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ORTHOGONAL EDGE COLORINGS OF GRAPHS

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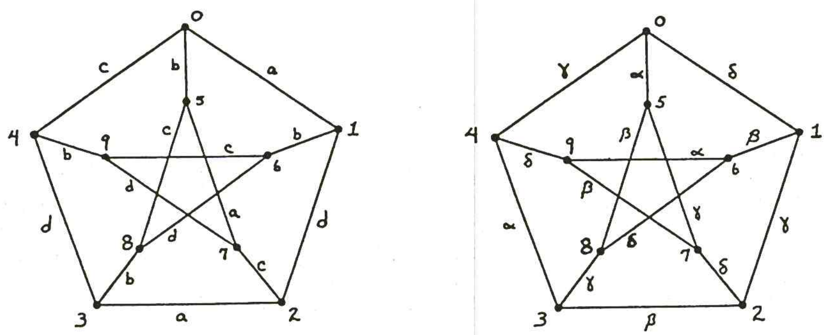
Abstract

We study proper edge colorings of graphs. Two colorings are orthogonal if any two edges which receive the same color in one coloring receive different colors in the other coloring. Let G be a graph which admits a pair of orthogonal colorings using n and m colors. We derive some necessary and some sufficient conditions on G, n and m. A relationship with certain combinatorial designs is discussed, in particular Room squares, Howell designs and orthogonal Latin squares are special cases of orthogonal colorings. Some open problems are presented.

1. Introduction

As usual in [9], whose notation and terminology we generally follow, a graph G contains no loops or multiple edges, is finite and undirected, and has p vertices and q edges. An <u>edge coloring</u> of a graph is a partitioning of the edges into color classes such that no two edges in the same color class are adjacent. For brevity, a <u>coloring</u> of G means an edge coloring below. Observe that two colorings are distinct if there exists a pair of edges which receives the same color in one coloring yet receives different colors in the other coloring. We say two colorings are <u>orthogonal</u> if every pair of edges which receives the same color in one of the colorings receives different colors in the other coloring. Equivalently, note that two colorings are orthogonal if and only if every pair of colors (one from the first coloring and one from the second coloring) determines at most one edge. As an example we give two orthogonal

4-colorings of the Petersen graph P in Figure 1.1. Note that edges 05, 16, 38 and 49 all receive color b in coloring (a) and receive colors α, β, γ and δ respectively in (b).



(a) (b)

Orthogonal colorings of P

Figure 1.1

	a	b	c	d
α		05	69	34
β	23	16	58	79
γ	57	38	04	12
δ	01	49	27	68

Tabular presentation of the orthogonal colorings of the preceding figure.

Figure 1.2

We immediately present another way to record a pair of orthogonal colorings. Let A be a $k_1 \times k_2$ rectangular array, where the rows of A are indexed by the k_1 colors of one coloring and the columns by the k_2 colors in the second coloring. Place edge e in cell (i,j) if and only if e receives color i in the first coloring and color j in the second. Thus every edge is in some cell. Note that since each coloring is proper, no vertex will appear twice in a row or column (partially row and column latin). Also note that since the colorings are orthogonal, each cell is either empty or assigned a unique edge.

We formalize the above discussion.

Proposition 1.1. The existence of two orthogonal colorings of a graph G using k_1 and k_2 colors is equivalent to the existence of a $k_1 \times k_2$ array V which satisfies the properties

i. each cell of V is empty or contains an edge of G,

ii. each edge in G is in exactly one cell of V,

iii. each vertex of G occurs at most once in any row or column of V. ∎

Figure 1.2 displays the 4 x 4 square constructed by use of Proposition 1.1 from the two orthogonal colorings of the Petersen graph given in Figure 1.1.

Using Proposition 1.1, we note that orthogonal colorings of certain types of graphs correspond to interesting types of combinatorial designs. For example, a pair of orthogonal n-colorings of $K_{n,n}$ corresponds to a pair of orthogonal Latin squares of side n. By reversing the described construction, we see that these colorings exist for all $n \neq 2,6$. Similarly, a pair of orthogonal (2n-1) colorings of K_{2n} yields a Room square of side 2n-1. These are also known to exist for all $2n - 1 \geq 7$, see [11]. Recently, necessary and sufficient conditions for the existence of a Howell design H(s,2n) (where $n \leq s \leq 2n-1$) were proven, see [1]. In our terminology, this implies (with four exceptional cases) the existence of an s-regular graph on 2n vertices, which admits a pair of orthogonal s-colorings for every $n \leq s \leq 2n-1$. We will discuss Latin squares, Room squares and Howell designs in Section 5.

