Orthogonal One-Factorization Graphs

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ABSTRACT

An orthogonal one-factorization graph (OOFG) is a graph in which the vertices are one-factorizations of some underlying graph H, and two vertices are adjacent if and only if the one-factorizations are orthogonal. An arbitrary finite graph, G, is realizable if there is an OOFG isomorphic to G. We show that every finite graph is realizable as an OOFG with underlying graph K_n for some n. We also discuss some special cases.

1. INTRODUCTION

All graphs in this paper will be assumed to be finite simple graphs. Let H be a graph. A *one-factor* (perfect matching) of H is a spanning subgraph which is regular of degree 1. A *one-factorization* $\mathscr{F}$ of H is a partition of the edges of H into one-factors. Two one-factorizations $\mathscr{F}_1$, $\mathscr{F}_2$ of H are *orthogonal* if given any one-factors $F_1 \in \mathscr{F}_1$ and $F_2 \in \mathscr{F}_2$ then F_1 and F_2 have at most one edge in common. Given any graph, H, and a collection of one-factorizations $\mathscr{F}_1, \ldots, \mathscr{F}_n$ of H, we construct a graph G by letting the vertices of G be $\mathscr{F}_1, \ldots, \mathscr{F}_n$ and $(\mathscr{F}_i, \mathscr{F}_j)$ is an edge of G if $\mathscr{F}_i$ and $\mathscr{F}_j$ are orthogonal one-factorizations of H. Call G an *orthogonal-one-factorization graph* (OOFG) with underlying graph H. In this article we will discuss OOFG with underlying graphs K_n and $K_{n,n}$ (the complete graph on n points and the complete bipartite graph with bipartitions both of size n, respectively).

If G is an arbitrary finite graph, say that G is *realizable* as an OOFG with underlying graph H if there is an OOFG (with underlying graph H) isomorphic to G.

OOFG's with underlying graph $K_{n,n}$ have been studied by Lindner et

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al. [9]. These graphs were termed *orthogonal Latin square graphs*, since there is a natural connection between orthogonal one-factorizations of $K_{n,n}$ and orthogonal Latin squares of order n. Phrased in our terminology their main results can be stated as

Theorem 1.1 (Lindner et al. [9]). Every finite graph is realizable as an OOFG with underlying graph $K_{n,n}$ for some n.

Theorem 1.2 (Lindner et al. [9]). Let G be any graph. There exists an integer $v = v(G)$ such that for all $n \geqslant v$, G is realizable as an OOFG with underlying graph $K_{n,n}$.

In this paper we prove the analogous results for OOFG with underlying graph K_n. We first need the connection between one-factorizations of K_n and certain Latin squares.

Definition 1.3. Two Latin squares R and C, of side n, are *orthogonal symmetric* Latin squares if they satisfy the following three properties:

(1) R and C are both symmetric;

(2) R and C are both idempotent (i.e., $R(i, i) = C(i, i) = i$);

(3) If R and C have (i, j) entries, a and b, respectively, where $i \leqslant j$, then there are not numbers k and m, $k < m$, for which R and C have (k, m) entries a and b, respectively, unless $k = i$ and $m = j$.

Two Latin squares satisfying property (3) above are as close as possible to being orthogonal (in the usual sense of Latin squares) without sacrificing symmetry, thus the term orthogonal symmetric Latin squares. This definition was originally given by Gross et al. [7]. Also note that condition 1 above implies that n is necessarily odd.

Let $\mathscr{F}$ be a one-factorization of K_{n+1} with one-factors $F_1, \ldots, F_n$. There is a one-to-one correspondence between one-factorizations of K_{n+1} and $n \times n$ symmetric idempotent Latin squares. This correspondence is given by $L(i, j) = k$ if and only if edge $\{i, j\}$ is in F_k, also $L(i, i) = i$. The following theorem is easy to check and was originally noticed by Horton [8] and Nemeth [11].

Theorem 1.4. Two one-factorizations $\mathscr{F}_1$ and $\mathscr{F}_2$ of K_{n+1} are orthogonal if and only if their corresponding $n \times n$ symmetric idempotent Latin squares are orthogonal symmetric Latin squares.

