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Room Squares and Related Designs

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1 INTRODUCTION AND HISTORY

Let S be a set of $n+1$ elements called *symbols*. A *Room square* of side n (on symbol set S) is an $n \times n$ array, F, that satisfies the following properties:

1. Every cell of F either is empty or contains an unordered pair of symbols from S.
2. Each symbol of S occurs once in each row and column of F.
3. Every unordered pair of symbols occurs in precisely one cell of F.

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$\infty 0$			15		46	23
34	$\infty 1$			26		50
61	45	$\infty 2$			30	
	02	56	$\infty 3$			41
52		13	60	$\infty 4$		
	63		24	01	$\infty 5$	
		04		35	12	$\infty 6$

Figure 1.1 A Room square of side seven.

It is immediate that n must be odd for a Room square of side n to exist. In Figure 1.1 we present a Room square of side seven.

Room squares were named after T. G. Room who published a paper in 1955 [200] in which he proved that Room squares of sides three and five do not exist and constructed a Room square of side seven. The history of these squares, however, goes back much further. In 1850, Kirkman also presented a Room square of side seven in [128] and used it to solve the well-known "15 Schoolgirls Problem." Concerning the Room square, he remarked in that paper:

> It will be found difficult to imitate this arrangement with more than eight things.

Since it took over 120 years for the Room square problem to be solved, we would concur with Kirkman's opinion.

Cayley also constructed a Room square of side seven in 1863 in [41]. Howell and Whitfield constructed Room squares for use as schedules for duplicate bridge tournaments at the end of the nineteenth century. Some of these schedules are presented in [35], [93], [105], [167], and [170].

Of course, Cayley and Kirkman were well-known mathematicians. Howell seems to be a more enigmatic figure. Howell rotations and Howell designs, which we discuss in Sections 10 and 11, respectively, were named after Howell. However, Paul Smith makes the following remark in his PhD thesis concerning Howell [221, p. 12]:

> Although every duplicate bridge player knows what a Howell movement is, I have been unable to identify Edwin C. Howell except in allusions by later writers. Gruenther states that he was a professor of mathematics at MIT but a search of the archives at that institution which Professors Rota and Spencer conducted in response to a question by me turned up no trace of him in any capacity.

degree in 1883. However, in his varied career, he was never a university professor. Whitfield seems to be less well known than Howell among current researchers in design theory, despite the fact that he anticipated some later results by over 70 years [221, p. 48]. Other interesting historical information can be found in the PhD thesis by Paul Smith [221].

One application of Room squares is to the construction of round-robin tournaments. A Room square of side n can be used to schedule a tournament with $n+1$ teams. If the rows of the square index the rounds, and the columns index the locations, then the resulting tournament will have the following properties:

1. Every team plays every other team exactly once during the tournament.
2. Every team plays in exactly one game in each round.
3. Every team plays at every location exactly once.

Several connections exist regarding the Room square of side 7 to other mathematical structures. An interesting elementary discussion is given in Chapter 11 of Martin Gardner's book "Time Travel and Other Mathematical Bewilderments" [88].

After Room's 1955 paper, the problem of constructing Room squares attracted the interest of the mathematical community. Some early results can be found in [22], [23], [38], and [267]. As of 1968, a Room square of order n was known to exist only for the following values of n: $n = 2^j - 1$ ($j \geq 3$) [23]; $n = 11, 19, 23$ [22]; $n = 9$ [267], and $n = 13$ [191].

In 1968, Stanton and Mullin introduced the starter-adder technique in [224], and constructed Room squares of all odd sides n, $7 \leq n \leq 47$. From that point on, progress was rapid as several researchers made significant contributions, notably, Horton, Mullin, Nemeth, Stanton, and Wallis. The spectrum of Room squares was determined by 1973. In March of that year, W. D. Wallis presented a Room square of side 257 at the Fourth Southeastern Conference of Combinatorics, Graph Theory, and Computing at Boca Raton, Florida [254], [255] (Dillon and Morris independently constructed a Room square of side 257 in [55]). This was the last unknown side. Shortly thereafter, a condensed existence proof was given by Mullin and Wallis [189], which we record as the following:

Theorem 1.1. *A Room square of side n exists if and only if n is odd and $n \neq 3$ or 5.*

We present two proofs of this theorem in Section 3.

We refer the interested reader to the paper [189] and to the book by Wallis, Street, and Wallis [266] for most of the results on Room squares which appeared before 1974. Other early papers include the following: [39], [54], [118], [155], [171], [180], [183], [188], [220], [226], and [250]. Also, a survey was writ-

We should mention two recent textbooks on combinatorial designs that contain results on Room squares: Wallis [263] and Anderson [21]. There is also a survey written by Wallis in the mid-1980s [247]; as of February 1991, this survey has not been published.

In order to understand some of the generalizations of Room squares, it is important to mention some of the combinatorial objects that are equivalent to Room squares. We begin with a graph-theoretic interpretation. Let G be a graph. A *one-factor* of G is a set of edges that partition the vertex set of G (i.e., a perfect matching). A *one-factorization* of G is a set of one-factors that partition the set of edges of G. Two one-factorizations of G, $\mathcal{F}$ and $\mathcal{G}$, are said to be *orthogonal* if, for any one-factors $f \in \mathcal{F}$ and $g \in \mathcal{G}$, f and g contain at most one common edge. The following theorem proved in [190] relates orthogonal one-factorizations and Room squares:

Theorem 1.2. *The existence of a Room square of side n is equivalent to the existence of two orthogonal one-factorizations of the complete graph K_{n+1}.*

In Figure 1.1, each row of the Room square is a one-factor of K_8, and the union of these one-factors forms a one-factorization. Similarly, the columns form a one-factorization, and the two one-factorizations are orthogonal, since any cell of the Room square contains at most one edge (pair of symbols).

We will keep returning to this idea of orthogonality because it is the glue that bonds all the generalizations of Room squares that we survey. Another equivalence to Room squares can be phrased in terms of a certain type of Latin square. A Latin square L is *idempotent* if $L(i,i) = i$ for all i, and L is *symmetric* if $L(i,j) = L(j,i)$ for all i, j. Suppose that L and M are idempotent, symmetric Latin squares of the same order. We say that L and M are *orthogonal-symmetric* Latin squares if, for any two elements x and y, there exists at most one ordered pair (i,j) with $i < j$ such that $L(i,j) = x$ and $M(i,j) = y$.

We note that orthogonal-symmetric Latin squares are not orthogonal Latin squares, but they are as "orthogonal" as possible, given that they are symmetric. The following equivalence was first proved by Bruck [38] in terms of quasigroups:

Theorem 1.3. *The existence of a Room square of side n is equivalent to the existence of two orthogonal-symmetric Latin squares of order n.*

One of the central problems concerning Latin squares is to find as many pairwise orthogonal Latin squares (of a given order) as possible. This naturally leads us to study the existence of sets of pairwise orthogonal-symmetric Latin squares. Hence, for any odd integer $n > 1$, we define $\nu(n)$ to be the maximum number of pairwise orthogonal-symmetric Latin squares of order n. From Theorem 1.3, we see that $\nu(n) \geq 2$ if and only if there exists a Room

Pairwise orthogonal-symmetric Latin squares of order n give rise to a higher dimensional generalization of a Room square called a Room d-cube. A *Room d-cube* of side n is a d-dimensional array, each cell of which either is empty or contains an unordered pair of symbols, such that each two-dimensional projection is a Room square of side n. The higher dimensional generalizations of Theorems 1.2 and 1.3 are given in [114]. We should note that, although [114] was not published until 1981, it was written about ten years earlier.

Theorem 1.4. *The existence of the following are equivalent:*

1. *A Room d-cube of side n.*
2. *d pairwise orthogonal-symmetric Latin squares of order n.*
3. *d pairwise orthogonal one-factorizations of K_{n+1}.*
4. $\nu(n) \geq d$.

This survey is organized as follows: First, we discuss the tools used to construct Room squares and Room d-cubes. These include starters (Section 2.1) and frames (Section 3). In Section 2.2, we discuss hill-climbing algorithms for constructing starters and Room squares. In Section 4, we consider Room squares with subsquares. In Section 5, we study Room d-cubes with $d > 2$, and in Section 6, we investigate nonisomorphic designs. Then we study several special types of Room square, such as skew (Section 7), perfect (Section 8), maximum empty subarray (Section 9), and balanced (Section 10). We then survey generalizations of Room squares. Section 11 deals with Howell designs, Section 12 discusses orthogonal Steiner triple systems, and Section 13 concerns orthogonal edge-colorings of graphs. Houses are studied in Section 14, orthogonal one-factorization graphs in Section 15, and balanced tournament designs in Section 16. We conclude with a section on miscellaneous topics and with a list of open problems.

2 DIRECT CONSTRUCTIONS

2.1. Starters

The most useful technique for the direct construction of Room squares has been the technique of orthogonal starters (or equivalently, the starter-adder method). This technique was introduced in the mathematical literature in 1968 by Stanton and Mullin in [224]. However, P. Smith in [221, p. 48] observes that Whitfield in fact employed the same method in the 1890's. Let G be an additive abelian group of odd order g. A *starter* in G is a set of unordered pairs $S = \{\{s_i, t_i\} : 1 \leq i \leq (g-1)/2\}$ which satisfies the following two properties:

1. $\{s_i : 1 \leq i \leq (g-1)/2\} \cup \{t_i : 1 \leq i \leq (g-1)/2\} = G \setminus \{0\}$

Let $S = \{\{s_i, t_i\} : 1 \leq i \leq (g-1)/2\}$ and $T = \{\{u_i, v_i\} : 1 \leq i \leq (g-1)/2\}$ be two starters in G. Without loss of generality, we may assume that $s_i - t_i = u_i - v_i$, for all i. Then S and T are said to be *orthogonal* starters if $u_i - s_i = u_j - s_j$ implies $i = j$, and if $u_i \neq s_i$ for all i.

Example 2.1. Two orthogonal starters in $\mathbf{Z}_7$ are

$$S_1 = \{\{2,3\},\{4,6\},\{1,5\}\},$$
$$S_2 = \{\{3,4\},\{6,1\},\{5,2\}\}.$$

These two orthogonal starters were used to generate the Room square of side 7 presented in Figure 1.1. The first starter appears in the first row of the square, while the second starter appears in the first column.

The following theorem, due to Horton [114], gives the connection between orthogonal starters and Room d-cubes.

Theorem 2.1 ([114]). *The existence of d pairwise orthogonal starters in an abelian group of order n implies the existence of a Room d-cube of side n (i.e.,* $\nu(n) \geq d$*).*

If $S = \{\{s_i, t_i\} : 1 \leq i \leq (g-1)/2\}$ is a starter, then $-S = \{\{-s_i, -t_i\} : 1 \leq i \leq (g-1)/2\}$ is also a starter. In any abelian group G of odd order, the set of pairs $P = \{\{x, -x\} : x \in G\}$ is a starter, called the *patterned* starter. A starter $S = \{\{s_i, t_i\} : 1 \leq i \leq (g-1)/2\}$ is said to be *strong* if $s_i + t_i = s_j + t_j$ implies $i = j$, and $s_i + t_i \neq 0$ for any i. Note in Example 2.1 that S_1 is strong and S_2 is patterned.

The following result concerning strong starters is easy to prove:

Theorem 2.2 ([114]). *If there exists a strong starter S in an abelian group of odd order n, then the starters S, −S and P are pairwise orthogonal; hence, there exists a Room 3-cube of side n (i.e.,* $\nu(n) \geq 3$*).*

Strong starters have also found many other applications in the construction of various types of designs. These include Howell designs [3], [67], Kirkman squares [208], [240], Kotzig factorizations [48], [115], and hamiltonian path-balanced tournament designs [116].

If $S = \{\{s_i, t_i\} : 1 \leq i \leq (g-1)/2\}$ is a starter, then a set $A = \{\{a_i\} : 1 \leq i \leq (g-1)/2\}$ is defined to be an *adder* for S if the elements in A are nonzero and distinct, and the set $S + A = \{\{s_i + a_i, t_i + a_i\} : 1 \leq i \leq (g-1)/2\}$ is again a starter. If A is an adder for S, then S and $S + A$ are in fact orthogonal starters. Conversely, if S and T are orthogonal starters, then $T = S + A$, where A is an adder for S. Hence, any statement regarding orthogonal starters can be made in terms of starters and adders. However, we will usually use the

We note that Whitfield constructed adders for patterned starters in $\mathbf{Z}_n$ for $n = 7, 11, 13, 15, 17$, and 19. These are presented in the 1897 edition of Mitchell's *Duplicate Whist* [170]. As well, according to Gruenther [105], Howell found starters and adders in $\mathbf{Z}_n$ for all odd n, $7 \leq n \leq 29$. (These starters and adders are presented in [221].)

In Example 2.1, the adder $\{1,2,4\}$ would carry S_1 into S_2. We can also see from Figure 1.1 how to construct the Room square from the starter and adder. Think of the rows and columns of the Room square in Figure 1.1 as being indexed by the elements in $\mathbf{Z}_7$ so that $\{\infty, x\}$ appears in cell (x,x), $0 \leq x \leq 6$. The pairs in the starter S_1 will all appear in row 0. More precisely, the pair $\{s_i, t_i\}$ will be placed in cell $(0, -a_i)$. Thus, $\{2,3\}$ is placed in cell $(0,6)$, $\{4,6\}$ is placed in cell $(0,5)$, and $\{1,5\}$ is placed in cell $(0,3)$. Then the cells in the other rows are filled in by cyclically shifting the first row: The pair $\{x+s_i, x+t_i\}$ is placed in cell $(x, x-a_i)$, $0 \leq x \leq 6$.

A starter $S = \{\{s_i, t_i\} : 1 \leq i \leq (g-1)/2\}$ is said to be *skew* if $\{\pm(s_i + t_i) : 1 \leq i \leq (g-1)/2\} = G \backslash \{0\}$. Note that a skew starter is strong. A skew starter gives rise to a special type of Room square called a skew Room square. These are studied further in Section 7. Similarly, we can define two starters $S = \{\{s_i, t_i\} : 1 \leq i \leq (g-1)/2\}$ and $T = \{\{u_i, v_i\} : 1 \leq i \leq (g-1)/2\}$ to be *skew-orthogonal* starters if $u_i - s_i = \pm(u_j - s_j)$ implies $i = j$, and if $u_i \neq s_i$ for all i, where $s_i - t_i = u_i - v_i$, for all i. The adder for two skew-orthogonal starters is called a *skew* adder. In Example 2.1, S_1 is skew and S_1 and S_2 are skew-orthogonal.