Our purpose is to prove several existence results for orthogonal colorings of graphs. In some cases graphs satisfying certain parameter sets are constructed, in other cases we treat the graph as given. Section 2 examines orthogonal colorings using different numbers of colors, i.e., colorings where the corresponding arrays are rectangular. Section 3 concentrates on orthogonal colorings using the same number of colors, i.e., the arrays are square. Section 4 examines some higher dimensional analogues using three or more colorings. Section 5 gives small examples. We conclude in Section 6 with some interesting variations and open problems.

2. Rectangles

Suppose that we are given a graph G and integers k_1 and k_2. In this section we prove some necessary and some sufficient conditions for G to admit a pair of orthogonal colorings using k_1 and k_2 colors.

Two necessary conditions come to mind immediately. First, G must have an edge coloring in each of the desired number of colors. Since the colorings are to be proper, we require $k_1, k_2 \geq \Delta$, the maximum degree. Recall that χ', the <u>chromatic index</u>, is the minimum number of colors needed to edge color G. By the theorem of Vising $\chi' = \Delta$ or $\Delta + 1$. A graph is said to be in class 1 or class 2, respectively. So slightly strengthening our earlier observation, we actually need $k_1, k_2 \geq \chi'$. For the second observation, recall that in a pair of orthogonal colorings each pair of colors determines at most one edge. Thus we orthogonal colorings each pair of colors determines at most one edge. Thus we need $k_1 k_2 \geq q$.

It is interesting to examine which necessary condition is stronger for a given graph. For purposes of discussion, suppose that G is an r-regular class 1 graph and suppose that $k_2 \geq k_1 = \chi' = r$. In this case, if $r > p/2$, then each of the k_1 color classes contains strictly fewer than k_1 edges, i.e. $k_2 \geq \chi'$ implies that $k_1 k_2 > q$. If, on the other hand, $r < p/2$, then each of the k_1 color classes contains more than k_1 edges, i.e., $k_1 k_2 > q$ implies that $k_2 > \chi'$. In particular, if r is small in relation to p, then k_2 exceeds k_1 by a large amount. In the third case, if $r = p/2$, then $k_2 = k_1$ if and only if $k_1 k_2 = q$.

We now turn our attention to some sufficient conditions on G, k_1, and k_2 which give the desired colorings. A coloring is called <u>balanced</u> if the number of edges in color class i differs from that in color class j by at most 1 for any i and j. The following theorem is an easy consequence of that of Folkman and Folkerson [7].

<u>Theorem 2.1</u> There exists a balanced k coloring of G for all $k \geq \chi'$. ∎

We will use the coloring given by Theorem 2.1 for the first of our two colorings. We now begin our construction of the second of our two colorings. The following is an easy corollary of P. Hall's Theorem (see [8]).

<u>Lemma 2.2</u> Let H be a bipartite graph with vertex set $V_1 \cup V_2$ and set $p_i = |V_i|$ and $n = \min\{p_1, p_2\}$. Suppose that $\deg(v_1) + \deg(v_2) \geq n$ for all $v_1 \in V_1, v_2 \in V_2$. Then H has a matching of size n. ∎

Theorem 2.3 If k_1, k_2 are integers with $k_2 \geq 2k_1(\Delta-1)$, $k_1 \geq \chi'$, and $k_1 k_2 \geq q$, then G has a pair of orthogonal colorings using k_1 and k_2 colors respectively.

Proof. By Theorem 2.1 since $k_1 \geq \chi'$ we can find a balanced coloring of G using k_1 colors, say $a_1, \ldots, a_{k_1}$. Since $q \leq k_1 k_2$ each class a_i has at most k_2 edges. We will construct an orthogonal coloring by successively coloring the edges in color class a_i for $i = 1, \ldots, k_1$.