By using the above theorem it is possible to show that a graph G is realizable as an OOFG with underlying graph K_{n+1} by finding a set of symmetric Latin squares of side n with orthogonality relations realizing G. We will exploit this fact throughout the remainder of this article. It should be noted that a pair of orthogonal symmetric Latin squares of

order n is equivalent to a Room square of order n. We will make further use of this in Section 3. For background on orthogonal symmetric Latin squares the reader is referred to [7], [8], or [13].

2. MAIN THEOREMS

For the remainder of this article we will assume that all OOFG's use underlying graph K_n. We say that $G \in \mathrm{OOFG}(n)$ if G is realizable as an OOFG with underlying graph K_n. Define the *Spectrum* of G ($Spec(G)$) to be the set of all n such that $G \in \mathrm{OOFG}(n)$. Our first main theorem will show that every graph G is realizable as an OOFG, while our second main theorem will deal with $\mathrm{Spec}(G)$. We first need a preliminary result.

Lemma 2.1. For any positive integer v, there exists an integer $v' = v'(n)$ such that there exist v pairwise orthogonal Latin squares of order n and v pairwise orthogonal symmetric Latin squares of order n for all odd $n \geqslant v'$.

Proof. Gross et al. [7, Theorem 4], there is an integer v_0 such that for all odd $n \geqslant v_0$ there exist v pairwise orthogonal symmetric Latin squares of order n. By Chowla et al. [1], there exists an integer v_1 such that for all $n \geqslant v_1$ there exist v pairwise orthogonal Latin squares of order n. Thus the lemma follows by letting $v' = \max\{v_0, v_1)$. ∎

Theorem 2.2. Every graph is realizable as an OOFG with underlying graph K_t, for some t.

Proof. Let $G = (V, E)$ be a graph with vertex set $V = \{x_1, x_2, \ldots, x_v\}$. Using Lemma 2.1 let $L_1, \ldots, L_v$ be v pairwise orthogonal symmetric Latin squares of order m, let $M_1, \ldots, M_v$ be v mutually orthogonal Latin squares of order n, and let $N_1, \ldots, N_v$ be v mutually orthogonal symmetric Latin squares of order n, for some m and n.

For each i, $1 \leqslant i \leqslant v$, construct $\overline{L}_i$ as given in Figure 1.

Here $M_i(a, b)$ (also $N_i(a, b)$) is the Latin square $M_i(N_i)$ with entry x replaced by $(x, L_i(a, b))$, where $L_i(a, b)$ is the entry in cell (a, b) of L_i. This construction is essentially the same as the usual direct product of Latin squares altered slightly to preserve the symmetric property.

It is easy to see that the squares $\overline{L}_1, \ldots, \overline{L}_v$ form a set of pairwise orthogonal symmetric Latin squares of side mn. Thus K_v is realizable as an $\mathrm{OOFG}(mn + 1)$ by Theorem 1.4. We wish to alter the $\overline{L}_i$'s in order to delete some of the edges of the K_v and thus obtain G.

If $\{x_i, x_j) \in E$, do not alter L_i or L_j.

If $\{x_i, x_j\} \notin E$, we can assume $i < j$. Change $\overline{L}_i$ as follows: Replace

$$\overline{L}_i = \begin{array}{|c|c|c|c|c|}
\hline
N_i(1,1) & M_i(1,2) & M_i(1,3) & \cdots & M_i(1,m) \\
\hline
M_i^T(2,1) & N_i(2,2) & M_i(2,3) & \cdots & M_i(2,m) \\
\hline
M_i^T(3,1) & M_i^T(3,2) & N_i(3,3) & \cdots & M_i(3,m) \\
\hline
\vdots & \vdots & \vdots & \ddots & \vdots \\
\hline
M_i^T(m,1) & M_i^T(m,2) & M_i^T(m,3) & \cdots & N_i(m,m) \\
\hline
\end{array}$$

FIGURE 1

$M_i(i, j)$ by $M_j(i, j)$ and replace $M_i^T(j, i)$ by $M_j^T(j, i)$, where $M_j(i, j)$ is the Latin square M_j with each entry x replaced by $(x, L_i(i, j))$.

