,12 27,36 22,32 20,31 3,15 17,30 41,8 24,39 29,45 21,31
16,34 46,18 23,43 5,26 11,33 2,25

T_1: 10,11 26,27 39,42 31,35 2,7 45,4 22,29 16,24 19,28 15,25 1,12 38,3 30,43 46,13 6,21 36,5 17,31
14,32 37,9 20,40 23,44 33,8 18,41

T_2: 8,10 19,21 39,42 24,28 11,16 34,40 22,29 1,9 6,15 26,36 2,13 33,45 37,3 32,46 12,27 4,20 14,31
25,43 35,7 18,38 23,44 30,5 41,17

orthogonal λ-starters of $2K_{50}$

S_1: 29,30 16,17 28,31 44,48 8,13 34,40 36,43 18,26 38,47 9,19 35,46 42,5 2,15 10,24 12,27 39,6 3,20
7,25 22,41 33,4 11,32 1,23 14,37 21,45

S_2: 38,40 41,43 32,35 46,1 8,13 23,29 17,24 37,45 10,19 15,25 7,18 30,42 47,11 34,48 21,36 12,28 3,20
4,22 44,14 6,26 33,5 9,31 16,39 27,2

T_1: 23,24 45,46 15,18 48,3 30,35 32,38 47,5 43,2 22,31 40,1 28,39 8,20 12,25 13,27 11,26 17,33 4,21
41,10 36,6 14,34 37,9 7,29 42,16 44,19

T_2: 29,31 5,7 33,36 15,19 1,6 18,24 37,44 45,4 13,22 20,30 21,32 39,2 46,10 12,26 42,8 9,25 35,3 16,31
28,47 23,43 27,48 38,11 40,14 17,41

orthogonal λ-starters of $2K_{52}$

S_1: 48,49 28,29 30,33 42,46 8,13 31,37 3,10 12,20 36,45 22,32 5,16 6,18 25,38 39,2 11,26 7,23 35,1
9,27 21,40 50,19 34,4 43,14 24,47 44,17 41,15

S_2: 11,13 45,47 1,4 16,20 26,31 12,18 25,32 42,50 48,6 14,24 10,21 37,49 33,46 29,43 23,38 44,9 17,31
35,2 22,41 19,39 7,28 5,27 36,8 30,3 15,40

T_1: 16,17 38,39 50,2 9,13 10,15 36,42 14,21 27,35 49,7 34,44 32,43 28,40 11,24 12,26 4,19 41,6 29,41
5,23 18,37 25,45 33,3 30,1 8,31 47,20 48,22

T_2: 8,10 34,36 42,45 14,18 38,43 15,21 41,48 27,35 16,25 44,3 49,9 7,19 24,37 12,26 31,46 17,33 39,1
11,29 4,23 32,1 50,20 6,28 30,2 40,13 22,47

orthogonal λ-starters of $2K_{54}$

S_1: 39,40 47,48 5,8 28,32 21,26 37,43 50,4 15,23 10,19 45,2 11,22 6,18 25,38 51,12 27,42 20,36 14,31
34,52 41,7 24,44 49,17 13,35 46,16 9,33 29,1 30,3

S_2: 49,51 46,48 42,45 16,20 6,11 23,29 14,21 26,34 19,28 30,40 43,1 27,39 5,18 47,8 9,24 15,31 38,1
32,50 3,22 17,37 44,12 41,10 13,36 33,4 35,7 52,25

T_1: 20,21 29,30 35,38 4,8 49,1 9,15 33,40 31,39 18,27 7,17 23,34 43,2 19,32 50,11 37,52 6,22 46,10
47,12 26,45 28,48 3,24 36,5 44,14 42,13 16,41 25,51

T_2: 18,20 9,11 24,27 37,41 52,4 7,13 43,50 21,29 17,26 35,45 22,33 46,5 2,15 30,44 39,1 16,32 19,31
49,14 42,8 28,48 38,6 12,34 40,10 23,47 31,3 25,51

orthogonal λ-starters of $2K_{56}$

S_1: 13,14 3,4 23,26 2,6 5,10 37,43 46,53 21,29 7,16 38,48 9,20 35,47 32,45 19,33 27,42 34,50 8,25
52,15 22,41 31,51 28,49 17,39 1,24 30,54 11,36 18,44 40,12

S_2: 2,4 40,42 38,41 45,49 46,51 44,50 32,39 54,7 10,19 26,36 11,22 5,17 15,28 21,35 18,33 53,14 20,37
13,31 48,12 27,47 3,24 8,30 6,29 1,25 9,34 52,23 16,43

T_1: 20,21 39,40 10,13 48,52 37,42 23,29 31,38 11,19 18,27 41,51 53,9 3,15 46,4 16,30 7,22 33,49 26,43
54,17 50,14 36,1 24,45 25,47 44,12 8,32 35,5 2,28 34,6

T_2: 29,31 54,1 34,37 24,28 9,14 10,16 6,13 38,46 32,41 52,7 12,23 47,4 45,3 25,39 20,35 26,42 27,44
48,11 2,21 40,5 15,36 51,18 30,53 50,19 8,33 17,43 22,49

orthogonal λ-starters of $2K_{58}$

S_1: 15,16 46,47 39,42 14,18 17,22 3,9 12,19 56,7 54,6 38,48 44,55 20,32 24,37 35,49 11,26 45,4 10,27
25,43 34,53 21,41 29,50 8,30 36,2 28,52 33,1 5,31 13,40 23,51

S_2: 13,15 23,25 29,32 2,6 11,16 56,5 43,50 41,49 10,19 45,55 33,44 14,26 17,30 4,18 39,54 20,36 21,38
42,3 28,47 46,9 31,52 12,34 35,1 27,51 40,8 22,48 37,7 53,24

T_1: 51,52 43,44 17,20 12,16 49,54 41,47 8,15 24,32 19,28 25,35 26,37 33,45 1,14 53,10 3,18 11,27 22,39
46,7 23,42 36,56 9,30 48,13 40,6 38,5 34,2 29,55 4,31 50,21

T_2: 51,53 29,31 33,36 26,30 55,3 19,25 34,41 1,9 47,56 13,23 21,32 8,20 15,28 40,54 48,6 52,11 18,35
44,5 24,43 7,27 38,2 39,4 46,12 49,16 17,42 45,14 10,37 22,50

orthogonal λ-starters of $2K_{60}$

S_1: 15,16 43,44 32,35 36,40 7,12 54,1 21,28 26,34 39,48 52,3 22,33 51,4 25,38 50,5 57,13 2,18 20,37
24,42 10,29 11,31 46,8 56,19 45,9 41,6 30,55 47,14 49,17 58,27 53,23

S_2: 29,31 39,41 16,19 17,21 4,9 8,14 49,56 40,48 26,35 32,42 7,18 34,46 47,1 22,36 57,13 11,27 45,
2,20 33,52 44,5 50,12 15,37 28,51 58,23 30,55 43,10 38,6 25,53 54,24

T_1: 8,9 19,20 3,6 26,30 35,40 11,17 15,22 4,12 48,57 13,23 25,36 32,44 39,52 41,55 58,14 50,7 28,4
31,49 27,46 33,53 56,18 42,5 24,47 37,2 29,54 34,1 16,43 10,38 51,21

T_2: 23,25 32,34 28,31 3,7 44,49 55,2 50,57 10,18 15,24 42,52 26,37 48,1 4,17 5,19 30,45 22,38 54,1
33,51 8,27 21,41 14,35 36,58 56,20 29,53 47,13 39,6 43,11 40,9 46,16