Suppose those edges of G in color classes $a_1, \ldots, a_{i-1}$ have already been assigned a color in the second coloring, and let V_1 be the set of edges in color class a_i. Let V_2 be the set of colors in the second coloring, say $b_1, \ldots, b_{k_2}$. We form a bipartite graph H on the vertex set $V_1 \cup V_2$ by joining $e \in V_1$ to $b_j \in V_2$ if and only if e is not adjacent (in G) to an edge which has already received color b_j. Observe that a proper assignment of colors to the edges in V_1 is just a matching in H of size $|V_1|$. To find this matching we will use Lemma 2.2 (note that $|V_1| \leq k_2 = |V_2|$). Let $e \in V_1$. To bound $\deg_H(e)$ we note that e is adjacent in G to at most $2(\Delta-1)$ other edges. Thus there are at least $k_2 - 2(\Delta-1)$ colors available for e, i.e., $\deg_H(e) \geq k_2 - 2(\Delta-1)$. Next, let $b_j \in V_2$. There are at most $k_1 - 1$ edges in color class b_j. Each of these are incident with at most $2(\Delta-1)$ edges in V_1. Thus at least $|V_1| - (k_1-1)2(\Delta-1)$ of the edges of V_1 can receive color b_j, i.e., $\deg_H(b_j) \geq |V_1| - (k_1-1)2(\Delta-1)$.

Thus

$$\deg_H(e) + \deg_H(b_j) \geq [k_2 - 2(\Delta-1)] + [|V_1| - (k_1-1)2(\Delta-1)]$$
$$= |V_1| + k_2 - 2k_1(\Delta-1).$$

Since by hypothesis $k_2 \geq 2k_1(\Delta-1)$, Lemma 2.2 gives the desired matching. ∎

The most interesting case in graph colorings is to color r-regular graphs in r colors. We state the following corollary of Theorem 2.3.

Corollary 2.4 Every class 1 r-regular graph on 2n vertices has a pair of orthogonal colorings using r and n colors respectively, provided that

$$n \geq 2r\,(r-1). \quad \blacksquare$$

We believe that the bound in Corollary 2.4 can be considerably improved.

3. Squares

We now examine orthogonal colorings which use the same number of colors. We first establish a result which holds for any graph G. Let $\chi_2'(G)$ denote the minimum k such that G has a pair of orthogonal k-colorings.

Theorem 3.1 For any graph G,

$$\lceil\sqrt{q}\,\rceil \leq \chi_2'(G) \leq 2\,(\Delta-1) + \lceil\sqrt{q}\,\rceil.$$

Proof. Set $k = \chi_2'(G)$. The lower bound follows immediately from the observation that each pair of colors determines at most one edge, so $k^2 \geq q$.

The upper bound is not much harder. Suppose that we have 2 sets of $2(\Delta-1) + \lceil\sqrt{q}\,\rceil$ colors available (one for the first coloring and one for the second). We will color the edges of G one by one. We need to show that when coloring the i'th edge e, there exists a pair of colors (one from each coloring) such that no edge adjacent to e has received either of these colors, and this pair of colors has not been previously assigned to an edge. There are at most $2\,(\Delta-1)$ edges adjacent to e. Thus there are at least $\lceil\sqrt{q}\,\rceil$ colors available in each coloring which have not been assigned to an edge adjacent to e. Hence we get at least $\lceil\sqrt{q}\,\rceil^2$ pairs of such colors. Since at most q-1 edges have been previously assigned a pair of colors, we can find an 'unused' pair of colors in this set. Thus we can color the i'th edge, and the upper bound is proven. ∎

It should be clear from the proof of (3.1) that the upper bound is probably far from tight, i.e., that $\chi_2'(G)$ is usually much closer to the lower bound. The authors know of only a few graphs in which the lower bound is not equal to χ_2', namely $K_{2,2}$, $K_{6,6}$, $K_2 \times K_3$, $K_2 \times K_5$, $K_2 \times K_7$ among those with $\Delta \le p/2$, and K_6 and K_8 for those with $\Delta > p/2$. Note that in the former case the cardinality of the edge set in each graph is a perfect square. Also, a pair of orthogonal 9-colorings of $K_2 \times K_9$ can be constructed from a skew Room Square of side 9, so the apparent sequence $K_2 \times K_{2n+1}$ of these exceptions does not generalize. A characterization of graphs with $\chi_2'(G)$ not equal to the lower bound would be interesting, but is expected to be extremely difficult.