Denote the resulting Latin squares by $\hat{L}_i$, $1 \leq i \leq v$. We claim that each $\hat{L}_i$ is a symmetric Latin square, and that $\hat{L}_i$ and $\hat{L}_j$ are orthogonal symmetric Latin squares if and only if $\{x_i, x_j\} \in E$.

First of all, each $\overline{L}_i$ is symmetric and any replacement clearly retains the symmetry. If $\{x_i, x_j\} \notin E$, then $\hat{L}_i$ and $\hat{L}_j$ are not orthogonal symmetric by construction. If $\{x_i, x_j\} \in E$, then $\hat{L}_i$ and $\hat{L}_j$ are unchanged from $\overline{L}_i$ and $\overline{L}_j$ in the (i, j) position. Thus if the squares $M_i(a, b)$ and $M_i^T(b, a)$ are changed in $\overline{L}_i$, the squares $M_j(a, b)$ and $M_j^T(b, a)$ remain as they were in $\overline{L}_j$. Since $M_i(a, b)$ and $M_i^T(a, b)$ are replaced by the squares $M_r(a, b)$ and $M_r^T(a, b)$, where M_r and M_j are orthogonal, this does not effect the orthogonality of the resulting squares $\hat{L}_i$ and $\hat{L}_j$.

By Theorem 1.4, let $\mathcal{F}_1, \ldots, \mathcal{F}_v$ be the one-factorizations of K_{mn+1} equivalent to $\hat{L}_1, \ldots, \hat{L}_v$, respectively. Then, the OOFG with vertices $\mathcal{F}_1, \ldots, \mathcal{F}_v$ is the required one-factorization graph realizing G. ∎

The following theorem is standard and is given here without proof (see [8] or [6]).

Theorem 2.3. Let G be a graph and let $x_1 + 1, x_2 + 1, \ldots, x_t + 1$ lie in Spec(G). If there is a pairwise balanced design of order v with block sizes in $\{x_1, x_2, \ldots, x_t\}$, then $v + 1 \in$ Spec(G). (i.e. Spec(G) is PBD closed).

For our main theorem we will use the results of Wilson on PBD closed sets. Let D be a set of positive integers. Let $\alpha(D) = gcd\{k(k - 1) \mid k \in D\}$ and $\beta(D) = gcd\{k - 1 \mid k \in D\}$. Wilson's Theorem [14] states that there is an integer v_0, such that if $v > v_0$ and $v(v - 1) \equiv 0 \bmod \alpha(D)$ and $v - 1 \equiv 0 \bmod \beta(D)$, then there exists a pairwise balanced design of order v with block sizes all in D. ∎

Theorem 2.4. Let G be a graph. There exists an integer $v_0 = v_0(G)$ such that if $v \geqslant v_0$, then $v \in \mathrm{Spec}(G)$.

Proof. Let $|V(G)| = k$. Use Lemma 3.1 to choose an integer v' such that for all odd $v \geqslant v'$, there exist k pairwise orthogonal Latin squares of order v and k pairwise orthogonal symmetric Latin squares of order v. By Theorem 2.2, $mn + 1$ Spec(G) for all odd $m, n \geqslant v'$. Let $D = \{mn \mid m \geqslant v', n \geqslant v', m, n \text{ odd}\}$. Then $\alpha(D) = 2$ and $\beta(D) = 2$. Wilson's Theorem states that there is a v_0 such that for all $v \geqslant v_0$, v odd, there is a pairwise balanced design of order v with block sizes in D. Theorem 2.3 establishes the result.

The proofs of Theorem 2.2 and 2.4 are similar to those in [9].

3. SPECIAL CASES

In this section we discuss the spectra of some specific graphs. We will draw heavily upon results on Room squares and symmetric Latin squares.

A *Room square* of side n is an $n \times n$ array of cells, whose entries are chosen from a set S of $n + 1$ symbols which satisfies the following conditions: (i) every cell is either empty or contains an unordered pair of distinct symbols from S, (ii) each symbol occurs exactly once in each row and each column of the array, and (iii) every unordered pair of symbols occurs precisely once in the array.

It is easy to see that a Room square R of side n is equivalent to a pair of orthogonal one-factorizations of K_{n+1}, where one one-factorization is obtained from the rows of R and one from the columns.