Other special types of starters have been studied in the literature. These include frame starters, λ-starters, balanced starters, partitionable starters, and symmetric starters. Some of these types of starters will be discussed further in this section and in Section 10.

We now discuss some specific constructions for starters. The most important class of strong starters are the Mullin-Nemeth starters, which were discovered in 1969 [175]. Suppose that q is a prime power that can be written in the form $q = 2^k t + 1$, where $t > 1$ is odd (it is an easy exercise in number theory to show that the only prime powers that cannot be so written are the Fermat primes, and 9). Let ω be a primitive element in the field GF(q), and define C_0 to be the multiplicative subgroup of GF$(q)\backslash\{0\}$ of order t. C_0 has cosets $C_i = \omega^i C_0$, $0 \leq i \leq 2^k - 1$, which are often referred to as *cyclotomic classes*. Define $\Delta = 2^{k-1}$, $H = \bigcup_{i=0}^{\Delta-1} C_i$. Then $T = \{\{x, \omega^\Delta x\} : x \in H\}$ is the Mullin-Nemeth starter. Mullin-Nemeth starters have the following properties:

Theorem 2.3 ([175]). *Let q be a prime power such that $q = 2^k t + 1$, where $t > 1$ is odd. Then the Mullin-Nemeth starter is a skew starter (and hence also a strong starter).*

In the case $q \equiv 3 \pmod 4$ (so $k = 1$ and $H = C_0$), Horton [114] generalized the Mullin-Nemeth starters as follows. For any $a \in C_0$ define $T_a = \{\{x, ax\}$:

obtained are all strong starters, and are pairwise orthogonal. We record this as follows:

Theorem 2.4 ([114]). *Let $q \equiv 3 \pmod 4$ be a prime power. Then any starter T_a ($a \in C_1$) is a strong starter. Further, T_a and T_b are orthogonal if $a, b \in C_1$, $a \neq b$. Hence, $\nu(q) \geq (q-1)/2$.*

In 1979, Dinitz [56] presented a class of starters that generalize both the Mullin-Nemeth starters and the Horton starters. As before, let q be a prime power such that $q = 2^k t + 1$, where $t > 1$ is odd, and again define the cyclotomic classes C_i, $0 \leq i \leq 2^k - 1$. Define $C_i^a = (1/(a-1))C_i$, and define $\Delta = 2^{k-1}$. For each $a \in C_\Delta$, construct a starter $S_a = \{\{x, ax\} : x \in \bigcup_{i=0}^{\Delta-1} C_i^a\}$. The following theorem describes the main properties of the Dinitz starters:

Theorem 2.5 ([56]). *Let q be a prime power such that $q = 2^k t + 1$, where $t > 1$ is odd. Then any starter S_a ($a \in C_\Delta$) is a strong starter. Further, S_a and S_b are orthogonal if $a, b \in C_\Delta$, $a \neq b$. Hence, $\nu(q) \geq t$.*

Other infinite classes of starters of the same orders have been given by Gross [101] and Gross and Leonard [102].

In the starter S_a defined above, the quotient of the elements in any pair in the starter is equal to a (or a^{-1}). These starters are thus referred to as *one-quotient* starters. A starter in which the quotients of the elements of the pairs in the starter take one of r possible values (or their inverses) is called an *r-quotient* starter. In [59], there is a construction for 2^{k-1}-quotient starters in GF(q) when $q = 2^k t + 1$ is a prime power and $t > 1$ is odd. These starters are discussed further in Section 5.

The Mullin-Nemeth strong starters yield strong starters in any field GF(q) when q is odd, except for $q = 9$ and when q is a Fermat prime. In 1974, Chong and Chan ([43], [42]) gave a construction for strong starters in GF(q) when $q > 5$ is a Fermat prime. It was later observed by Dinitz [57] that the Chong-Chan construction can be modified to provide a strong starter in the ring $\mathbf{Z}_{16t^2+1}$. Lins and Schellenberg gave a short proof of the existence of these starters in [165]. Hence, for all odd prime powers q, $q \neq 3$, 5, or 9, there exists an abelian group of order q that admits a strong starter. Finally, we note that there is no strong starter in GF(3), GF(5), or GF(9).

The following three theorems provide examples of strong starters in groups not of prime power order:

Theorem 2.6 ([113]). *Let G be an abelian group of order relatively prime to 6, and suppose there is a strong starter in G. Then there is a strong starter in the direct sum $G \oplus \mathbf{Z}_5$.*

Theorem 2.7 ([98]). *Let G and H be abelian groups of odd order, where also H has order relatively prime to 3. Suppose that there exist strong starters in G*

Theorem 2.8 ([102]). *Let G be an abelian group of odd order, and let H be a subgroup of G. Suppose that there exist strong starters in H and in G/H. Suppose also that there is a permutation π of H such that $\pi + I$ and $\pi - I$ are also permutations of H, where I is the identity permutation. Then there is a strong starter in G.*

It is still an open question as to whether there exists a strong starter in every cyclic group of odd order exceeding 9. The authors certainly do not hesitate to conjecture that this is true. It has been shown by computer that there is a strong starter in every odd order cyclic group $\mathbf{Z}_n$, where $7 \leq n \leq 999$, $n \neq 9$. This result was proved in [67] using the hill-climbing algorithm described in [68] (see Section 2.2). More generally, Horton [117] has conjectured that any abelian group of odd order has a strong starter, with the exceptions of the groups $\mathbf{Z}_3$, $\mathbf{Z}_5$, $\mathbf{Z}_9$, and $\mathbf{Z}_3 \times \mathbf{Z}_3$.

The strongest general existence result for strong starters is the following:

Theorem 2.9 ([117]). *Let G be an abelian group of order n where $n > 5$ and n is relatively prime to 6. Then there is a strong starter in G.*

There are few general nonexistence results regarding starters. Probably the most interesting is the following result of Wallis and Mullin.

Theorem 2.10 ([265]). *Suppose that G is an abelian group of order* $3 \pmod 6$ *in which the 3-Sylow subgroup is cyclic. Then there is no skew starter in G.*

Theorem 2.10 precludes the existence of a skew starter in $\mathbf{Z}_{15}$, for example. However, there do exist two skew-orthogonal starters in $\mathbf{Z}_{15}$.

Example 2.2 ([181]). Two skew-orthogonal starters in $\mathbf{Z}_{15}$ are

$$S_1 = \{\{1,2\},\{3,5\},\{7,10\},\{9,13\},\{6,11\},\{8,14\},\{12,4\}\},$$
$$S_2 = \{\{11,12\},\{4,6\},\{5,8\},\{13,2\},\{9,14\},\{1,7\},\{3,10\}\}.$$

An important generalization of the starter is the frame starter, defined in [66] (a more general concept, the partial starter, was defined earlier in [13]). Let G be an additive abelian group of order g, and let H be a subgroup of order h of G, where $g - h$ is even (i.e., g and h are both even or both odd). A *frame starter* in $G \backslash H$ is a set of unordered pairs $S = \{\{s_i, t_i\} : 1 \leq i \leq (g-h)/2\}$ such that the following two properties are satisfied:

1. $\{s_i : 1 \leq i \leq (g-h)/2\} \cup \{t_i : 1 \leq i \leq (g-h)/2\} = G \backslash H$.
2. $\{\pm(s_i - t_i) : 1 \leq i \leq (g-h)/2\} = G \backslash H$.

starters, replacing $\{0\}$ by H and $(g-1)/2$ by $(g-h)/2$ in each definition. Frame starters will be used to generate a generalization of Room squares called a *frame*, which we discuss in Section 3.

Example 2.3. A skew frame starter in $\mathbf{Z}_{10}\backslash\{0,5\}$ is

$$S_1 = \{\{3,4\},\{7,9\},\{1,8\},\{2,6\}\}$$

Several constructions for infinite classes of frame starters are known. We mention some of them now.

Theorem 2.11 ([241], [66]). *If $q \equiv 1 \pmod 4$ is a prime power and $n \geq 1$, then there is a skew frame starter in $(GF(q) \times (\mathbf{Z}_2)^n) \backslash(\{0\} \times (\mathbf{Z}_2)^n)$.*

Theorem 2.12 ([66]). *Let $q \equiv 1 \pmod 4$ be a prime power such that $q = 2^k t + 1$, where $t > 1$ is odd. Then there exist t orthogonal frame starters in $(GF(q) \times (\mathbf{Z}_2)^n)\backslash(\{0\} \times (\mathbf{Z}_2)^n)$ for all $n \geq 1$.*

Theorem 2.13 ([13]). *If $p \equiv 1 \pmod 6$ is a prime, $p \geq 19$, then there is a strong frame starter in $(GF(q) \times \mathbf{Z}_3)\backslash(\{0\} \times \mathbf{Z}_3)$.*

The following two theorems provide some nonexistence results for certain infinite classes of frame starters:

Theorem 2.14 ([66], [7]). *Let G be an abelian group of order g, and let H be a subgroup of G of order $h \equiv 2 \pmod 4$. If $g/h \equiv 2$ or $3 \pmod 4$, then there does not exist a frame starter in $G\backslash H$.*

Theorem 2.15 ([66]). *Let G be an abelian group of order g, and let H be a subgroup of G of order $h \equiv 1 \pmod 2$. If $g/h = 5$, then there does not exist a strong frame starter in $G\backslash H$.*

2.2. Hill-Climbing Algorithms for Room Squares and Starters

In the study of Room squares and related designs it is often essential to construct "small" examples of the required designs. Often, the smallest cases can be constructed by hand, but this becomes impossible for subsequent cases. For larger cases, it is necessary to use a computer. However, the traditional methods of exhaustive search, such as backtracking algorithms, usually prove to be infeasible for orders just slightly larger than those which can be done by hand.

In this section, we present extremely fast and effective computer algorithms is an example of what is called a *hill-climbing algorithm* (or a local search algorithm). For a general discussion of hill-climbing algorithms and their use in the construction of combinatorial designs, the reader is referred to [236].

The first algorithm we present is the hill-climbing algorithm for finding one-factorizations in K_n. This algorithm was presented by Dinitz and Stinson in [72]. In order to use a hill-climbing approach, we formulate the problem as an optimization problem. We represent a one-factorization of K_n as a set $\mathcal{F}$ of pairs, each having the form $(f_i, \{x,y\})$, where $1 \le i \le n-1$ and x and y are distinct vertices of K_n. There will be $(n^2-n)/2$ such pairs and the following properties will be satisfied:

1. Every edge $\{x,y\}$ of K_n occurs in a unique pair $(f_i, \{x,y\})$.
2. For every f_i and every vertex x, there is a unique pair of the form $(f_i, \{x,y\})$.

Define a *partial one-factorization* of K_n to be a set $\mathcal{F}$ of pairs, each of which has the form $(f_i, \{x,y\})$, where $1 \le i \le n-1$ and x and y are distinct vertices in K_n, that satisfies the following properties:

1. Every edge $\{x,y\}$ of K_n occurs in *at most* one pair $(f_i, \{x,y\})$.
2. For every f_i and every vertex x, there is *at most* one pair of the form $(f_i, \{x,y\})$.

Let $\mathcal{F}$ be a partial one-factorization of K_n. The f_i's are called *partial one-factors*. A vertex x is said to be *live* if there is some f_i such that for all vertices $y \neq x$, $(f_i, \{x,y\}) \notin \mathcal{F}$ (i.e., such that x does not occur in the partial one-factor f_i). A partial one-factor f_i is *live* if it does not occur in exactly $n/2$ pairs in $\mathcal{F}$ (i.e., if it is not a [complete] one-factor of K_n). We say that x does not occur in f_i if there is no point y such that $(f_i, \{x,y\}) \in \mathcal{F}$. Finally, we say that x and y do not occur together if there is no partial one-factor f_i such that $(f_i, \{x,y\}) \in \mathcal{F}$.

Define the cost $c(\mathcal{F})$ of a partial one-factorization $\mathcal{F}$ to be $(n^2-n)/2 - |\mathcal{F}|$, where $|\mathcal{F}|$ denotes the number of pairs in $\mathcal{F}$. Clearly, $\mathcal{F}$ is a one-factorization if and only if $c(\mathcal{F}) = 0$.

Two heuristics H_1 and H_2 will be used in this algorithm.

Heuristic H_1

```
1.   Choose any live point x
2.   Choose any partial one-factor f_i such that x does not occur in f_i
3.   Choose any point y such that x and y do not occur together
4.   If y does not occur in f_i, then
       replace F by F ∪ {(f_i, {x,y})}
```

Heuristic H_2

1. Choose any live partial one-factor f_i
2. Choose any two points x and y such that x and y do not occur in f_i
3. **If** x and y do not occur together, **then**
   replace $\mathcal{F}$ by $\mathcal{F} \cup \{(f_i, \{x,y\})\}$
   else there is a pair in $\mathcal{F}$ of the form $(f_j, \{x,y\})$ $(j \neq i)$
   replace $\mathcal{F}$ by $\mathcal{F} \cup \{(f_i, \{x,y\})\} \backslash \{(f_j, \{x,y\})\}$

Note that if we apply either heuristic H_1 or H_2 then we obtain a new partial one-factorization in which the cost either remains the same or decreases by one. Also observe that both heuristics can always be performed as long as $c(\mathcal{F}) \neq 0$.

The hill-climbing algorithm for one-factorizations is now given below.

Hill-climbing algorithm to construct a one-factorization of K_n

1. **While** $c(\mathcal{F}) \neq 0$, **do**
2. choose r =1 or 2 at random with equal probability
   perform H_r

There is no guarantee that this algorithm will terminate in finite time. However, in practice, in over ten million attempts, it has always found the desired one-factorization. It is also extremely fast. When implemented in C on a SPARCstation 1, it can construct a one-factorization of K_{20} in about 0.13 seconds, one of K_{40} in about 0.67 seconds and one of K_{60} in about 1.67 seconds.

Since a Room square of side $n-1$ is equivalent of a pair of orthogonal one-factorizations of K_n, we can use the above algorithm with some small modifications to make Room squares. We first construct a one-factorization $\mathcal{F}$; then we construct a one-factorization $\mathcal{G}$ which is orthogonal to $\mathcal{F}$. In terms of the resulting Room square, we first determine the rows (say), and then we attempt to "sort out" the columns. This algorithm was also first given by Dinitz and Stinson in [72].

In order to modify the algorithm to construct a one-factorization $\mathcal{G}$ orthogonal to a given one-factorization $\mathcal{F}$, we will maintain the array R, in which the rows are indexed by the one-factors of $\mathcal{F}$ and the columns are indexed by the (partial) one-factors of $\mathcal{G}$. At any stage of the algorithm, $R(f_j, g_i) = \{x,y\}$ if $(f_j, \{x,y\}) \in \mathcal{F}$ and $(g_i, \{x,y\}) \in \mathcal{G}$; and $R(f_j, g_i)$ is empty, otherwise.