We now turn our attention to orthogonal colorings in which the graph is not fixed beforehand. We choose in this case to study only regular graphs. Define a V-square, V(k, r, v), as an r-regular graph on v vertices together with a pair of orthogonal k-colorings. Which triples (k, r, v) are the parameters of a V-square? Recall that under the correspondence with arrays given in Proposition 1.1, we obtain a k x k array, each cell empty or filled with an unordered pair such that each symbol occurs r times and no symbol occurs twice in a row or column. Figure 1.2 is thus a V(4,3,10) whose underlying graph is the Petersen graph.

Some necessary conditions are immediate. We first note that $k \ge r$. Also, at least one of v, r must be even. Finally, a simple counting argument shows that $k^2 \ge rv/2 = q$. A triple (k,r,v) satisfying these three necessary conditions will be called a legal triple. We conjecture that, with only a few small exceptions, every legal triple is the parameter set for a V-square, i.e., these necessary conditions are sufficient.

An interesting special case of this conjecture is the external case, when $2k^2 = rv$. We say such a square is crowded. Note that every cell in a crowded array is filled, i.e., every pair of colors is assigned to some edge. The existance of crowded V-squares was shown in [2] where the following theorem was proven.

Theorem 3.2 There exists a V(k,r,v) for all triples with $k \geq r$ and $2k^2 = vr$ except for a V(2,2,4). ▮

As further evidence towards our conjecture we offer:

Theorem 3.3 Let (k,r,v) be a legal triple with $k \geq 4r(r-1)$. Then there exists a V(k,r,v).

Proof. Break the k x k array into r x k rectangles and one k_1 x k rectangle as shown in Figure 3.1, where $r < k_1 \leq 2r$. We will fill in each rectangle with suitable edges on disjoint graphs. For the r x k rectangles we note that it is easy to find a class 1 r-regular graph on 2k vertices, for since $k \geq 2r(r-1)$ the rectangles exist by Corollary 2.4. Note that each of these rectangles has no empty cells. We now proceed to fill in the k_1 x k rectangle. Let $\alpha = k^2 - rv/2$ = the number of empty cells in the array. We may assume $\alpha \leq rk$, since if $\alpha \geq rk$ we may leave one r x k rectangle completely empty.

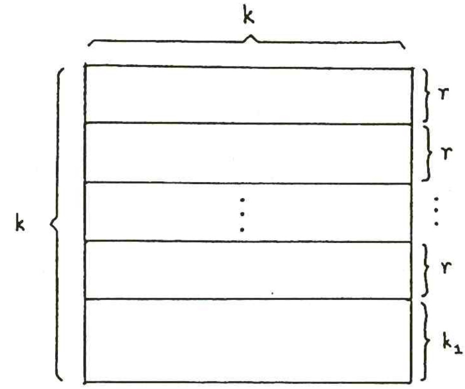


Figure 3.1

Set $v_1 = 2\ (k_1 k - \alpha)/r$. Since $k_1 \equiv k \pmod r$ and $2(k^2 - \alpha) \equiv 0 \pmod r$, v_1 is an integer. Morever, v_1 and r are not both odd. Let G be any r-regular graph on v_1 vertices. We have $\chi'(G) \leq r + 1 \leq k_1$. Also, $k_1 \leq 2r$ which implies that $2\ k_1\ (r-1) \leq 4r\ (r-1) \leq k$ (the last inequality follows by hypothesis). The desired rectangle now exists by Theorem 2.3. ∎

Corollary 3.4 For each fixed r the necessary conditions are sufficient except for at most finitely many triples.