Using results on Room squares and Room cubes we have the following:

Theorem 3.1. (i) If G is a single point, then Spec$(G) = \{n \mid n \geqslant 2, n$ even$\}$,

(ii) If $G = K_2$, then Spec$(G) = \{n \mid n \geqslant 8, n$ even$\}$,

(iii) If $G = K_3$, then Spec$(G) = \{n \mid n \geqslant 8, n$ even$\}$.

Proof. (i) K_n has a one-factorization for all $n \geqslant 2$, n even.

(ii) In [10], Mullin and Wallis prove the existence of Room squares of side n for all odd $n \geqslant 7$. The result follows from the remark preceding this theorem.

(iii) A Room cube of side n is a three-dimensional array with the property that each two-dimensional projection is a Room square of side n. The existence of a Room cube of side n is equivalent to the existence of 3 pairwise orthogonal one-factorizations of K_{n+1}. In [4], Dinitz and Stinson prove existence of Room cubes of side n for all odd $n > 7$. Thus the result. ∎

A sub-Room square is defined in the obvious way (see [5] or [11]) and the following lemma easily follows.

Lemma 3.2. The existence of a Room square of side n with a sub-Room square of side t, is equivalent to the existence of two orthogonal one-factorizations of K_{n+1} containing two orthogonal sub-one-factorizations of K_{t+1}.

The following lemma, rephrased by use of Lemma 3.2 is proved in [5].

Lemma 3.3. Let $K = \{n \mid n = 24, 30, 32, 36, 40, 44, 48, 50$ and all even $n \geqslant 56\}$. Then if $s \in K$, there exist a pair of orthogonal one-factorizations of K_n containing a pair of orthogonal sub-one-factorizations of K_8.

Theorem 3.4. Let C_6 be the cycle with 6 vertices, then $K \subseteq \mathrm{Spec}(C_6)$.

Proof. Let $n \in K$, then by Lemma 3.3 there exist two orthogonal one-factorizations F' and G' of K_n which contain orthogonal sub-one-factorizations of K_8. From F' and G' delete the edges contained in the sub-one-factorizations of K_8 and call the remaining partial one-factorizations F and G, respectively. Let A, B, and C be three pairwise orthogonal one-factorizations of K_8 (existence is assured by [4]). Define $F \cup A$ to be the one-factorization of K_n obtained by adjoining A to F. Here we mean to add a matching from A to one of the "short" matchings in F. $F \cup B$, $F \cup C$, $G \cup A$, $G \cup B$, and $G \cup C$ are all defined similarly. Each object so defined is a one-factorization of K_n. A realization of C_6 as an OOFG(n) is given in Figure 2. Thus for all $n \in K$, we have $n \in \mathrm{Spec}(C_6)$. ∎

Corollary 3.5. Let P_n denote the path on n vertices, then $K \subseteq \mathrm{Spec}(P_i)$ for $i = 3, 4$, and 5.

Proof. From the realization of C_6 above, delete one, two, or three adjacent one-factorizations to obtain P_5, P_4, or P_3, respectively.

There are two other graphs we wish to discuss. They are $K_{1,3}$, the star with four vertices, and C_4. It is our goal to determine their spectra

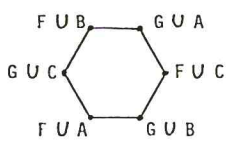

FIGURE 2

exactly, not just find a subset of the spectra as we have done for C_6. To make our first bound on the spectra we will draw heavily upon the known results concerning Room squares with subsquares. We will then complete the spectra by use of special strong starters.

Lemma 3.6. If there exist four pairwise orthogonal one-factorizations of K_{t+1} and if there exists a Room square of side n with a sub-Room square of order t, then $n + 1 \in \mathrm{Spec}(K_{1,3})$ and $n + 1 \in \mathrm{Spec}(C_4)$.