... heuristics H_1 and H_2 to construct $\mathcal{G}$ as we did to orthogonality is never violated. The modified H_1 we call OH_1, and likewise the modified H_2 we call OH_2.

Heuristic OH_1

1. Choose any live point x
2. Choose any partial one-factor g_i such that x does not occur in g_i
3. Choose any point y such that x and y do not occur together (in $\mathcal{G}$)
4. Let f_j be the one-factor of $\mathcal{F}$ which contains the edge $\{x,y\}$
5. **If** $R(f_j,g_i)$ is not empty, **then**
   OH_1 fails
   else if y does not occur in g_i, **then**
   replace $\mathcal{G}$ by $\mathcal{G} \cup (g_i,\{x,y\})$
   define $R(f_j,g_i) = \{x,y\}$
   else there is a pair in $\mathcal{G}$ of the form $(g_i,\{z,y\})$ $(z \neq x)$
   replace $\mathcal{G}$ by $\mathcal{G} \cup (g_i,\{x,y\}) \backslash (g_i,\{z,y\})$
   define $R(f_j,g_i) = \{x,y\}$

Heuristic OH_2

1. Choose any live partial one-factor g_i
2. Choose any two points x and y such that x and y do not occur in g_i
3. Let f_j be the one-factor of $\mathcal{F}$ which contains the edge $\{x,y\}$
4. **If** $R(f_j,g_i)$ is not empty, **then**
   OH_2 fails
   else if x and y do not occur together, **then**
   replace $\mathcal{G}$ by $\mathcal{G} \cup (g_i,\{x,y\})$
   define $R(f_j,g_i) = \{x,y\}$
   else there is a pair in $\mathcal{G}$ of the form $(g_k,\{x,y\})$ $(k \neq i)$
   replace $\mathcal{G}$ by $\mathcal{G} \cup (g_i,\{x,y\}) \backslash (g_k,\{x,y\})$
   define $R(f_j,g_i) = \{x,y\}$
   define $R(f_j,g_k)$ to be empty

As we noted earlier, there are times when neither heuristic can be performed. There are also times when succesive uses of these heuristics can lead to an infinite loop. In order to address these problems we define a threshold function $T : \mathbf{Z}^+ \to \mathbf{Z}^+$. When the number of iterations of the heuristics exceeds the value of the threshold, the algorithm is terminated. It can then be restarted with a different random seed. In practice, the threshold function $T(n) = 100n$ has proved suitable.

The hill-climbing algorithm for Room squares is as follows. A discussion of the success rate of this algorithm and timings are given in [72]. It was noted there that this algorithm succeeds in finding Room squares on average in one out of every ten tries. Since its run time is very quick, it has proved to be ex-

Hill-climbing algorithm to construct a Room square of side $n-1$

```
1.  Use the hill-climbing algorithm to construct F, a one-factorization of K_n
2.  Number-of-iterations is initialized to be 0
3.  While (number-of-iterations < T(n)) and (c(G) ≠ 0), do
       choose r =1 or 2 at random with equal probability
       perform OH_r
       increment number of iterations
```

The hill-climbing algorithm can easily be modified to find objects related to Room squares. In subsequent sections of this survey we will mention some of these modifications. In particular, variations of the algorithm have beeen used to construct Room frames [74], [65] (see Section 3), Room squares with subsquares [74], [65] (see Section 4), and Howell designs with subdesigns [64] (see Section 11).

The next algorithm which we will discuss is the hill-climbing algorithm for finding strong frame starters. This algorithm was originally described for strong starters in [68], but it is trivial to generalize it to the case of strong frame starters. (We note that a discussion of this algorithm also appears in [173], [49], and [236].) We will present the hill-climbing algorithm for strong frame starters in a quite different manner than the original presentation in [68]. The original description of the algorithm incorporated extra heuristics which do not effectively decrease the run time of the algorithm. This version we give here evolved from that original one but is more efficient, easier to read, and easier to implement.

Let G be an additive abelian group of order g, and let H be a subgroup of G of order h, where $g-h$ is even. We define a *partial* strong frame starter to be a set of n unordered pairs $S = \{\{s_i, t_i\},\ 1 \le i \le n\}$ that satisfies the following properties:

1. The s_i's and t_i's are distinct elements of $G \backslash H$.
2. The differences $\pm(s_i - t_i)$ are distinct elements of $G \backslash H$.
3. The sums $s_i + t_i$ are distinct elements of $G \backslash H$.

Note that $1 \le n \le (g-h)/2$. We will employ similar terminology to that used for the one-factorization algorithm. Given a partial strong frame starter, S, an element $x \in G\backslash H$ is *live* if there is no pair of the form $\{x,y\} \in S$. A difference $d \in G\backslash H$ is *live* if there is no pair $\{x,y\} \in S$ such that $y - x = \pm d$; and a sum $s \in G\backslash H$ is *live* if there is no pair $\{x,y\} \in S$ such that $y + x = s$. Note that no element $x \in H$ can be a live element, difference or sum. Define the *cost* $c(S)$ of a partial strong frame starter S to be $(g-h)/2 - |S|$, where $|S|$ denotes the number of pairs in S. Clearly, S is a strong frame starter if and only if $c(S) = 0$. Again, two heuristics H_1 and H_2 are defined that will be

Heuristic H_1

1.	Choose any live element x
2.	Choose any live difference d
3.	Let $y = x + d$ or let $y = x - d$ (with equal probability)

4.	**If** $x + y$ is a live sum, **then**
	  if y is a live element, **then**
	    $S = S \cup \{\{x,y\}\}$
	  else there exists a pair $\{u,y\} \in S$
	    $S = S \cup \{\{x,y\}\} \setminus \{\{u,y\}\}$

Heuristic H_2

1.	Choose two live elements x and y
2.	Let $d = y - x$

3.	**If** $x + y$ is a live sum, **then**
	  if d is a live difference, **then**
	    $S = S \cup \{\{x,y\}\}$
	  else there exists a pair $\{u,v\} \in S$ with $v - u = d$
	    $S = S \cup \{\{x,y\}\} \setminus \{\{u,v\}\}$

As in the Room square algorithm, there may be times when neither heuristic can be performed. There are also times when successive uses of these heuristics may lead to an infinite loop. Again, we define a threshold function T, which is used as in the previous algorithm.

The hill-climbing algorithm for strong frame starters is now given. In practice, this algorithm is also extremely fast and effective. It had been observed that strong starters of orders exceeding 60 are impossible to find in a reasonable amount of time using backtracking algorithms [50]. In contrast, the hill-climbing algorithm finds strong starters in $\mathbf{Z}_{101}$ in an average of 0.38 seconds, in $\mathbf{Z}_{1001}$ in an average of 2.62 seconds and in $\mathbf{Z}_{10001}$ in an average of 28.1 seconds when implemented in C on a SPARCstation 1. The algorithm also has been modified to find objects related to strong frame starters including orthogonal frame starters and strong starters. Applications include finding Room cubes of dimension 5 [61] (see Section 5), balanced Room squares [239] (see Section 10), and λ-squares [63] (see Section 13).

Hill-climbing algorithm for a strong frame starter in $G \setminus H$

1.	Number-of-iterations is initialized to be 0
2.	**While** (number-of-iterations $< T(g)$) **and** ($c(S) \neq 0$), **do** choose r =1 or 2 at random with equal probability

3 ROOM FRAMES

In this section, we discuss a generalization of Room squares known as *Room frames*. These designs are of fundamental importance in recursive constructions for Room squares. Let S be a set, and let $\{S_1,\ldots,S_n\}$ be a partition of S. An $\{S_1,\ldots,S_n\}$-*Room frame* is an $|S|\times|S|$ array, F, indexed by S, that satisfies the following properties:

1. Every cell of F either is empty or contains an unordered pair of symbols of S.
2. The subarrays $S_i \times S_i$ are empty, for $1 \le i \le n$ (these subarrays are referred to as *holes*).
3. Each symbol $x \notin S_i$ occurs once in row (or column) s, for any $s \in S_i$.
4. The pairs in F are those $\{s,t\}$, where $(s,t) \in (S\times S)\setminus\bigcup_{i=1}^{n}(S_i\times S_i)$.

As is usually done in the literature, we will refer to a Room frame simply as a *frame*. The *type* of a frame F is defined to be the multiset $\{|S_i| : 1 \le i \le n\}$. We usually use an "exponential" notation to describe types: a type $t_1^{u_1}t_2^{u_2}\ldots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \le i \le k$.

In Figure 3.1, a frame of type $2^5 4^1$ based on symbol set $\{1,\ldots,9,a,b,c,d,e\}$ is depicted.

Frames of type 1^n and Room squares of side n are equivalent, as follows:

			ac		3*d*			8*b*	7*e*		46	59	
		7*b*		9*d*		6*e*	4*c*			8*a*	35		
9*c*				8*e*		5*b*		7*d*		16			2*a*
	9*e*				*ab*			6*c*	1*d*	57		28	
		8*c*	1*e*			*ad*	9*b*			23		47	
4*d*	3*c*	*ae*							2*b*		89		17
3*b*		5*d*		2*c*	4*e*						1*a*		69
			2*d*	1*b*				3*e*	5*c*	49		6*a*	
5*e*	4*b*					1*c*	6*d*				27		38
	8*d*		6*b*		7*c*		2*e*					13	45
68	5*a*			37	29			14					
	67	19	58			24	3*a*						
			79	4*a*	18			25	36				
7*a*		26				39	15		48				

Theorem 3.1. *A frame of type* 1^n *is equivalent to a Room square of side* n.

Proof. Suppose that F is a Room square of side n on symbol set S. Let ∞ be any symbol in S. By a suitable permutation of rows and columns of F, we can place the cells containing ∞ on the main diagonal (such a Room square is said to be *standardized*). Call this standardized square F'. Index the rows and columns of F' so that $\{\infty, x\}$ is the content of cell (x,x) of F, for all $x \in S \backslash \{\infty\}$. Now delete the pairs from all the cells on the main diagonal of F', constructing a frame of type 1^n.

Conversely, suppose that F is a frame of type 1^n on symbol set S, and let ∞ be any symbol not in S. If we place the pair $\{\infty, x\}$ in cell (x,x), for all $x \in S$, we obtain a (standardized) Room square of side n. □

We will first discuss the existence of frames of type t^u. Such frames can be constructed from orthogonal frame starters in the same way that Room squares can be constructed from orthogonal starters.

Theorem 3.2 ([66]). *Suppose that a pair of orthogonal frame starters in* $G \backslash H$ *exist, where* $|G| = g$ *and* $|H| = h$. *Then there exists a frame of type* $h^{g/h}$.

The first Room frame constructed in the literature was one of type 2^5; it was presented by Wallis in [258]. We give an example of a frame of type 2^5 (different from Wallis' example) in Figure 3.2. This frame was constructed from the frame starter presented in Example 2.3. Note that the rows and columns of this frame are indexed by the elements in $\mathbf{Z}_{10}$. The five holes are the subarrays indexed by $\{i, 5+i\} \times \{i, 5+i\}$, $i = 0,1,2,3,4$, and as such, the holes have not been drawn as contiguous 2×2 subarrays. Clearly, one could apply a suitable permutation of rows and columns so that the holes would be contiguous. If this were done, however, the automorphism would be less apparent.

Next, we describe three recursive constructions for frames. The first of these employs group-divisible designs. We refer to this construction as the fundamental frame construction (or FFC).

A *group-divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$, that satisfies the following properties:

1. $\mathcal{G}$ is a partition of X into subsets called *groups*.
2. $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point.
3. Every pair of points from distinct groups occurs in a unique block.

The *group-type* (or *type*) of a GDD $(X, \mathcal{G}, \mathcal{A})$ is the multiset $\{|G| : G \in \mathcal{G}\}$.

						79	34	26	18
29							80	45	37
48	30							91	56
67	59	41							02
13	78	60	52						
	24	89	71	63					
		35	90	82	74				
			46	01	93	85			
				57	12	04	96		
					68	23	15	07	

Figure 3.2 A frame of type 2^5.

Construction 3.3 (Fundamental Frame Construction, [231]). *Let $(X, \mathcal{G}, \mathcal{A})$ be a GDD having type T, and let $w : X \to \mathbf{Z}^+ \cup \{0\}$ (we say that w is a weighting). Suppose that for every $A \in \mathcal{A}$, there is a frame having type $\{w(x) : x \in A\}$. Then there is a frame having type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

A *pairwise balanced design* (or PBD) is a pair $(X, \mathcal{A})$ that satisfies the following properties:

1. $\mathcal{A}$ is a set of subsets of X (called blocks).
2. Every pair of points occurs in a unique block.

$(X, \mathcal{A})$ is a (v, K)-PBD if $|X| = v$ and $|A| \in K$ for all $A \in \mathcal{A}$. A set of positive integers K is said to be *PBD-closed* if $v \in K$ whenever a (v, K)-PBD exists. We have the following useful corollary of the fundamental frame construction.

Corollary 3.4 ([66]). *Let $t \geq 1$. The set $U_t = \{u \geq 4$: there exists a frame of type $t^u\}$ is PBD-closed.*

Proof. Let $(X, \mathcal{A})$ be a (v, U_t)-PBD. Then $(X, \{x : x \in X\}, \mathcal{A})$ is a GDD. Give every point weight t and apply FFC. □

If T is the type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ and m is an integer, then mT is defined to be the type $mt_1{}^{u_1} mt_2{}^{u_2} \ldots mt_k{}^{u_k}$. The following recursive construction is referred to as the *inflation construction*. It essentially "blows up" every cell into a pair of orthogonal Latin squares.

Construction 3.5 (Inflation Construction, [231]). *Suppose that there is a*

The third construction is a doubling construction. We need some special ingredients, which we now define. Let F be an $\{S_1, \ldots, S_n\}$-frame. We say that F is *skew* if for any pair of cells (s,t) and (t,s), where $(s,t) \in (S \times S) \setminus \bigcup_{i=1}^{n}(S_i \times S_i)$, precisely one is filled. As one would expect, the frame arising from a pair of skew-orthogonal frame starters (or a skew frame starter) is a skew frame. As an example, we note that the frame in Figure 3.2 is skew.

The second ingredient is a pair of orthogonal partitioned incomplete Latin squares. Let S be a set, and let $\{S_1, \ldots, S_n\}$ be a partition of S. Define an $\{S_1, \ldots, S_n\}$-*partitioned incomplete Latin square* (or PILS) to be an $|S| \times |S|$ array, F, indexed by S, which satisfies the following properties:

1. Every cell of F either is empty or contains a symbol from S.
2. The subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these are holes).
3. Each symbol $s \in S_i$ occurs once in row (or column) r, for any $r \notin S_i$.