Proof. If $k \geq 4r(r-1)$, the desired V-square exists by Theorem 3.3. For each fixed $k < 4r(4-1)$ there are only finitely many values for v with $2k^2 - rv \geq 0$, the V-squares may or may not exist for these values. ∎

4. Sets of Orthogonal Colorings

Here we will discuss sets of pairwise orthogonal edge colorings of a graph

G. An edge coloring is trivial if all edges receive different colors. Clearly a trivial edge coloring is orthogonal to any other edge coloring. A natural question to ask is, given a graph G, what is the maximum number of pairwise orthogonal nontrivial edge colorings of G? In general this number is hard to determine although we will conjecture a lower bound. To motivate the general conjecture we will prove the next statement.

Theorem 4.1 Let G be an r-regular graph on 2n vertices. If $C_1, C_2, \ldots, C_N$ is a largest possible set of pairwise orthogonal nontrivial edge colorings of G with C_1 using r colors, then $N \geq n-1$.

Proof. Since this set is maximal and pairwise orthogonal, each pair of nonadjacent edges must receive the same color in exactly one of the colorings C_i. G has $\binom{nr}{2} - 2n\binom{r}{2}$ pairs of nonadjacent edges. Let $\alpha(C_i)$ be the number of pairs of edges in G which receives the same color in coloring C_i. Thus $\alpha(C_1) = r\binom{n}{2}$ and we have

$$\binom{nr}{2} - 2n\binom{r}{2} = \sum_{i=1}^{N} \alpha(C_i) = r\binom{n}{2} + \sum_{i=2}^{N} \alpha(C_i).$$

If $i \geq 2$ then C_i can have at most r edges in any one color class since C_i is orthogonal to C_1. Let r_i be the number of colors in coloring C_i, and let s_j, $1 \leq j \leq r_i$ be the number of edges in color class j. Thus $\alpha(C_i) = \sum_{j=1}^{r_i} \binom{s_j}{2}$, where $s_j \leq r$ and $\sum_{j=1}^{r_i} s_j = nr$. It follows that $\alpha(C_i) \leq n\binom{r}{2}$ for $2 \leq i \leq N$. Thus we have

$$\binom{nr}{2} - 2n\binom{r}{2} \leq r\binom{n}{2} + \sum_{i=2}^{N} n\binom{r}{2} \leq r\binom{n}{2} + (N-1)\,n\binom{r}{2}.$$

This simplifies to $N \geq n-1$. ∎

We give two examples where this bound is tight. The first is $K_{n,n}$ where n is a prime power. There exist n-1 orthogonal 1-factorizations (n-colorings) of $K_{n,n}$ corresponding to the n-1 mutually orthogonal Latin squares of side n which exist since n is a prime power. Note that in this case $r = n = \chi'$, and each coloring has exactly n colors.

The second example is C_{2n}, the cycle with 2n edges. Here of course $\chi'(C_{2n}) = 2$ so the first coloring C_1 is the unique 2 coloring of C_{2n}. If the edges are labeled $e_0, e_1, \ldots, e_{2n-1}$, then define n-2 more colorings $C_2, \ldots, C_{n-1}$ by $C_i(e_{2j}) = j$, $C_i(e_{2j+2i-1}) = j$ for $0 \leq j \leq n-2$, $2 \leq i \leq n-1$. It is easy to check that $C_1, C_2, \ldots, C_{n-1}$ are n-1 orthogonal nontrivial edge colorings of C_{2n} with C_1 using two colors.

Theorem 4.1 and these examples lead to our conjecture.

<u>Conjecture 4.2</u> If G is r regular on 2n vertices and if $C_1 \ldots, C_N$ is a largest set of orthogonal nontrivial colorings, then $N \geq n-1$.

Large sets of pairwise orthogonal colorings of G, where each coloring uses exactly $\chi'(G)$ colors, have been studied extensively when $G = K_{n,n}$ or $G = K_n$. When $G = K_{n,n}$, any t orthogonal colorings enclosing n colors correspond to t mutually orthogonal Latin squares of order n (see Brouwer [5]). When $G = K_n$, any t orthogonal colorings of G each using n-1 colors correspond to a t-dimensional Room square of order n. (See for example Dinitz [6]).