Proof. Analogous to the proof of Theorem 3.4, we note that by hypothesis there exist two orthogonal one-factorizations of K_{n+1} (F' and G', say) containing orthogonal sub-one-factorizations of K_{t+1}. Again let F and G be F' and G' minus the sub-one-factorizations of K_{t+1}. Let A, B, C, D be four mutually orthogonal one-factorizations of K_{t+1}. Then, using the notation of Theorem 3.4, a realization of $K_{1,3}$ as an OOFG($n + 1$) is given in Figure 3a and a realization of C_4 as an OOFG($n + 1$) is given in Figure 3b. ∎

Lemma 3.7. There exist four pairwise orthogonal one-factorizations for K_{n+1} if $n = 9, 11, 13, 15, 17$.

Proof. For $n = 11, 13, 15, 17$ see [2]. Four orthogonal one-factorizations for K_{10} have recently been discovered. See [6] for a description of the result. ∎

Lemma 3.8. If $n \geqslant 45$ or $n = 35, 37, 39$, or 41 there exists a Room square of order n with a sub-Room square of order t for t equaling one of 9, 11, 13, 15, or 17.

Proof. This lemma can be deduced from results presented in [5]. In that paper most known results on Room squares with subsquares are included. The particular orders in this lemma can all be constructed by use of theorems presented in that paper. ∎

Combining Lemmata 3.6, 3.7 and 3.8 we have

Lemma 3.9. If $n \geqslant 46$ or $n = 36, 38, 40$, or 42, then $n \in \mathrm{Spec}(K_{1,3})$ and $n \in \mathrm{Spec}(C_4)$.

FIGURE 3a

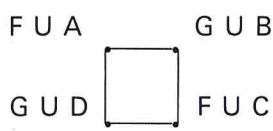

FIGURE 3b

In order to complete our analysis of the above spectra, we need one more combinatorial object, the strong starter.

Let G be an abelian group of odd order r (in our case we can use $G = Z_r$); write $r = 2s + 1$. A *starter* S in G is defined to be a set of unordered pairs of elements of G, $S = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_s, y_s\}\}$ satisfying the property that $\{x_1, x_2, \ldots, x_s, y_1, \ldots, y_s\}$ and $\{\pm(y_1 - x_1), \pm(y_2 - x_2), \ldots, \pm(y_s - x_s)\}$ consist of all nonzero elements of G taken once. S is a *strong starter* if, in addition, $x_1 + y_1, x_2 + y_2, \ldots, x_s + y_s$ are all distinct and nonzero.

Given a starter S in a group G, $|G| = r$, one can form a one-factorization of K_{r+1}. Let $S' = \{\{0, \infty\}, \{x_1, y_1\}, \ldots, \{x_s, y_s\}\}$ and $S' + g = \{\{g, \infty\}, \{x_1 + g, y_1 + g\}, \ldots, \{x_s + g, y_s + g\}\}$. We see that for all $g \in G$, $S' + g$ is a one-factor of K_{r+1} (with symbol set $\{\infty\} \cup G$) and $\cup S' + g$ is a one-factorization of K_{r+1}.

Suppose $S = \{\{x_i y_i\}\}$ and $T = \{\{z_i, w_i\}\}$ are starters in G. Then without loss of generality $y_i - x_i = w_i - z_i$ for $1 \leqslant i \leqslant s$. S and T are *orthogonal starters provided* $x_i - z_i = x_j - z_j$ implies $i = j$. There is no difficulty in showing that the two ideas of orthogonality coincide—in other words, two starters are orthogonal if and only if their associated one-factorizations are orthogonal.

Let $S = \{\{x_i, y_i\}\}$ be a strong starter. Then S and $-S = \{\{-x_i, -y_i\}\}$ are orthogonal starters. Also, if S and T are orthogonal starters, then $-S$ and $-T$ are also a pair of orthogonal starters. One particular starter is the patterned starter, defined as $P = \{\{x, -x\} \mid x \in G\}$. It is not hard to show that S is a strong starter if and only if S and P are orthogonal. All this information on starters can be found in [13].

In order to complete the spectrum of C_4, we will find strong starters S and T in Z_{n-1} such that S is orthogonal to T but not to $-T$. For notation let $\overline{S}$, $\overline{T}$, $\overline{-S}$ and $\overline{-T}$ be the one-factorizations of K_n obtained from S, T, $-S$ and $-T$, respectively. Then C_4 is realizable as the OOFG given in Figure 4.