Suppose F and G are $\{S_1, \ldots, S_n\}$-PILS on symbol set S. We say that F and G are *orthogonal* PILS (or OPILS) if for every $(s,t) \in (S \times S) \setminus \bigcup_{i=1}^{n}(S_i \times S_i)$, there is a unique cell (x,y) such that $(F(x,y), G(x,y)) = (s,t)$.

We define the *type* of PILS in the usual manner. The doubling construction is as follows:

Construction 3.6 (Doubling Construction, [231]). *Suppose that there exists a skew frame of type T and a pair of OPILS of type T. Then there exists a frame of type $2T$.*

We observe that the frame produced by the doubling construction will not be skew. The following is a useful corollary of the doubling construction:

Corollary 3.7 ([69]). *Suppose that there is a skew frame of type 1^n. Then there is a frame of type 2^n.*

Proof. OPILS of type 1^n are equivalent to a pair of orthogonal Latin squares having a common transversal. These exist for all positive integers $n \neq 2$, 3, or 6 [36]. But a frame of type 1^n does not exist if $n = 2$, 3, or 6. □

We illustrate the doubling construction by presenting a frame of type 2^7 in Figure 3.3. This frame was first constructed in [69]. The rows and columns are indexed $0,1,2,3,4,5,6,a,b,c,d,e,f,g$. The holes are indexed by $\{0,a\} \times \{0,a\}$, $\{1,b\} \times \{1,b\}$, etc.

We now indicate how to use these constructions by sketching the proof of existence of frames of type 2^u given in [69]. First, it is easy to see that there is no frame of type 2^2 or 2^3. Less obvious is the fact that a frame of type 2^4

	62	45	eg	13	cd	fb							
gc		03	56	fa	24	de							
ef	ad		14	60	gb	35							
46	fg	be		25	01	ac							
bd	50	ga	cf		36	12							
23	ce	61	ab	dg		40							
51	34	df	02	bc	ea								
								$6c$	$e5$	$4g$	$b3$	$2d$	$f1$
							$g2$		$0d$	$f6$	$5a$	$c4$	$3e$
							$4f$	$a3$		$1e$	$g0$	$6b$	$d5$
							$e6$	$5g$	$b4$		$2f$	$a1$	$0c$
							$1d$	$f0$	$6a$	$c5$		$3g$	$b2$
							$c3$	$2e$	$g1$	$0b$	$d6$		$4a$
							$5b$	$d4$	$3f$	$a2$	$1c$	$e0$	

Figure 3.3 A frame of type 2^7.

Define $F_2 = \{5,6,\ldots,20,22,23,24,27,28,29,32,33,34,39\}$. Recall that $U_2 = \{u \geq 4 :\text{there exists a frame of type } 2^u\}$. Suppose that we can prove that $F_2 \subseteq U_2$. We know from Corollary 3.4 that U_2 is PBD-closed. Hanani has proved in [106] that there is a (v,F_2)-PBD for all $v \geq 5$. Hence, it would then follow that $U_2 = \{v \geq 5\}$. So, we have reduced the existence question for frames of type 2^u to the problem of constructing a small number of these frames.

What can we prove using the tools described thus far? First, $\{5,9,13,17\} \subseteq U_2$ by using strong frame starters constructed in Theorem 2.11 (or, see [182]). It can also be shown that $\{8,12,16,20,24,28,32\} \subseteq U_2$ using strong frame starters constructed in [69] by computer. Now, Theorem 2.14 tells us that we cannot find a frame starter in $\mathbf{Z}_{2u}\setminus\{0,u\}$ if $u \equiv 2$ or $3 \pmod 4$, so the remaining values must be handled by other methods. Corollary 3.7 will handle the values $u = 7,11,15,19,23,27,29,33$, and 39, given skew frames of type 1^u for these u. These skew frames exist by Theorem 2.3 for $u = 7,11,19,23,27$, and 29. For $u = 15,33$, and 39, skew-orthogonal starters in $\mathbf{Z}_u$ were presented in [181].

It remains to consider the values $u = 6,10,14,18,22$ and 34. These were handled by a technique introduced in [69] called *orthogonal intransitive frame starters*. We do not describe the construction in detail but present in Figure 3.4 a frame of type 2^6 constructed by this method. This frame has Z_{10} in its automorphism group. The rows and columns are indexed $0,1,2,3,4,5,6,7,8,9,$

		$4Y$		$3X$		79		26		81	
			$5Y$		$4X$		80		37		92
48				$6Y$		$5X$		91		03	
	59				$7Y$		$6X$		02		14
13		60				$8Y$		$7X$		25	
	24		71				$9Y$		$8X$		36
$9X$		35		82				$0Y$		47	
	$0X$		46		93				$1Y$		58
$2Y$		$1X$		57		04				69	
	$3Y$		$2X$		68		15				70
67		89		01		23		45			
	78		90		12		34		56		

Figure 3.4 A frame of type 2^6.

In fact, it is proved in [69] that one can always construct by this method a frame of type 2^{q+1} whenever $q \equiv 1 \pmod 4$ is a prime power. So, $u = 6, 10, 14, 18$ are done, leaving only $u = 22$ and 34. These were also done by intransitive starters.

We summarize known existence results of frames of type t^u in the next two theorems.

Theorem 3.8 ([69], [65]). *There exists a frame of type t^u if any of the following conditions hold:*

1. $u = 4$, *and t is divisible by* 4, 6, *or* 10.
2. $u = 5$, *and t is divisible by* 2, 3, 5, *or* 7.
3. $u \geq 6$ *is even, and t is even.*
4. $u \geq 7$ *is odd.*

Theorem 3.9 ([69]). *There does not exist a frame of type t^u if any of the following conditions hold:*

1. $u = 2$ *or* 3.
2. $u = 4$, *and* $t = 2$ ([229]).
3. $u = 5$, *and* $t = 1$.
4. *u is even, and t is odd.*

Some attention has also been paid to frames with two hole sizes. Many computational results are presented in [65]. It is also observed there that the set $U_{s,t} = \{u :$for every $0 \leq a \leq u$, there exists a frame of type $s^a t^{u-a}\}$ is PBD-closed. Then, using PBD constructions, it is shown that $u \in U_{2,4}$ for all $u \geq 48$.

The next construction we mention is a filling-in-holes construction. This produces Room squares from frames. We present a basic version of this construction now; more general versions will be discussed in the Section 4.

Construction 3.10 (Filling-in-Holes Construction). *Suppose that there is a frame of type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$, and let $w = 0$ or 1. For $1 \leq i \leq k$, suppose there is a Room square of side $t_i + w$. Then there is a Room square of side $w + \sum_{i=1}^{k} t_i u_i$.*

The last topic of discussion in this section is two existence proofs for Room squares. The first uses frames as building blocks, and is based on [74]. This proof also makes use of the filling-in-holes construction. The only ingredients we need for the first existence proof are the following:

1. A Room square of side n, for $7 \leq n \leq 59$, n odd.
2. A frame of type $2^i 4^{6-i}$, for $0 \leq i \leq 6$.
3. A transversal design $TD(6, m)$, for all m such that $\gcd(m, 6) = 1$.

The Room squares of sides up to 59 all can be constructed by strong starters, for example, except side 9, that can be obtained from orthogonal starters. The frames of type $2^i 4^{6-i}$ were found in [74]. The necessary transversal designs all exist by MacNeish's theorem [166].

Given these ingredients, we proceed by induction as follows. Let $n \geq 61$, and assume that a Room square of side n' exists for all odd n', $7 \leq n' < n$. Find an integer m such that $\gcd(m, 6) = 1$ and $12m + 1 \leq n \leq 24m + 1$. Take a transversal design $TD(6, m)$, and give every point weight 2 or 4 so that the sum of the weights of all the points is $n - 1$. Apply FFC, using the frames of type $2^i 4^{6-i}$ as input frames. Fill in the holes of the resulting frame with Room squares, using $w = 1$ in the filling-in-holes construction. The necessary Room squares all exist by the induction assumption; thus, we obtain a Room square of side n. This completes the proof of Room square existence.

We are emphasizing frame techniques because of their applicability to many other problems involving Room squares, some of which we shall encounter in other places in this survey. However, the original "condensed" existence proof for Room squares given by Mullin and Wallis in [189] is already very short and elegant. We feel it would also be of interest to present a sketch of this proof.

This proof depends on the Mullin-Nemeth strong starters [175] and the Chong-Chan strong starters [43]. Also required are Room squares of sides 9 and 15. Then two recursive constructions will finish the job. This first of these is the direct product construction of Stanton and Horton [222] that states that square of side n_1n_2 (see Section 4). The second is the following powerful construction of Wallis.

Theorem 3.11 ([252], [254]). *If a Room square of side n_1 exists, $n_1 \geq n$, and n is odd, then a Room square of side nn_1 exists.*

The Mullin-Wallis existence proof for Room squares is also an inductive proof. Any odd positive integer v has a prime power factorization $v = 3^{a_3}5^{a_5}7^{a_7}$ .... First, suppose that $a_3 = a_5 = 0$. If v is a prime power, then a strong starter of order v exists (and hence a Room square exists) unless $v = 9$, but a Room square of side 9 exists also. If v is not a prime power, then write $v = v_1v_2$, where Room squares of sides v_1 and v_2 exist (by induction), and apply the direct product construction. Next, suppose $a_3 + a_5 = 1$. Then Theorem 3.11 gives a Room square of side v (using $n = 3$ or 5). Next, suppose that $a_3 + a_5 = 2$. Then v can be written as $v = v_1v_2$, where $v_1 = 9$, 15, or 25. Since a Room square of order v_1 exists, we have one of order v (using the direct product construction, unless $v = v_1$, in which case we're already done). Finally, if $a_3 + a_5 \geq 3$, then we can use Theorem 3.11 with $n = 3$ or 5. This completes the proof.

4 ROOM SQUARES WITH SUBSQUARES

Suppose that F is a Room square of side n on symbol set S. A square $s \times s$ subarray G of F is said to be a *Room subsquare* of side s if it is itself a Room square of side s on a subset $T \subseteq S$ containing $s+1$ symbols. In particular, any filled cell of a Room square is a Room subsquare of side 1. In view of Theorem 1.1, no Room square can contain a Room subsquare of side 3 or 5. However, we can construct Room squares *missing* subsquares of these sides. We have the following formal definition:

Let S be a set of $n+1$ symbols, and let T be a subset of S of cardinality $s+1$. An (n,s)-IRS (*incomplete Room square*) is an $n \times n$ square array F that satisfies the following:

1. Every cell of F either is empty or contains an unordered pair of symbols of S.
2. There is an empty $s \times s$ subarray G of F.
3. Each symbol of $S \backslash T$ occurs once in each row and column of F.
4. Each symbol of T occurs once in each row and column not meeting G, but not in any row or column meeting G.
5. The pairs in F are precisely those $\{x,y\}$ where $(x,y) \in (S \times S) \backslash (T \times T)$.

			48			37	6*X*			59
			69			5*X*	38			47
				39	4*X*			57	68	
67	8*X*		3*Y*			04	15	29		
58	79			4*Y*	03				2*X*	16
9*X*		78		06	5*Y*		24		13	
			05	7*X*	89	6*Y*		14		23
	46	3*X*		25		19	7*Y*			08
	35	49	1*X*		26			8*Y*	07	
34		56			17	28		0*X*	9*Y*	
			27	18			09	36	45	*XY*

Figure 4.1 An (11,3)-incomplete Room square.

a Room square of side n containing a subsquare of side s. We also observe that existence of an (n,s)-IRS is equivalent to existence of a Room frame of type $1^{n-s}s^1$.

In Figure 4.1, we present an $(11,3)$-IRS that implies the existence of a frame of type $1^8 3^1$. The symbol set is $\{0,1,2,3,4,5,6,7,8,9,X,Y\}$, and the missing subsquare on symbols set $\{0,1,2,3\}$ is in the upper left corner.

For which ordered pairs (n,s) does there exist an (n,s)-IRS? Of course, n and s must be odd, and it was shown in [50] that $n \geq 3s+2$ is a necessary condition for existence. In the remainder of this section, we discuss the research done on this question.

First, we mention some more general filling-in-holes constructions that utilize IRS.

Construction 4.1 (Filling in Holes).

1. *Suppose that there is a frame of type $t_1^{u_1}t_2^{u_2}\dots t_k^{u_k}$, and let $w \geq 1$ be odd. For $1 \leq i \leq k$, suppose that there is a $(t_i + w, w)$-IRS. Then there is a $(w + \sum_{i=1}^{k} t_i u_i, w)$-IRS.*
2. *Suppose that there is a frame of type $t_1^{u_1}t_2^{u_2}\dots t_k^{u_k}$, where $u_k = 1$, and let $w \geq 1$ be odd. For $1 \leq i \leq k-1$, suppose that there is a $(t_i + w, w)$-IRS. Then there is a $(w + \sum_{i=1}^{k} t_i u_i, w + t_k)$-IRS.*

If we combine the inflation construction with the filling-in-holes constructions in various ways, we obtain a class of constructions called the product constructions. The simplest form is the direct product. This result was first stated by Bruck [38] in 1963, but the construction he gave was incorrect, as observed by Mullin and Nemeth [174]. A correct construction was given by

Construction 4.2 (Direct Product, [222]). *Suppose that there exist Room squares of sides u and v. Then there exists a Room square of side uv that contains Room subsquares of sides u and v.*

Proof. A Room square of side u is equivalent to a frame of type 1^u. Applying the inflation construction, we get a frame of type v^u. Then we apply the filling-in-holes construction with $w = 0$. □

A generalization known as the singular direct product was then given by Horton, Mullin and Stanton in 1971 [119].