Another application of sets of orthogonal colorings is in the area of message authentication. Brickell [4] defines an orthogonal multi-array, OMA $(k, n, r_1, \ldots, r_k)$ to be an array $A = (a_{ij})$ with n^2 rows and k columns satisfying:

i. a_{ij} is an r_j - subset of the set $\{1,2,\ldots,nr_j\}$, and

ii. Given integers j_1 and j_2 with $1 \le j_1 < j_2 \le k$ and integers s_1, s_2 with $1 \le s_i \le nr_{j_1}$ and $1 \le s_2 \le nr_{j_2}$, there exists exactly one row i such that $s_1 \in a_{ij_1}$ and $s_2 \in a_{ij_2}$.

It is then proven that the existence of an OMA$(k,n,r_1,\ldots,r_k)$ implies the existence of a doubly perfect Cartesian authentication system with k messages and probability of failure of the system equaling 1/n. It is easy to show that the existence of k orthogonal edge colorings of any n-regular graph on 2n vertices with each coloring using exactly n colors implies the existence of an OMA (k + 1, n, 1,1,1,...,1,2). As a nice example, Brickell constructs an OMA (4,6,1,1,1,2) by finding three orthogonal colorings of a 6 regular graph on 12 vertices. Incidentally, the graph is the dodecahedron together with edges joining the pairs of antipodal vertices.

Another area for further study is in sequences of pairwise orthogonal colorings of a graph. Let $|C_i|$ denote the number of colors in coloring C_i. We say that $C_1, C_2,\ldots, C_n$ form a <u>greedy sequence</u> of colorings provided that:

i. No C_i is trivial,

ii. The number of colors used in C_1 is $\chi'(G)$,

iii. For $1 < t \le n$, C_t uses the minimum number of colors possible for C_t to be orthogonal to colorings using $|C_1|, |C_2|,\ldots,|C_{t-1}|$ colors.

Let M(G) be the number of colorings in a maximal greedy sequence of colorings. Theorem 4.1 proves that if G is r-regular on n vertices and $\chi'(G) = r$, then $M(G) \ge n-1$.

We have worked out the greedy sequence of colorings for some small graphs. We present our results without proofs in the following table.

Graph	M(G)	$C_1, C_2, \ldots, C_{M(G)}$
C_{2n}	n-1	2, n, n, n,...,n
C_5	3	3, 3,-4
C_7	4	3, 3, 4, 6
C_9	5	3, 3, 5, 5, 8
K_5	4	5, 6, 6, 8

5. Some Small Examples

We have noted that some design theoretic objects (orthogonal Latin squares, Howell designs, Room squares) can be regarded as orthogonal edge colorings of certain graphs. For each of these objects the spectrum of existent orders is known. In this section we will construct the "closest" approximations to the nonexistent orders.

I. Orthogonal Latin squares

As observed before, a pair of orthogonal Latin squares, OLS, of order n is equivalent to a pair of orthogonal edge colorings of $K_{n,n}$ where each coloring uses n colors. It is well known that OLS exist for all orders n except n= 2 or n = 6. Thus by Proposition 1.1 we cannot place the edges of $K_{2,2}$ in a 2 x 2 array where the rows correspond to one coloring and the columns correspond to the other. The best that can be done is two orthogonal colorings of $K_{2,2}$ using three colors each. Similarly, there exist two orthogonal colorings of $K_{6,6}$ where one color uses six colors and the other uses seven (G.H.J. Van Rees, private communication). These colorings are exhibited by use of Proposition 1.1 in the following figure.

$0\bar{0}$	$1\bar{1}$	
		$0\bar{1}$
		$1\bar{0}$

$K_{2,2}$ in a 3 x 3 array

$6\bar{5}$	$2\bar{2}$		$5\bar{6}$	$4\bar{4}$	$1\bar{1}$	$3\bar{3}$
$1\bar{2}$	$6\bar{3}$	$4\bar{5}$		$2\bar{6}$	$3\bar{4}$	$5\bar{1}$
$4\bar{6}$	$3\bar{5}$	$6\bar{1}$	$1\bar{3}$		$5\bar{2}$	$2\bar{4}$
	$1\bar{6}$	$5\bar{3}$	$6\bar{4}$	$3\bar{1}$	$2\bar{5}$	$4\bar{2}$
$5\bar{4}$		$3\bar{6}$	$2\bar{1}$	$6\bar{2}$	$4\bar{3}$	$1\bar{5}$
$2\bar{3}$	$4\bar{1}$	$1\bar{4}$	$3\bar{2}$	$5\bar{5}$	$6\bar{6}$	

$K_{6,6}$ in a 6 x 7 array

Orthogonal colorings of two complete bigraphs.