Theorem 3.10. $\mathrm{Spec}(C_4) = \{n \mid n \geqslant 8, n \text{ even}\}$, except possibly $n = 12$.

Proof. For order $n - 1 = 13, 15, 17, 18, 21, 23, 25, 27, 31, 33$, and 43, strong starters S and T have been found which satisfy the condition that S is orthogonal to T but not to $-T$. These starters are listed in Appendix 1. They were found by a modification of the computer algorithm

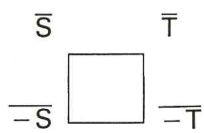

FIGURE 4

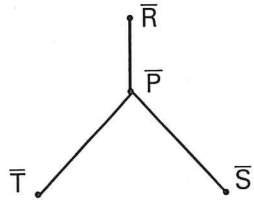

FIGURE 5

presented in [3]. Thus from the discussion above, $n \in \mathrm{Spec}(C_4)$ if $n =$ 12, 14, 16, 18, 20, 22, 24, 26, 28, 32, 34, or 44.

Now, in conjunction with Lemma 3.9, we have that if $n \geqslant 12$, n even, then $n \in \mathrm{Spec}(C_4)$.

If $n < 8$, then there exist no pair of orthogonal one-factorizations of K_n, so of course $n \notin \mathrm{Spec}(C_4)$. Finally, one-factorizations of K_8 and K_{10} realizing C_4 have been found thus completing the proof. The pesky case of $n = 12$ remains unsolved, although we do not hesitate to conjecture that $12 \in \mathrm{Spec}(C_4)$. ∎

In order to complete the spectrum of $K_{1,3}$ we will find strong starters R, S, and T such that none of these three strong starters are orthogonal to each other. Then, remembering that R, S, and T are orthogonal to P, the patterned starter, we have that $K_{1,3}$ is realizable as the OOFG given in Figure 5.

Theorem 3.11. $\mathrm{Spec}(K_{1,3}) = \{n \mid n \geqslant 8,\ n \text{ even}\}$.

Proof. The proof is similar to that of Theorem 3.10. Strong starters R, S, T of order $n - 1 =$ 13, 15, 17, 19, 21, 23, 25, 27, 31, 33, and 43 are given in Appendix 2, one-factorizations of K_8 and K_{10} and K_{12} realizing $K_{1,3}$ have been found and use of Theorem 3.9 completes the proof. ∎

Corollary 3.12. $\mathrm{Spec}(P_3) = \{n \mid n \geqslant 8,\ n \text{ even}\}$.

Proof. Delete one vertex from the $K_{1,3}$ of Theorem 3.11. ∎

4. CONCLUSION

We have defined orthogonal one-factorizations graphs and proven that every graph is realizable as an orthogonal one-factorization graph. We have also considered some special cases including C_6, C_4, and $K_{1,3}$.

Further research in this area could include determining the spectrum of other graphs and studying orthogonal one-factorization graphs on underlying graphs other than $K_{n,n}$ or K_n. It would also be satisfying to see orthogonal one-factorizations of K_{12} realizing C_4.

APPENDIX 1

Order 13

S =	9, 10	5, 7	1, 4	12, 3	6, 11	2, 8
T =	4, 5	9, 11	12, 2	6, 10	3, 8	1, 7

Order 15

S =	11, 12	4, 6	7, 10	1, 5	13, 3	8, 14	2, 9
T =	8, 9	12, 14	2, 5	3, 7	11, 1	4, 10	6, 13

Order 17

S =	13, 14	9, 11	1, 4	6, 10	2, 7	16, 5	8, 15	12, 3	
T =	9, 10	14, 16	3, 6	7, 11	13, 1	2, 8	15, 5	4, 12	

Order 19

S =	2, 3	15, 17	7, 10	12, 16	1, 6	8, 14	4, 11	5, 13	9, 18
T =	11, 12	7, 9	14, 17	16, 1	3, 8	4, 10	18, 6	13, 2	15, 5

Order 21

S =	4, 5	8, 10	17, 20	14, 18	1, 6	9, 15	12, 19	3, 11	7, 16
	13, 2								
T =	9, 10	2, 4	13, 16	7, 11	19, 3	12, 18	20, 6	14, 1	8, 17
	5, 15								