Construction 4.3 (Singular Direct Product, [112], [119]). *Suppose that there is a Room square of side u and a Room square of side v containing a Room subsquare of side w, where $v - w \neq 6$. Then there exists a Room square of side $u(v-w)+w$ that contains Room subsquares of sides u, v and w.*

Proof. A Room square of side u is equivalent to a frame of type 1^u. Noting that $v - w \neq 6$, we apply the inflation construction, obtaining a frame of type $(v-w)^u$. Then we apply the filling-in-holes construction. □

A further generalization known as the singular indirect product was discovered by Mullin in 1980 [172]. A *transversal design* $TD(k,n)$ can be defined to be a k-GDD of type n^k. It is well known that a $TD(k,n)$ exists if and only if $k-2$ mutually orthogonal Latin squares of order n exist. However, Mullin's construction employs a transversal design with a "hole." An *incomplete transversal design* $TD(k,n) - TD(k,m)$ is a quadruple $(X, Y, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

1. X is a set of cardinality kn.
2. $\mathcal{G} = \{G_i : 1 \leq i \leq k\}$ is a partition of X into k groups of size n.
3. $Y \subseteq X$, $|Y| = km$, and $|Y \cap G_i| = m$, for $1 \leq i \leq k$.
4. $\mathcal{A}$ is a set of $n^2 - m^2$ blocks of size k, each of which intersects each group in a point.
5. Every pair of points x, y from distinct groups, such that at least one of x, y is in $X \setminus Y$, occurs in a unique block of $\mathcal{A}$.

A $TD(k,n) - TD(k,m)$ can be thought of as a $TD(k,n)$ from which the blocks of a $TD(k,m)$ have been removed.

Construction 4.4 (Singular Indirect Product, [172]). *Suppose that there is a Room square of side u, and a Room square of side v containing a Room subsquare of side w. Let $0 \leq a \leq w$, and suppose that there is an incomplete transversal design $TD(4, v-a) - TD(4, w-a)$. Finally, suppose that there is a Room square of side $u(w-a)+a$. Then there exists a Room square of side*

$a \neq 3(w-a)$, then the resulting Room square also contains a Room subsquare of side v.

Other generalizations of the product constructions are discussed in [172] and [231].

Other "early" constructions for Room squares containing subsquares were given by Wallis; see [253], [256], [260]. The first general existence result was proved by Wallis [259] in 1974. In that paper, it was first proved that for any odd $s \geq 7$, there is a constant n_s such that an (n,s)-IRS exists if n is odd, $n \geq n_s$. At that time, no specific upper bounds could be placed on the constants n_s.

Concrete upper bounds were first given by Stinson in 1981 [232] using frame techniques. It was shown there that $n_s \leq \max\{s+644, 6s+9\}$ for any odd $s \geq 1$. (That paper also provided the first known examples of $(n,3)$-IRS and $(n,5)$-IRS.) Further improvements were made in [235], [75], and [74]. The following theorem summarizes the best known general existence results for (n,s)-IRS. Since existence of an (n,s)-IRS requires $n \geq 3s+2$, these results are very close to best possible.

Theorem 4.5.

1. ([74],[65]) *For odd s, $3 \leq s \leq 15$, there is an (n,s)-IRS if and only if $n \geq 3s+2$ is odd.*
2. ([74]) *For all odd $s \geq 37$ and all odd $n \geq (7s-5)/2$, there is an (n,s)-IRS.*
3. ([74]) *For all odd $s \geq 127$ and all odd $n \geq 3s+240$, there is an (n,s)-IRS.*
4. ([244]) *For all odd $s \geq 393$, there is an (n,s)-IRS if and only if $n \geq 3s+2$ is odd.*

We present in Table 4.1 the upper bounds on n_s for $17 \leq s \leq 35$, updated from [74].

The results in Theorem 4.5 also utilize the filling-in-holes constructions, but make use of frames constructed recursively from FFC. In this way one obtains more flexibility than in the product constructions.

The extremal case of $(3s+2,s)$-IRS is especially interesting. In [259], Wallis conjectured that a $(23,7)$-IRS does not exist and offered a prize of \$10.00 for a proof or disproof. In fact, no $(3s+2,s)$-IRS were known to exist until Dinitz constructed an $(11,3)$-IRS on the computer in 1980. This square was first presented in [231]; we have reproduced it in Figure 4.1. Using the $(11,3)$-IRS,

TABLE 4.1 Bounds on the existence of Room squares with subsquares

s	$n_s \leq$	s	$n_s \leq$	s	$n_s \leq$	s	$n_s \leq$
17	67	19	69	21	71	23	71
25	95	27	97	29	99	31	101

Stinson proved in [231] that there exists a $(3s+2,s)$-IRS for all $s \equiv 3 \pmod 8$. In 1982, Wallis disproved his own conjecture by constructing a $(23,7)$-IRS in [262] (we do not know if Wallis paid himself the \$10.00). Soon after, Wallis constructed $(3s+2,s)$-IRS for all odd $s \geq 3$.

Theorem 4.6 ([261]). *For all odd $s \geq 3$, there is a $(3s+2,s)$-IRS.*

In view of Theorems 4.5 (parts 1 and 4) and 4.6, it seems extremely probable that an (n,s)-IRS exists if and only if n and s are odd, $n \geq 3s+2$, with the single exception $(n,s) \neq (5,1)$.

Finally, we note that Lindner and Rosa proved in [164] that a partial Room square can be (finitely) embedded in a Room square, but they gave no concrete upper bounds on the size of the square obtained.

5 ROOM d-CUBES OF DIMENSION EXCEEDING TWO

Recall that a Room d-cube of side n is a d-dimensional array, each two-dimensional projection of which is a Room square of side n. We have also pointed out that a Room d-cube of side n is equivalent to d pairwise orthogonal-symmetric Latin squares, or POSLS, of order n. The maximum number of POSLS of order n is denoted $\nu(n)$. The existence theorem for Room squares tells us that $\nu(n) \geq 2$ for all odd $n \geq 7$.

The first discussion of lower bounds for $\nu(n)$ was given by Gross, Mullin, and Wallis in 1973 in [104]. Since then, further results have appeared in [31], [56], [57], [58], [59], [61], [66], [70], [76], [101], [114], and [117].

Several results concerning $\nu(n)$ were proved in [104], one of which is the following:

Theorem 5.1. *$\nu(n) \to \infty$ as $n \to \infty$.*

Gross, Mullin, and Wallis also showed that $\nu(q) \geq 1 + (t-1)/(2^{\alpha-1})$ when $q = 2^\alpha t + 1$ is a prime power and t is odd. However, this bound was improved by Dinitz [56] to $\nu(q) \geq t$ under the same conditions (as we stated in Theorem 2.5).

A list of lower bounds on $\nu(n)$ for $n \leq 101$ was also presented in [104]. This list was later updated in [31] and [58]. We present the current world records in Table 5.1. Finally, we note that a longer list of lower bounds, for $n \leq 999$, was given in [66].

Many of the records in Table 5.1 arise from a special type of starter defined by Dinitz in [59], which we now describe. Suppose that $q = 2^\alpha t + 1$ is an odd prime power, where t is odd. Let ω be a primitive element in GF(q), and let C_0 be the (unique) subgroup of G^* of order t and index 2^α, where G^*

TABLE 5.1 Lower bounds on $\nu(n)$

n	$\nu(n) \geq$	n	$\nu(n) \geq$	n	$\nu(n) \geq$	n	$\nu(n) \geq$
1	$= \infty$	27	13	53	17	79	39
3	$= 1$	29	13	55	5	81	5
5	$= 1$	31	15	57	5	83	41
7	$= 3$	33	5	59	29	85	5
9	$= 4$	35	5	61	21	87	5
11	5	37	15	63	5	89	11
13	5	39	5	65	5	91	5
15	4	41	9	67	33	93	5
17	5	43	21	69	5	95	5
19	9	45	5	71	35	97	5
21	5	47	23	73	9	99	5
23	11	49	5	75	5	101	31
25	7	51	5	77	5		

A starter S in GF(q) is said to be a Δ-quotient coset starter (or Δ-QCS) if the following property is satisfied: For all pairs $\{x,y\},\{x',y'\} \in S$, if $x,x' \in C_i$ for some i, then $y/x = y'/x'$.

For any $z \in \mathrm{GF}(q)$, $z \neq 1$, define $C_i^z = (1/(z-1))C_i$. Given a list of Δ field elements from GF(q), say, $A = (a_0,\ldots,a_{\Delta-1})$, define $S(A) = \{\{x, a_i x\} : x \in C_i^{a_i} : 0 \leq i \leq \Delta - 1\}$. Note that if $S(A)$ is a starter, then it is a Δ-QCS. The conditions for $S(A)$ to be a starter were given in [73] and are stated in the following theorem:

Theorem 5.2. *Suppose that $q = 2^\alpha t + 1$ is an odd prime power, where t is odd. Denote $\Delta = 2^{\alpha-1}$. Then $S(A)$ is a Δ-quotient coset starter in GF(q) if and only if the following conditions are satisfied:*

1. $a_i \notin C_0$, $0 \leq i \leq \Delta - 1$.
2. $(a_j - 1)/(a_i - 1) \notin C_{(j-i) \bmod 2^\alpha}$, $0 \leq i,\ j \leq \Delta - 1$, $i \neq j$.
3. $(a_j - 1)/(a_j a_i - a_j) \notin C_{(j-i) \bmod 2^\alpha}$, $0 \leq i,\ j \leq \Delta - 1$, $i \neq j$.
4. $(a_i a_j - a_i)/(a_i - 1) \notin C_{(j-i) \bmod 2^\alpha}$, $0 \leq i,j \leq \Delta - 1$, $i \neq j$.
5. $(a_i a_j - a_i)/(a_j a_i - a_j) \notin C_{(j-i) \bmod 2^\alpha}$, $0 \leq i,\ j \leq \Delta - 1$, $i \neq j$.

We remark that the Dinitz starters S_a described in Section 2.1 are a special case of Δ-QCS, since S_a is the Δ-QCS $S(a,a,\ldots,a)$.

Using Δ-QCS, the following lower bounds were obtained in [59]. These are all improvements over the bounds from Theorem 2.5.

Theorem 5.3 ([59]). $\nu(13) \geq 5$, $\nu(25) \geq 7$, $\nu(29) \geq 9$, $\nu(37) \geq 15$, $\nu(41) \geq 9$

We have mentioned that $\nu(n) \geq 2$ for all odd $n \geq 7$. In fact, Dinitz and Stinson in 1981 [70] proved that $\nu(n) \geq 3$ for all odd $n \geq 7$. Then, in 1987, Dinitz proved the following:

Theorem 5.4 ([61]). *$\nu(n) \geq 5$ for all odd $n \geq 11$, except possibly for $n = 15$.*

This result is proved by a combination of direct and recursive methods. Direct methods were required for most values of n up to 355. The main observation is that the existence of two strong starters S and T in $\mathbf{Z}_n$, such that S and T are orthogonal and S and $-T$ are orthogonal, implies the existence of five orthogonal starters, S, $-S$, T, $-T$, and P (the patterned starter). Hence, a Room 5-cube of side n can then be produced. The necessary starters were constructed by a suitable modification of the hill-climbing algorithm presented in Section 2.2 [68].

The main recursive construction for Room d-cubes uses a d-dimensional generalization of Room frames. In general, we can define a d-dimensional Room frame of type T to be a d-dimensional array, each 2-dimensional projection of which is a Room frame of type T. If there exist d orthogonal frame starters in $G \backslash H$, where $|G| = g$ and $|H| = h$, then we can obtain a d-dimensional Room frame of type $h^{g/h}$. Hence, Theorem 2.12 provides us with infinite classes of d-dimensional Room frames for each $d \geq 2$.

The recursive constructions for Room 5-cubes used in [61] depended heavily on 5-dimensional Room frames of type 2^u, $u = 12, 13, 16, 17, 20$, and 21. These were also obtained from orthogonal frame starters found by computer search.

Most of the lower bounds in Table 5.1 arise from orthogonal starters, via Theorem 2.5, Theorem 5.3, or Theorem 5.4. The result $\nu(9) = 4$ is due to Dinitz and Wallis [76].

The only known general upper bound on $\nu(n)$ is the trivial bound $\nu(n) \leq n-2$ [114]. Gross, Mullin, and Wallis conjectured that $\nu(n) \leq (n-1)/2$ for all n. On the other hand, Luc Teirlinck conjectured that $\nu(n) = n-2$ for all sufficiently large n. This is probably the most interesting open conjecture concerning Room squares and their generalizations.

6 NONISOMORPHIC ROOM SQUARES

Let F_1 and F_2 be Room squares of side n on symbol set S. We say that F_1 and F_2 are *isomorphic* if we can obtain one from the other by any combination of permuting the rows, permuting the columns and permuting the symbols in S. We say that F_1 and F_2 are *equivalent* if they are isomorphic, or if F_1 is isomorphic to the transpose of F_2.

More generally, we can define *equivalent* Room d-cubes of side n. The

TABLE 6.1 Values $\mathrm{IR}_d(n)$, $n < 10$

n	d	$\mathrm{IR}_d(n)$	Reference
3	1	1	
3	2	0	
5	1	1	
5	2	0	[200]
7	1	6	[89], [90]
7	2	6	[251], [95], [96]
7	3	1	[266]
7	4	0	[266]
9	1	396	[89], [90]
9	2	257630	[28]
9	3	267	[27]
9	4	1	[76]
9	5	0	[76]

S, having one factors $f_1,\ldots,f_n$. If π is a permutation of S, then define $\mathcal{F}^\pi$ to be the one-factorization having one-factors $f_1^\pi,\ldots,f_n^\pi$, where $f_i^\pi = \{\{x^\pi,y^\pi\} : \{x,y\} \in f_i\}$, $1 \le i \le n$. Now, let $\mathsf{F} = \{\mathcal{F}_1,\ldots,\mathcal{F}_d\}$ and $\mathsf{G} = \{\mathcal{G}_1,\ldots,\mathcal{G}_d\}$ be two sets of d pairwise orthogonal one-factorizations of K_{n+1}. We say that F and G are *equivalent* if there is some permutation π of the vertex set S such that $\{\mathcal{F}_1^\pi,\ldots,\mathcal{F}_d^\pi\} = \{\mathcal{G}_1,\ldots,\mathcal{G}_d\}$. Note that this definition of equivalent coincides with the previous definition when $d = 2$.

Define $\mathrm{IR}_d(n)$ to be the number of inequivalent sets of d pairwise orthogonal one-factorizations of K_{n+1}, on the same vertex set S. When $d = 1$, $\mathrm{IR}_d(n)$ just counts the number of inequivalent (or nonisomorphic) one-factorizations of K_{n+1}. Also, define $\mathrm{NR}(n)$ to be the number of nonisomorphic Room squares of side n on symbol set S. It is obvious that $\mathrm{NR}(n) \le 2\mathrm{IR}_2(n)$.

All numbers $\mathrm{IR}_d(n)$ have been determined for $n < 10$. These are presented in Table 6.1.

For $n > 10$, no exact values of $\mathrm{IR}_d(n)$ are known. Cameron proved in [40] that $\ln \mathrm{IR}_1(n) \sim n^2/(2\ln n)$ for sufficiently large odd n. Gross proved some general lower bounds in [97]. Dinitz and Stinson proved a lower bound on $\mathrm{NR}(n)$ in [71], which we state below in Theorem 6.1. For $d > 2$, no general lower bounds on $\mathrm{IR}_d(n)$ are known. We conjecture that for any fixed d, $\mathrm{IR}_d(n) \to \infty$ as $n \to \infty$.