Figure 5.1

II. Room squares

A Room square of order 2n-1 is equivalent to two orthogonal colorings of K_{2n} where each coloring uses exactly 2n-1 colors. It is known that Room squares exist for all positive odd orders $2n-1 \neq 3,5$. Thus it is impossible to put K_4 in a 3 x 3 array or K_6 in a 5 x 5 array. Below we give K_4 in a 5 x 4 array and K_6 in a 6 x 5 array.

12			
	13		
		14	23
34			
	24		

12	34	56		
54	16	23		
36	52	14		
			13	24
			26	15
			53	46

Orthogonal colorings of two complete graphs.

Figure 5.2

The graph K_{2n-1} for $n \geq 4$ has two orthogonal colorings each using 2n-1 colors. These colorings can be obtained by deleting a symbol from the Room square resulting from orthogonal colorings of K_{2n}. This is best possible since K_{2n-1} has (2n-1) (n-1) edges and at most n-1 edges can receive the same color in any coloring. Below we give K_3 in a 3 x 3 array and K_5 in a 6 x 5 array.

12		
	13	
		23

12				35
	13	25		
		14	23	
34			15	
				24
	45			

Figure 5.3

Orthogonal colorings of two additional complete graphs.

III. Howell Designs

A Howell design H(s,2n) is equivalent to two orthogonal colorings of an s-regular graph G on 2n vertices with the property that each coloring uses exactly s colors.

Howell designs are thus a generalization of both Room squares ($G = K_{2n}$) and orthogonal Latin squares ($G = K_{n,n}$). The orders for which Room squares do not exist are also orders for which Howell designs do not exist. Also, the Howell design H(2,4) does not exist. It is known [1] that there is exactly one other order, H(5,8), for which a Howell design does not exist. Thus it is impossible to find two orthogonal 5-colorings of <u>any</u> 5-regular graph on eight vertices. In Figure 5.4 we exhibit two orthogonal colorings of such a graph where one coloring uses five colors (rows) and the other uses six (columns).

		37	48	15	26
		16	25	47	38
46	35			28	17
27	18			36	45
13	24	58	67		

Figure 5.4

6. Conclusion

We have introduced the topic of orthogonal edge colorings of a graph and explored some necessary and some sufficient conditions. We take this opportunity to offer several variations on this theme.

First, there is nothing special about the role of edge colorings in the definition of orthogonality. In particular, let π_1 and π_2 be two partitions of a set. We say π_1 is orthogonal to π_2 if any two elements which lie in the same part in π_1 lie in different parts in π_2. Two orthogonal edge colorings are thus just two orthogonal partitionings of the edge set into independent sets of edges.

Many other examples of orthogonal partitions of a graph have been studied. A balanced tournament design [14] is a pair of orthogonal partitions of the edges of K_{2n} with the first partition consisting of a (2n-1) edge coloring of K_{2n} and the second partition consisting of exactly n parts where each part is a graph G with $\deg_G(v) = 1$ or 2 for all $v \in V(G)$. A Kotzig factorization [9] is a pair of orthogonal partitions of the edge of the graph K_{2n+1} with the first partition being a Hamiltonian decomposition and the second a near one-factorization. It has been noted that Room squares and Howell designs are examples of orthogonal partitions of a graph. It follows that most of the generalizations of these objects can also be considered in this manner. See

Rosa [12] for many examples of generalized Room squares.

Of course, the underlying graph can be a hypergraph. An example which has been studied is that of a Kirkman square [14]. Here the edges of the underlying hypergraph correspond to the blocks in a resolvable (v,k,1) design, and each partition of the edges consists of (v-1)/(k-1) parts where each part is a set of v/k independent edges.

Some specific small graphs have been studied extensively. In [3] a list of all sets of three orthogonal 1-factorizations of K_{10} is given. Recently, Rosa and Stinson [13] list all pairs of orthogonal 1-factorizations of every regular graph on fewer than 12 vertices.

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