Order 23

S =	6, 7	18, 20	19, 22	13, 17	11, 16	3, 9	1, 8	4, 12	5, 14
	15, 2	10, 21							
T =	5, 6	14, 16	9, 12	17, 21	19, 1	4, 10	13, 20	22, 7	2, 11
	8, 18	15, 3							

Order 25

S =	18, 19	22, 24	4, 7	5, 9	11, 16	2, 8	6, 13	15, 23	12, 21
	10, 20	17, 3	14, 1						
T =	3, 4	11, 13	16, 19	8, 12	17, 22	20, 1	2, 9	7, 15	14, 23
	21, 6	24, 10	18, 5						

Order 27

S =	17, 18	13, 15	2, 5	7, 11	19, 24	22, 1	3, 10	6, 14	16, 25
	21, 4	9, 20	23, 8	26, 12					
T =	21, 22	16, 18	26, 2	9, 13	10, 15	6, 12	23, 3	20, 1	5, 14
	7, 17	24, 8	19, 4	25, 11					

Order 31

S =	9, 10	23, 25	13, 16	17, 21	24, 29	14, 20	1, 8	30, 7	2, 11
	18, 28	26, 6	3, 15	22, 4	5, 19	12, 27			
T =	11, 12	19, 21	30, 2	6, 10	17, 22	26, 1	27, 3	15, 23	5, 14
	28, 7	13, 24	8, 20	16, 29	4, 18	25, 9			

Order 33

S =	28, 29	8, 10	3, 6	15, 19	20, 25	17, 23	11, 18	22, 30	4, 13
	24, 1	27, 5	2, 14	32, 12	7, 21	16, 31	26, 9		
T =	23, 24	14, 16	27, 30	1, 5	31, 3	6, 12	4, 11	18, 26	8, 17
	22, 32	9, 20	7, 19	15, 28	21, 2	10, 25	13, 29		

Order 43

S =	6, 7	19, 21	24, 27	5, 9	10, 15	28, 34	33, 40	36, 1	20, 29
	16, 26	12, 23	30, 42	38, 8	31, 2	39, 11	41, 14	18, 35	4, 22
	13, 32	17, 37	25, 3						
T =	8, 9	24, 26	18, 21	12, 16	29, 34	32, 38	39, 3	33, 41	28, 37
	17, 27	42, 10	1, 13	23, 36	6, 20	7, 22	31, 4	2, 19	40, 15
	11, 30	5, 25	14, 35						

APPENDIX 2

Order 13

R =	9, 10	5, 7	1, 4	12, 3	6, 11	2, 8
S =	3, 4	5, 7	9, 12	10, 1	6, 11	2, 8
T =	9, 10	4, 6	2, 5	7, 11	3, 8	8, 1

Order 15

R =	6, 7	10, 12	1, 4	14, 3	8, 13	5, 11	2, 9
S =	3, 4	7, 9	11, 14	2, 6	8, 13	10, 1	5, 12
T =	5, 6	7, 9	10, 13	12, 1	14, 4	2, 8	11, 3

Order 17

R =	11, 12	7, 9	13, 16	2, 6	15, 3	4, 10	1, 8	14, 5
S =	14, 15	4, 6	9, 12	16, 3	5, 10	7, 13	1, 8	11, 2
T =	15, 16	3, 5	10, 13	7, 11	1, 6	8, 14	2, 9	4, 12

Order 19

R =	3, 4	13, 15	18, 2	5, 9	6, 11	8, 14	10, 17	12, 1	7, 16
S =	17, 18	13, 15	2, 5	3, 7	9, 14	6, 12	4, 11	8, 16	1, 10
T =	4, 5	15, 17	9, 12	3, 7	11, 16	8, 14	18, 6	13, 2	1, 10

Order 21

R =	3, 4	11, 13	6, 9	18, 1	14, 19	10, 16	8, 15	20, 7	17, 5
	2, 12								
S =	19, 20	13, 15	3, 6	8, 12	4, 9	16, 1	11, 18	2, 10	5, 14
	7, 17								
T =	6, 7	12, 14	15, 18	20, 3	11, 16	4, 10	19, 5	1, 9	8, 17
	13, 2								