Theorem 6.1 ([71]). *For odd* $n \ge 153$, $NR(n) > 0.19e^{.04n^2}$.

We can sketch the proof of this theorem. First, construct a large number of distinct Room squares of a given side n (on a specified symbol set) using the fundamental frame construction with input frames of types 1^9 and $1^8 3^1$. Then divide by the maximum number of (distinct) squares that could be isomorphic ... obtaining a lower bound on $\mathrm{NR}(n)$. In this way, it is and s are odd, then

$$\mathrm{NR}(s) > \frac{(9!)^{n^2} 2^9 (n!)^{18}}{((11n)!)^2 (11n+1)!}.$$

Use of Stirling's formula then yields Theorem 6.1. We note that the constants in Theorem 6.1 can be improved, but it does not seem possible to prove by this method a result of the form $\mathrm{NR}(n) \geq c_1 n^{c_2 n^2}$. Nevertheless, we conjecture that $\mathrm{NR}(n) \geq c_1 n^{c_2 n^2}$ for positive constants c_1 and c_2.

Also of interest is the idea of isomorphic starters. Two starters S_1 and S_2 are said to be *isomorphic* if the one-factorizations they generate are isomorphic. Two starters S_1 and S_2 in an abelian group G are *equivalent* if $\alpha(S_1) = S_2$ for some permutation α which is an automorphism of the group G. (In the case $G = \mathbf{Z}_n$, α must have the form $\alpha(i) = mi \pmod n$ for some m relatively prime to n.) Clearly, equivalent starters are isomorphic, but the converse need not hold. The following results have been proved concerning equivalent and isomorphic strong starters:

Theorem 6.2 ([129], [196], [193]). *Suppose that S_1 and S_2 are isomorphic strong starters in the group $\mathbf{Z}_n$. Then S_1 and S_2 are equivalent if n satisfies one of the following conditions:*

1. *n is an odd prime.*
2. *$n = pq$, where p and q are odd primes, $p < q$ and $q \not\equiv 1 \pmod p$.*

Theorem 6.3 ([196]). *If $p > 5$ is an odd prime and $n \geq 2$, then there exist two strong starters in the group $\mathbf{Z}_{p^n}$ that are isomorphic but not equivalent.*

Theorem 6.4 ([196]). *If p and q are odd primes, $5 < p < q$ and $q \equiv 1 \pmod p$, then there exist two strong starters in the group $\mathbf{Z}_{pq}$ that are isomorphic but not equivalent.*

Finally, we mention an enumeration of distinct and inequivalent strong starters in small cyclic groups that was done by computer in [129]. We present these results in Table 6.2 on the following page.

7 SKEW ROOM SQUARES

In this section, we discuss skew Room squares. Let F be a Room square that has been standardized. We say that F is *skew* if, given any pair of cells located symmetrically with respect to the main diagonal, precisely one is empty. More generally, skew frames were defined in Section 3, and it is easily seen that a skew Room square of side n is equivalent to a skew frame of type 1^n.

We presented a skew Room square of side 7 in Figure 1.1, and a skew

TABLE 6.2 Enumeration of strong starters in cyclic groups

n	Distinct strong starters in $\mathbf{Z}_n$	Inequivalent strong starters in $\mathbf{Z}_n$
7	1	1
9	0	0
11	4	2
13	4	2
15	32	4
17	224	14
19	800	52
21	6600	555
23	27554	1267
25	158680	7934
27	1249650	69425

by Beaman and Wallis [29] in 1975 after a lengthy computer search. (It is in fact unique up to isomorphism.)

Skew Room squares were investigated by several researchers, mainly Mullin, Wallis, and Stinson. Wallis first proved in [257] that a skew Room square of side n exists for all odd $n > n_0$, where n_0 is some positive integer. Then the spectrum of skew Room squares was studied in the following papers, among others: [19], [30], [185], [186], and [227]. The spectrum was finally determined by 1981, when the "last" side was produced, namely, side 69, in [231]. A short existence proof was then given by Stinson [230]; a summary of this proof is presented in [233].

Theorem 7.1 ([233]). *There is a skew Room square of side n if and only n is odd and $n \neq 3$ or 5.*

∞1		49	37	28		56		
89	∞2				57	34		16
	58	∞3		69	24		17	
	36	78	∞4		19		25	
	79		12	∞5	38		46	
45					∞6	18	39	27
		26	59	13		∞7		48
67	14					29	∞8	35
23		15	68	47				∞9

As with Room squares, the proof is a combination of direct methods (based on skew starters and skew-orthogonal starters) and recursive constructions (using skew frames, in particular, those of types 4^4, $4^4 2^1$, 4^5, and $4^4 6^1$).

Other than type 1^u, the only class of skew frames of type t^u to be investigated are those of type 2^u. The following existence result is proved in [238] and [163]:

Theorem 7.2 ([238], [163]). *Let $u \geq 5$, $u \neq 6, 22, 23, 24, 26, 27, 28, 30, 34$, or 38. Then there is a skew Room frame of type 2^u.*

There is a generalization of skew Room squares that bears mentioning, namely, the idea of complementary Room squares [265]. Two standardized Room squares of order n, F and F', are said to be *complementary* if every off-diagonal cell is empty in precisely one of F and F'. More generally, let F and F' be $\{S_1, \ldots, S_n\}$-frames. We say that F and F' are *complementary* if every cell $(s,t) \in (S \times S) \setminus \bigcup_{i=1}^{n} (S_i \times S_i)$ is filled in precisely one of F and F'. We observe that if F is a skew frame, and F^T is obtained by transposing F, then F and F^T are complementary. However, there are cases where a pair of complementary frames are known to exist, whereas a skew frame is not known to exist. One example is type 2^6 [141].

One type of three-dimensional generalization of skew Room squares has been studied by Lamken and Vanstone in [147]. They study three-dimensional frames with the property that one two-dimensional projection is skew and the other two two-dimensional projections are complementary. Several constructions are given for these objects.

We will now discuss several applications of skew Room squares and skew frames to the construction of other designs. We have already seen one application of skew frames, namely, in the doubling construction of Section 3. We remark that in the doubling construction, as well as in some of the other constructions we are about to describe, the construction will also work with complementary frames.

Skew Room squares were first introduced by Wallis in [249] and [248] as an ingredient in certain doubling constructions for Room squares. In those papers, he proved that the existence of a skew Room square of side n implies the existence of a Room square of side $2n+1$ [249] and a Room square of side $2n-1$ [248] (neither of the resulting Room squares will be skew). Even though the spectrum of Room squares was subsequently determined by other methods, skew Room squares were studied extensively, mainly for their intrinsic interest. However, skew Room squares have since had numerous, surprising applications to the construction of various other types of combinatorial designs, and have become an important tool in diverse areas of design theory. We will survey some of these applications now.

Skew Room squares appear implicitly in a 1970 construction by Mullin

Chan-Dinitz skew starter in $\mathbf{Z}_g$ implies the existence of a self-orthogonal Latin square of order g.

The next application (chronologically) after [177] is the result of Giles [92].

Theorem 7.3 ([92]). *If there exists a skew Room square of side* s, *then there is a balanced incomplete block design with parameters* $(s, \binom{s}{2}, 2s-2, 4, 6)$.

Proof. Construct a skew frame F of type 1^s from the skew Room square. Then, for every filled cell (x, y) of F, construct a block $\{x, y\} \cup F(x, y)$. It is a simple verification that every pair of symbols occurs in six such blocks. □

Next, there are several applications to problems of nesting of Steiner triple systems [156], [160], nesting group-divisible designs [157] and partial Steiner triple systems [195], nesting cycle systems [161], [163], construction of almost resolvable cycle systems [109], construction of decompositions of K_n and $2K_n$ into wheels [47], and construction of BIBDs with block size four admitting a blocking set [110]. Constructions of this type are surveyed in [198] and [199]; see also [162]. These constructions all have a similar flavor; we state one of them without proof as an example. This is a construction for nested cycle systems. An *m-cycle system* of order n is a partition of the edges of K_n into cycles of length m. An m-cycle system can be *nested* if it is possible to assign a point x to every cycle C, such that the set of all edges xy $(y \in C)$ also form an edge-partition of K_n.

Theorem 7.4 ([163]). *Let* $m \geq 3$ *be odd, and suppose that there is a skew Room frame of type* 2^k. *Then there is an* m*-cycle system of order* $2^k m + 1$ *that can be nested.*

Skew Room squares and frames have also been very useful in the construction of partitioned balanced tournament designs (or, equivalently, maximum empty subarray Room squares). These are discussed further in Section 9. Lamken and Vanstone gave several constructions for these objects utilizing skew Room squares in [141] and [145].

We also note that Horton employed skew starters in a construction for hamiltonian path balanced tournament designs presented in [116]; see Section 16. Finally, encryption methods using skew Room squares are discussed in [209].

8 PERFECT ROOM SQUARES

A one-factorization $\mathcal{F}$ of K_{2n} is called *perfect* if the union of any two one-two known infinite families of P1Fs. If p is prime, then there is a P1F of K_{p+1} and a P1F of K_{2p}. There are currently only 18 other values of n for which a P1F of K_{2n} is known to exist [217]. The smallest unsettled case is whether a P1F of K_{52} exists.

Since a Room square of side n is equivalent to a one-factorization of K_{n+1}, this suggests the following definition. A *perfect* Room square is one in which both the row and column one-factorizations of K_{n+1} are perfect. More generally, we define $\nu_P(n)$ to be the maximum number of orthogonal P1Fs of K_{n+1}. In particular, $\nu_P(n) \geq 1$ if and only if a P1F of K_{n+1} exists.

In 1973, Anderson [2] constructed a Room square of side 15 such that one of the two one-factorizations obtained from it was perfect. However, the first perfect Room square was discovered in 1987 by Rosa (see [62]), who found a perfect Room square of side 11 (hence $\nu_P(11) \geq 2$). The two orthogonal P1Fs are those generated from the Dinitz starters S_7 and S_8 in GF(11).

Several other examples of orthogonal P1Fs generated from Dinitz starters have been found. We record these results in the following theorem:

Theorem 8.1 ([62]). $\nu_P(11) \geq 3$, $\nu_P(19) \geq 5$, $\nu_P(23) \geq 9$, $\nu_P(43) \geq 3$, $\nu_P(47) \geq 5$, $\nu_P(59) \geq 5$, $\nu_P(67) \geq 7$, $\nu_P(71) \geq 7$, $\nu_P(79) \geq 9$, *and* $\nu_P(83) \geq 17$.

Using 2-QCS (as described in Section 5), the following lower bounds were also found.

Theorem 8.2 ([62]). $\nu_P(29) \geq 5$, $\nu_P(37) \geq 7$, *and* $\nu_P(53) \geq 8$.

Finally, we mention one other lower bound which has not appeared in the literature. In [73], 24 different 2-QCS in GF(125) are presented, each of which generates a P1F of K_{126}. In fact, there exists a subset of 9 of these 24 P1Fs, any two of which are orthogonal. Hence, we have

Theorem 8.3. $\nu_P(125) \geq 9$.

Proof. Construct GF(125) from the polynomial $x^3 + x^2 + 2$, which is irreducible over $\mathbf{Z}_5$, and use x as the primitive element. The starters are $S(x^9, x^{41})$, $S(x^{18}, x^{38})$, $S(x^{23}, x^{91})$, $S(x^{34}, x^{58})$, $S(x^{42}, x^{78})$, $S(x^{46}, x^{42})$, $S(x^{66}, x^{34})$, $S(x^{81}, x^{79})$, and $S(x^{86}, x^{18})$. □

Finally, we mention a generalization that has not received significant study to date. A one-factorization $\mathcal{F}$ of K_{2n} is said to be *uniform* if for any one-factors $f_1, f_2, f_3, f_4 \in \mathcal{F}$, where $f_1 \neq f_2$ and $f_3 \neq f_4$, we have that $f_1 \cup f_2 \cong f_3 \cup f_4$. (Hence, a P1F is uniform.) Then define a *uniform* Room square to be one in which both the row and column one-factorizations are uniform. We observe that a Mullin-Nemeth starter will generate a uniform Room

9 MAXIMUM EMPTY SUBARRAY ROOM SQUARES

Suppose that a Room square of side $2n+1$ has an $s \times s$ subarray of empty cells; then $s \leq n$. If $s = n$, then we say that the Room square is a *maximum empty subarray* Room square, which we abbreviate to MESRS. We present an MESRS of side 9 in Figure 9.1.

Maximum empty subarray Room squares were first studied by Stinson [237]. It was shown there that there is no MESRS of side 7, but that MESRS of sides 9 and 11 exist. As well, two recursive constructions for MESRS were also described. The square of side 9 in Figure 9.1 was taken from [237]. In fact, there are precisely two nonisomorphic examples of MESRS of side 9 [216].

The problem was further investigated by Lamken and Vanstone [141], [145], [144] and by Lamken [134]. We summarize the existence results of these four papers in the following theorem:

Theorem 9.1 ([141], [145], [144], [134]). *Let $n \geq 9$ be odd, $n \neq 17, 21, 29, 51, 53$, 67, or 87. Then there is an MESRS of side n.*

MESRS are equivalent to several other types of arrays. The following equivalences were observed in [237] and [144]:

Theorem 9.2 ([237], [144]). *An MESRS of side $2n-1$ is equivalent to any of the following arrays:*

1. *A partitioned balanced tournament design PBTD(n).*
2. *A pair of almost disjoint Howell designs $H(n, 2n)$*
3. *A partitionable house of side $2n$.*

We note that balanced tournament designs will be discussed in Section 16, Howell designs in Section 11, and houses in Section 14.

37					28	59	4X	16
	56				1X	47	29	38
		2X			67	18	35	49
			48		39	26	17	5X
				19	45	3X	68	27
12	8X	57	69	34				
46	13	89	7X	25				
58	79	14	23	6X				
9X	24	36	15	78				

Finally, we mention that an application of MESRS to the construction of semiframes is given by Rees [197].

10 BALANCED HOWELL ROTATIONS

As was noted in the introduction to this chapter, Room squares can be used for the construction of round-robin tournaments. Round-robin duplicate bridge tournaments with fixed partnerships are often conducted using Howell rotations [93]. In a duplicate bridge tournament, every board has a north–south direction and an east–west direction. Partnership i is said to *compete* with partnership j on a particular board if they play that board in the same direction (in different rounds, of course). The scoring of the tournament is based on competing partnerships. Hence, it is desirable to have all pairs of teams compete equally often. This motivates the following definition:

Suppose that there are $2n$ teams. A tournament satisfying the following conditions is called a *complete balanced Howell rotation* for $2n$ teams, or a CBHR($2n$).

1. Every board is played by at most one pair of teams (one in each direction) in each round.
2. Every team plays every other team exactly once during the tournament.
3. Every team plays exactly one board in each round.
4. Every team plays each board exactly once.
5. Every team competes equally often against any other team.

Parker and Mood first brought these schedules to the attention of mathematicians in a 1955 paper [194]. Schellenberg gave a formulation of CBHR($2n$) in terms of Room squares in [210], [211]. Suppose we start with a Room square of side $2n-1$, and replace every unordered pair $\{x,y\}$ by one of the two ordered pairs (x,y) or (y,x). The resulting array is called an *ordered* Room square. From each row of the ordered Room square construct a pair of complementary blocks—one block consisting of the elements that occur as the first coordinate of an ordered pair in that row, and one block consisting of the elements that occur as the second coordinate. It is possible that the resulting set of $2(2n-1)$ blocks comprise a balanced incomplete block design having parameters $(2n, 2(2n-1), 2n-1, n, n-1)$. If so, the (ordered) Room square is said to be a *balanced* Room square and is denoted BRS($2n$). If we let rows, columns, and symbols of a BRS($2n$) correspond, respectively, to boards, rounds, and teams in a round-robin tournament, we obtain the following:

Theorem 10.1 ([210], [211]). *The existence of a complete balanced Howell ro-*

Figure 1.1 provides an example of a balanced Room square of side 7, if the pairs in each cell are ordered as given there.

In [194], it is shown that a necessary condition for the existence of a CBHR($2n$) is that $2n \equiv 0 \pmod 4$. In that paper, CBHR(8), CBHR(12), and CBHR(16) were constructed. The necessary condition was reproved in [34] by showing that existence of a CBHR($2n$) implies the existence of a Hadamard matrix of order $2n$.

Let G be an abelian group of order $2n-1$. Suppose we replace every pair $\{x,y\}$ in a starter in G by one of the two possible ordered pairs (x,y) or (y,x). Write the resulting ordered starter as $S = \{(s_i,t_i) : 1 \le i \le n-1\}$. Define $A_1 = \{s_i : 1 \le i \le n-1\}$ and $A_2 = \{t_i : 1 \le i \le n-1\}$, and for $j = 1,2$, define ΔA_j to be the multiset $\{\pm(x-y) : x,y \in Aj,\ x \ne y\}$. If the multiset union $\Delta A_1 \cup \Delta A_2$ contains every nonzero element of G equally often, then A_1 and A_2 are said to be a pair of *supplementary difference sets*, and the (ordered) starter is said to be *balanced*. If A_1 and A_2 are developed through G, then a $(2n, 2(2n-1), 2n-1, n, n-1)$-BIBD is constructed. Thus, the existence of a balanced starter and adder in G implies the existence of BRS($2n$). In [34], it was shown that a balanced starter and adder exists in GF(q) whenever $q \equiv 3 \pmod 4$ is a prime power; hence, we have the following result:

Theorem 10.2 ([34]). *A BRS($q+1$) exists for all prime powers $q \equiv 3 \pmod 4$.*

In [210], [211], Schellenberg gave a composition theorem for BRS which constructs BRS($4n$) from BRS($2n$) under certain conditions. In particular, if $2n-1$ is a prime power congruent to $3 \pmod 4$, then a BRS($4n$) exists.

Theorem 10.3 ([210], [211]). *A BRS($2q+2$) exists for all prime powers $q \equiv 3 \pmod 4$.*

Schellenberg's construction was modified by Hwang, Kang, and Yu [127] to give sufficient conditions for existence of a BRS($4n$) in terms of a special type of skew balanced starter of order $2n$. Let $S = \{(s_i,t_i) : 1 \le i \le n-1\}$ be a skew balanced starter in an abelian group G of order $2n-1$. We say that S is *symmetric* if $\{s_i : 1 \le i \le n-1\} = \{t_i : 1 \le i \le n-1\}$.

Example 10.1. A symmetric skew balanced starter in $\mathbf{Z}_{13}$ is

$$\{(2,4),(6,12),(5,10),(8,9),(11,1),(7,3)\}.$$

In [127], it is shown that the existence of a symmetric skew balanced starter (or, SSBS) in an abelian group of order $2n-1$ implies the existence of a BRS($4n$). Since then, symmetric skew balanced starters have been studied in ... given by Hwang [125] in due to Du and Hwang [81]:

Theorem 10.4 ([81]). *There exists a symmetric skew balanced starter in GF(q) for all prime powers $q = 2^\alpha t + 1$, $\alpha \geq 2$, $t \geq 3$, t odd. Hence, a BRS($2q+2$) exists for all such q.*

Du and Hwang proceed as follows: Suppose that $q = 2^\alpha t + 1$ is an odd prime power, where t is odd and $\alpha \geq 2$. Let ω be a primitive element in GF(q), and let C_0 be the (unique) subgroup of G^* of order t and index 2^α, where G^* denotes the multiplicative group GF(q)$\setminus\{0\}$. Denote the cosets of C_0 by C_i $(0 \leq i \leq 2^\alpha - 1)$, where $C_i = \omega^i C_0$, and denote $\Delta = 2^{\alpha-1}$. Suppose that $u, v \in C_j$, where j is odd, and suppose that $(u+1)(v+1)$ and $(u-1)(v-1)$ are both quadratic nonresidues. Then

$$\{(ux, x) : x \in C_2 \cup C_4 \cup \cdots \cup C_\Delta\} \cup \{(vx, x) : x \in C_{\Delta+2} \cup C_{\Delta+4} \cup \cdots \cup C_{2\Delta}\}$$

can be shown to be an SSBS in GF(q). Jacobi sums are then used to prove the existence of such u and v, provided $t \neq 1$. The existence of SSBS in GF(q) remains unresolved when q is a Fermat prime.

Two other papers that discuss CBHR are [122] and [124]; however, note that [122] contains an error ([9]). Another special type of starter that has applications to the construction of CBHR are the partitionable starters; see [125] and [77], for example.

Several researchers have also investigated geometric constructions for balanced Room squares. In 1982, Anderson [9] gave a simple construction for BRS(2^n) for all odd $n \geq 3$ involving a hyperplane and a pencil of lines in the projective space PG($n-1, 2$). Fuji-Hara and Vanstone [84] give the following construction for BRS using affine geometries. A *skew class* in the affine geometry AG($n, 2$) is a set of lines, no two parallel, that partitions the 2^n points. A *skew resolution* is a set of skew classes that partitions the set of lines in the geometry. It is shown in [84] that the existence of a skew resolution in AG($n, 2$) implies the existence of a BRS(2^n). They also exhibit a skew resolution of AG($4, 2$). Using a modification of the hill-climbing algorithm for strong starters [68], skew resolutions are found in [239] for AG($6, 2$), AG($8, 2$), and AG($14, 2$). Then, applying recursive constructions from [85], it is shown in [239] that there is a skew resolution in AG($m, 2$) if $m = n(2^i - 1)$, where $n = 4$, 6, 8 or 14, and $i \geq 2$; or if $m = (n-1)(2^i - 1) + 1$, where $n = 4$, 6, 8 or 14, and $i \geq 2$. As a result of these constructions, we have the following:

Theorem 10.5. *There exists a BRS(2^n) if n is odd, or if n is even and $4 \leq n \leq 18$.*

Summarizing Theorems 10.2–10.5, BRS(n) exist for all $n \equiv 0 \pmod 4$ such that $8 \leq n \leq 100$, with the two possible exceptions $n = 36$ and $n = 92$.

We have noted that a CBHR($2n$) exists only if n is even. In the case $2n \equiv 2 \pmod 4$, it is possible to construct tournaments that satisfy all the properties of a CBHR, except that every team opposes every other team exactly twice. This motivates the following definition: A CBHR$_\lambda$($2n$) is a tournament that satisfies properties 1, 3, 4, and 5 of a CBHR($2n$), as well as the following property $2'$:

$2'$. Every team plays every other team exactly λ times during the tournament.

We can also interpret a CBHR$_\lambda$($2n$) as a balanced Room square BRS$_\lambda$($2n$), where we include the parameter λ to indicate that every pair of symbols occurs in precisely λ cells.

So, if $2n \equiv 2 \pmod 4$, it is of interest to construct BRS$_2$($2n$). The following theorem of Hwang gives in infinite class of such BRS.

Theorem 10.6 ([121]). *There exists a BRS$_2$($q+1$) for all prime powers $q \equiv 1 \pmod 4$.*

This result was reproved in [18] using the language of starters, adders, and supplementary difference families. In [18], the authors also give a construction for some BRS$_2$($2n$) that can be decomposed into two Room squares of side $2n-1$. Moreover, the first examples of BRS$_2$($2n$) were obtained when $2n \equiv 2 \pmod 4$ and $2n-1$ is not a prime power. In particular, BRS$_2$($2n$) were found for $2n = 22, 34, 66$, and 70. The construction employs strong starters satisfying some additional properties, which were found by a modification of the hill-climbing algorithm.

At present, there are only five orders $2n \equiv 2 \pmod 4$ less than 100 for which existence of a BRS$_2$($2n$) is unknown; these are for $2n = 46, 58, 78, 86$, and 94.

We also note that a weaker version of CBHR, namely, balanced Howell rotations, have also recieved considerable study [107], [123], [78], [80]. These schedules drop the notion of "round" and hence do not correspond to Room squares. For a survey of CBHR and related designs in the context of round robin schedules, see Hwang [126].

11 HOWELL DESIGNS

Let S be a set of $2n$ elements called symbols. A *Howell Design* H($s, 2n$) (on symbol set S) is an $s \times s$ array, F, that satisfies the following properties:

1. Every cell of F either is empty or contains an unordered pair of symbols from S.

$a0$		13	$2b$
23	$a1$	$0b$	
	$3b$	$a2$	01
$1b$	02		$a3$

Figure 11.1 An H(4,6).

Note that a trivial necessary condition for the existence of an H($s,2n$) is that $s+1 \leq 2n \leq 2s$. Also, an H($2n-1,2n$) is just a Room square of side $2n-1$. We present an H(4,6) in Figure 11.1.

It is easy to see that two orthogonal one-factorizations of G, an s-regular graph on $2n$ vertices, give rise to an H($s,2n$), and conversely, the existence of an H($s,2n$) implies the existence of a pair of orthogonal one-factorizations of some s-regular graph on $2n$ vertices, G, which we call the *underlying graph* of the Howell design.

Howell designs were defined by Hung and Mendelsohn in 1974 in [120]. However, some particular Howell designs were constructed earlier for use in bridge tournaments. For example, an H(9,12) is presented in Beynon's 1943 book [35, p. 22]. Howell designs have been extensively studied since Hung and Mendelsohn's paper. See, for example, the following papers: [4], [5], [6], [8], [10], [12], [14], [15], [16], [67], [137], and [206]. The existence of Howell designs has been completely determined in [20] and [234] as follows:

Theorem 11.1 ([20], [234]). *Let s and n be positive integers, where n is even, and $s+1 \leq 2n \leq 2s$. Then there exists an $H(s,2n)$ if and only if $(s,2n) \neq (2,4)$, $(3,4), (5,6)$, or $(5,8)$.*

We note that a proof of nonexistence of H(2,4) is trivial. The ordered pairs (3,4) and (5,6) correspond to (nonexistent) Room squares of sides 3 and 5. The nonexistence of an H(5,8) is much more difficult to establish; a reasonably short proof is given in [207].

The existence proof uses a variety of direct and recursive constuctions. One special type of Howell design that is very useful in recursive constructions is the $*$-*design*. An $\mathrm{H}^*(s,2n)$ can be defined as an H($s,2n$) whose underlying graph contains an independent set of size $2n-s$ (which is the maximum possible). Hung and Mendelsohn [120] gave a starter-adder construction for $*$-designs, and proved that all designs $\mathrm{H}^*(n+k,2n)$ exist for $0 \leq k \leq 10$, with the exceptions of H(2,4), H(3,4), H(5,6), and H(5,8) (which do not exist, as noted above), and with the possible exceptions corresponding to $k=1$, n even.

Howell n-starter in G is a set

$$S = \{\{s_i, t_i\} : 1 \leq i \leq s - n\} \cup \{\{s_i\} : s - n + 1 \leq i \leq n\}$$

that satisfies the following two properties:

1. $\{s_i : 1 \leq i \leq n\} \cup \{t_i : 1 \leq i \leq s - n\} = G \backslash \{0\}$.
2. $(s_i - t_i) \neq \pm(s_j - t_j)$ if $i \neq j$.

If S is a Howell n-starter, then a set $A = \{\{a_i\} : 1 \leq i \leq n\}$ is defined to be an *adder* for S if the elements in A are nonzero and distinct, and the set

$$S + A = \{\{s_i + a_i, t_i + a_i\} : 1 \leq i \leq s - n\} \cup \{\{s_i + a_i\} : s - n + 1 \leq i \leq n\}$$

is again a Howell n-starter. If A is an adder for S, then an $\mathrm{H}^*(s,2n)$ exists [120]. In the case where the group G is cyclic, the resulting Howell design will be termed *cyclic*.

In 1977, Anderson [3] discovered a method that often permits the construction of many different cyclic $\mathrm{H}(s,2n)$ (for a given odd value of s) by altering a particular strong starter in $\mathbf{Z}_s$. He used this construction to show that an $\mathrm{H}(s,2n)$ exists for s odd, $3 \leq s \leq 51$, $s+1 \leq 2n \leq 2s$, $n \neq s-1$, with the single exception of $\mathrm{H}(5,6)$. In 1980, Dinitz and Stinson used the hill-climbing algorithm for strong starters in conjunction with Anderson's technique to extend the result to all odd $s < 1000$ [67]. Anderson was also able to prove several general results using the same construction. For example, he proved in a 1981 paper with Leonard [17] that an $\mathrm{H}(p,2n)$ exists if $p \equiv 5 \pmod 8$ is a prime, $p > 5$, $p+1 \leq 2n \leq 2p$, and $n \neq p-1$.

All the results in the previous paragraph omitted the class of designs $\mathrm{H}(n+1,2n)$, n even. (In fact, a cyclic $\mathrm{H}(n+1,2n)$ does not exist if n is even [120].) The existence of this class of designs was completed in 1981 by Schellenberg, Stinson, Vanstone, and Yates [213]. Another class that was studied at about the same time was the class of designs $\mathrm{H}(2m,2m+2)$; Schellenberg and Vanstone [215] proved that these designs exist, provided $m > 1$.