Order 23

R =	3, 4	11, 13	14, 17	15, 19	2, 7	22, 5	9, 16	10, 18	20, 6
	21, 8	1, 12							
S =	19, 20	12, 14	4, 7	13, 17	5, 10	3, 9	15, 22	16, 1	2, 11
	21, 8	18, 6							
T =	13, 14	9, 11	4, 7	22, 3	15, 20	12, 18	1, 8	17, 2	19, 5
	6, 16	10, 21							

Order 25

R =	16, 17	19, 21	10, 13	5, 9	2, 7	23, 4	24, 6	14, 22	11, 20
	8, 18	1, 12	3, 15						
S =	22, 23	17, 19	9, 12	11, 15	1, 6	24, 5	14, 21	2, 10	7, 16
	3, 13	18, 4	8, 20						
T =	22, 23	13, 15	7, 10	16, 20	9, 14	2, 8	24, 6	18, 1	3, 12
	11, 21	19, 5	17, 4						

Order 27

R =	5, 6	17, 19	25, 1	8, 12	9, 14	23, 2	4, 11	16, 24	13, 22
	10, 20	15, 26	18, 3	21, 7					
S =	12, 13	23, 25	17, 20	26, 3	2, 7	5, 11	14, 21	16, 24	1, 10
	9, 19	4, 15	6, 18	22, 8					
T =	14, 15	11, 13	20, 23	21, 25	5, 10	1, 7	12, 19	22, 3	26, 8
	6, 16	18, 2	24, 9	4, 17					

Order 31

R =	18, 19	8, 10	25, 28	17, 21	2, 7	23, 29	4, 11	6, 14	13, 22
	16, 26	1, 12	24, 5	27, 9	20, 3	15, 30			
S =	17, 18	8, 10	19, 22	11, 15	30, 4	27, 2	6, 13	20, 28	23, 1
	24, 3	5, 16	9, 21	25, 7	12, 26	14, 29			
T =	21, 22	26, 28	11, 14	20, 24	29, 3	17, 23	5, 12	30, 7	10, 19
	6, 16	4, 15	1, 13	27, 9	25, 8	18, 2			

Order 33

R =	3, 4	24, 26	31, 1	9, 13	15, 20	16, 22	10, 17	11, 19	18, 27
	30, 7	21, 32	2, 14	28, 8	25, 6	23, 5	29, 12		
S =	15, 16	18, 20	11, 14	3, 7	29, 1	6, 12	30, 4	17, 25	19, 28
	22, 32	24, 2	31, 10	8, 21	13, 27	23, 5	26, 9		
T =	15, 16	5, 7	18, 21	22, 26	24, 29	28, 1	10, 17	11, 19	30, 6
	2, 12	31, 9	13, 25	23, 3	27, 8	32, 14	4, 20		

Order 43

R =	20, 21	4, 6	31, 34	13, 17	32, 37	18, 24	1, 8	15, 23	39, 5
	19, 29	41, 9	2, 14	33, 3	26, 40	12, 27	22, 38	25, 42	36, 11
	16, 35	10, 30	7, 28						
S =	12, 13	28, 30	38, 41	36, 40	2, 7	17, 23	11, 18	1, 9	24, 33
	25, 35	37, 5	8, 20	14, 27	32, 3	4, 19	15, 31	22, 39	16, 34
	10, 29	6, 26	21, 42						
T =	4, 5	38, 40	18, 21	11, 15	34, 39	3, 9	19, 26	27, 35	22, 31
	23, 33	2, 13	8, 20	24, 37	14, 28	1, 16	25, 41	32, 6	42, 17
	36, 12	10, 30	29, 7						

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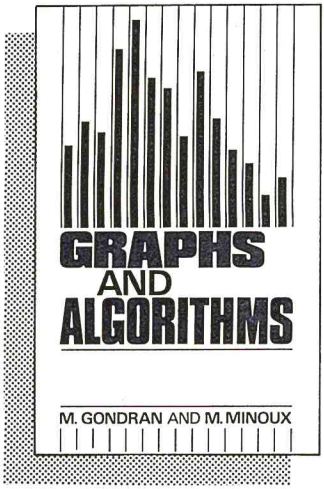

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