Shortly thereafter, several new recursive constructions were developed that lead to the completion of the spectrum of Howell designs of odd side in 1982 by Stinson [234]. Then the spectrum of Howell designs of even side was determined in 1984 by Anderson, Schellenberg, and Stinson [20]. The proof uses similar recursive constructions but also depends heavily on direct constructions for $\mathrm{H}^*(2q,2n)$ and $\mathrm{H}^*(4q,2n)$ for q an odd prime power, $q \neq 3,5$, or 9. We should also note that the existence proof for designs of even side can be somewhat simplified by a subsequent direct construction of Schellenberg [212] for designs $\mathrm{H}^*(6q,2n)$, q an odd prime power, $q \neq 3,5$, or 9.

Even though existence of Howell designs is completely determined, existence of [illegible] designs remains unresolved. We note that an $\mathrm{H}^*(6,12)$ does not of order 6), but Hung and Mendelsohn constructed an H(6,12). This is the only known case where an H(s, $2n$) exists but an H*(s, $2n$) does not exist.

A more difficult question is to ask which graphs are the underlying graphs of Howell designs. Of course, the underlying graph of an H($2n-1$, $2n$) is the complete graph K_{2n}. The underlying graph of an H($2m$, $2m+2$) is the *cocktail party graph* $K_{2m+2}-f$, where f is a one-factor. Also, an H(s, $2s$) with underlying graph $K_{s,s}$ is equivalent to a pair of mutually orthogonal Latin squares of order s. Existence or nonexistence of Howell designs with specified underlying graphs has been determined for all graphs on at most 10 vertices [207], [218], but the general problem seems hopeless at present.

We can also consider questions involving higher-dimensional designs. Generalizing to higher dimensions, we define a d-dimensional Howell design $H_d(s,2n)$ to be a d-dimensional array which satisfies property 1, such that each two-dimensional projection is an H(s, $2n$). We refer to an $H_3(s,2n)$ as a *Howell cube*. Clearly, an $H_d(s,2n)$ is equivalent to d mutually orthogonal one-factorizations of the underlying graph. Let $\nu(s,2n)$ denote the maximum value of d such that an $H_d(s,2n)$ exists.

Very little is known concerning upper bounds for $\nu(s,2n)$. It is very easy to see that $\nu(s,2n) \leq s-1$. This bound can be attained with equality for underlying graphs $K_{s,s}$ when s is a prime power (since $\nu(s,2s)$ is at least as large as the maximum number of mutually orthogonal Latin squares of order s, and $s-1$ MOLS of order s exist if s is a prime power). However, it seems unlikely that this bound can be met with equality if $s > n$. There is a *conjectured* upper bound, namely, $\nu(s,2n) \leq n-1$ [207]. Note that this conjectured bound is stronger than the bound $s-1$ if $s > n$, and if $s < n$, then $\nu(s,2n) \leq 1$, anyway. The two bounds agree if $s = n$. Also, there are infinitely many cases where $\nu(s,2n) \geq n-1$, as follows:

Theorem 11.2. *The following graphs on $2n$ vertices have at least $n-1$ orthogonal one-factorizations:*

1. K_{2n}, *if $2n-1 \equiv 3 \pmod 4$ is a prime power, or $2n = 10$.*
2. $K_{n,n}$, *if n is a prime power.*
3. K_{2n} *minus a one-factor, if $2n = 2^j + 2$, $j \geq 2$.*

Proof. Result 1 follows from Theorem 2.4 and the fact that $\nu(9) = 4$ (Table 5.1). The one-factorizations of the graphs in result 2 are equivalent to mutually orthogonal Latin squares, so this result is well-known. The result 3 is proved in [142]. □

Values for $\nu(s,2n)$, $2n \leq 10$, are as follows:

$\nu(2,4) = \nu(3,4) = \nu(5,6) = \nu(5,8) = 1$

TABLE 11.1 Values $NH_d(s,2n)$, $2n < 10$

s	$2n$	d	$NH_d(s,2n)$	Reference
2	4	1	1	
3	6	1	2	
3	6	2	1	
4	6	1	1	
4	6	2	1	
4	8	1	16	[207]
4	8	2	1	[207]
4	8	3	1	[207]
5	8	1	19	[207]
5	8	2	0	[120], [207]
6	8	1	13	[207]
6	8	2	3	[207]
5	10	1	3472	[207]
5	10	2	6	[207]
5	10	3	1	
5	10	4	1	
6	10	1	13277	[207]
6	10	2	18	[207]
7	10	1	14241	[207]
7	10	2	901	[207]
8	10	1	3192	[218]
8	10	2	18220	[218]
8	10	3	3	[218]
8	10	4	1	[218]

$\nu(4,8) = \nu(7,8) = 3.$
$\nu(5,10) = \nu(8,10) = \nu(9,10) = 4.$

Note that these values do not violate the conjectured upper bound.

It is also of interest to consider nonisomorphic Howell designs. Let $IR_d(G)$ denote the number of inequivalent sets of d pairwise orthogonal one-factorizations of a graph G. Also, let $NH_d(s,2n)$ denote the number of nonisomorphic d-dimensional Howell designs $H_d(s,2n)$. Evidently, $NH_d(s,2n) = \sum_G IR_d(G)$, where the sum is taken over all s-regular graphs G on $2n$ vertices. The nonisomorphic one-factorizations and (d-dimensional) Howell designs have been enumerated (for all d) for all graphs on at most 10 vertices [207], [218]. Values of $NH_d(s,2n)$, $2n \leq 10$, $s \leq 2n-2$, are presented in Table 11.1 (the values $NH_d(2n-1,2n) = IR_d(K_{2n})$ were already tabulated in Table 6.1).

An enumeration of one-factorizations and Howell designs for several interesting graphs on 12 and 14 vertices has been done in [219]. In particular, we should mention the existence of a Howell cube $H_3(6,12)$ due to Brickell [37] the only known case where $\nu(s, 2s)$ exceeds the maximum number of mutually orthogonal Latin squares of order s. Also, a Howell cube $H_3(7,12)$ is shown in [219].

We should also mention that Colbourn and Colbourn prove in [45] that the problem of determining the isomorphism of two Howell designs $H(s,2n)$ can be decided in time $O(n^{O(\log n)})$.

Other special types of Howell designs that have received study are $**$-designs, skew designs, complementary designs, and $*$-complementary designs. We briefly discuss $**$-designs and skew designs now. An $H^{**}(s,2n)$ is an $H(s,2n)$ that satisfies the following two additional properties:

1. The $H(s,2n)$ contains an empty $(s-n)\times(s-n)$ subarray.
2. There exists a transversal of the remaining n rows and n columns that forms a one-factor of the symbol set.

It is not difficult to see that an $H^{**}(2n-1,2n)$ is an MESRS of side $2n-1$, and conversely. Other $**$-designs proved useful in the existence proofs in [234] and [20].

An $H(2m,2m+2)$, say, H, is said to be *skew* if there exist two symbols a,b, where $\{a,b\}$ is not an edge of the underlying graph, such that the following properties are satisfied:

1. Denote the $2m$ cells of H that contain a by T_a, and denote the $2m$ cells of H that contain b by T_b. Then $T_a \cup T_b$ consists of the $2m$ cells on the diagonal of H (e.g., D), and $2m$ other cells which form a transversal of cells (e.g., D') of H, such that D' is symmetric with respect to D (i.e., a cell $(i,j)\in D'$ if and only if cell $(j,i)\in D'$).
2. Given any cell $(i,j)\notin D\cup D'$, precisely one of cell (i,j) and cell (j,i) is empty.

Note that the $H(4,6)$ in Figure 11.1 is skew. The following result on skew Howell designs was proved by Lamken and Vanstone [152]:

Theorem 11.3 ([152]). *There exists a skew $H(2m,2m+2)$ for all $m \geq 2$, with the exception $m=3$, and with the possible exceptions $m=5$ and 9.*

We defined subsquares of Room squares in Section 4. In an analogous fashion, we can define sub-Howell designs of Howell designs: If a subarray of an $H(s,2n)$ is itself an $H(t,2m)$, then we say that the subarray is a *sub-Howell design*. Howell designs containing sub-Howell designs were studied by Zhu [271] and by Dinitz and Lamken [64]. In [64], it is shown that there is an $H(2s-2,2s)$ containing a sub-$H(4,6)$ for all $s\geq 8$. Other results can be found in [64] as well.

Howell movement. The movement is said to be *complete* if $s = 2n - 1$ (i.e., the Howell design is a Room square); otherwise, it is *incomplete*. For the purposes of bridge tournaments, it is desirable that the cyclic Howell design have the property that the n filled cells in any row be adjacent. Such a design is defined to be *compact*. In terms of the starter-adder construction, it is equivalent to require that the adders used are precisely the elements $-i$, where $0 \leq i \leq n-1$.

The first constructions for compact cyclic Howell movements seem to be due to Sam Gold [221, p. 127]. He constructed several specific designs that are widely used in practice today. Some general existence and nonexistence results for compact Howell movements are discussed in [221]. We summarize these as follows:

Theorem 11.4 ([221], [51]). *If there exists a compact cyclic Room square of side* $2n-1$, *then* n *is odd.*

Theorem 11.5 ([221]). *Suppose that* $0 \leq t \leq 3$ *and* $n \geq 2t+1$. *Then there exists a compact cyclic* $H(n+t, 2n)$.

12 ORTHOGONAL STEINER TRIPLE SYSTEMS

A *Steiner triple system* of order n, or STS(n), can be defined to be a $(n,3,1)$-BIBD. The necessary and sufficient conditions for the existence of an STS(n) is that $n \equiv 1,3 \pmod 6$. Two STS(n) on the same point set, say, $(X, \mathcal{A})$ and $(X, \mathcal{B})$, are said to be *orthogonal* provided the following properties are satisfied:

1. $\mathcal{A} \cap \mathcal{B} = \emptyset$.
2. If $\{u,v,w\}$ and $\{x,y,w\} \in \mathcal{A}$, and $\{u,v,s\}$ and $\{x,y,t\} \in \mathcal{B}$, then $s \neq t$.

Example 12.1. Two orthogonal Steiner triple systems of order 7 are

$$\mathcal{A} = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\},$$

$$\mathcal{B} = \{\{1,2,7\},\{1,3,4\},\{1,5,6\},\{2,3,6\},\{2,4,5\},\{3,5,7\},\{4,6,7\}\}.$$

Orthogonal STS(n) will be denoted by OSTS(n). OSTS(n) can be used to construct a Room square of order n (or, equivalently, a pair of orthogonal one-factorizations of order $n+1$). Indeed, OSTS(n) were originally introduced in 1968 by O'Shaughnessy [191] (see also [192]) as a method of constructing Room squares. A Room square is obtained from OSTS(n) as follows. Let ∞ be a symbol not in X. We construct an array with rows and columns indexed by X. First place $\{\infty, x\}$ in cell (x,x) for all $x \in X$. Then, for every pair

OSTS(n) are known to exist if $n \equiv 1 \pmod 6$ is a prime power [176]. Also, the set OSTS $= \{n :$ there exists an OSTS(n)$\}$ is PBD-closed [153]. Define $P_{1,6}$ to be the set of prime powers congruent to 1 (mod 6). In [184] it was proved that there is an $(n, P_{1,6})$-PBD (and hence $n \in$ OSTS) if $n \equiv 1 \pmod 6$ and $n \geq 1927$. There remained 31 values of $n \equiv 1 \pmod 6$ less than 1927 for which an $(n, P_{1,6})$-PBD was not constructed. OSTS were constructed for two of these 31 values in [243], and the following theorem results.

Theorem 12.1 ([184], [243]). *If* $n \equiv 1 \pmod 6$, $n \geq 7$, *and* $n \neq 55, 115, 145, 205, 235, 265, 295, 319, 355, 391, 415, 445, 451, 493, 649, 655, 667, 697, 745, 781, 799, 805, 1243, 1255, 1315, 1585, 1795, 1819$, *or* 1921, *then there exist a pair of orthogonal Steiner triple systems of order n.*

Less is known regarding OSTS(n) for $n \equiv 3 \pmod 6$. First, there do not exist OSTS(9) [178]. The only small examples of OSTS(n) (i.e. $n < 100$) known to exist are $n = 15$ [91] and $n = 27$ [201].

Using recursive constructions, the following result was proved in [243]:

Theorem 12.2 ([243]). *For any* $n > 27363$, $n \equiv 3 \pmod 6$, *there exist a pair of orthogonal Steiner triple systems of order n. Further, a pair of orthogonal Steiner triple systems of order n exist for all* $n \equiv 3 \pmod 6$, $3 < n \leq 27363$, *with at most* 918 *possible exceptions.*

Gibbons has enumerated the existence of non-isomorphic OSTS(n) for $n \leq 15$ in [91]. For $n = 7$ and $n = 13$, there is a unique example (up to isomorphism). We have already mentioned that there is no OSTS(9). In the case $n = 15$, Gibbons showed that there are precisely 19 nonisomorphic OSTS(15). Some other papers that studied OSTS are [159], [169], [179] and [187].

Finally we mention a few interesting results that have been obtained regarding sets of d mutually orthogonal STS(n), $d > 2$ (of course, a set of d mutually orthogonal STS(n) can be used to construct a Room d-cube of side n). Sets of mutually orthogonal STS(n) were first studied by Gross in [99], [100]. In [99], he found six orthogonal STS(31). In [100], the following general results are proved:

Theorem 12.3 ([100]). *For all prime powers* $q \equiv 7 \pmod{12}$, $q > 174$, *there exist four mutually orthogonal STS*(q).

Theorem 12.4 ([100]). *For any positive integer d, there exists a positive integer* $Z(d)$, *such that there exist d mutually orthogonal STS*(v) *if* $v > Z(d)$ *and* $v \equiv 1 \pmod 6$.

Several other small examples of sets of mutually orthogonal STS are given

Finally, Zhu showed in [270] that there are three mutually orthogonal STS($2^n - 1$) if $\gcd(n, 6) = 1$. He also found six mutually orthogonal STS(127).

13 ORTHOGONAL EDGE-COLORINGS OF GRAPHS

Let G be a graph, let k be a positive integer, and let C be a set of k elements called *colors*. A *proper k-edge-coloring* of G is an assignment of one color (chosen from C) to each edge of G, so that no two edges having the same color are incident with a common vertex. Henceforth, we will refer to a proper k-edge-coloring simply as a *k-coloring*. Note that a one-factorization of a k-regular graph induces a k-coloring of that graph. Orthogonal colorings are studied in the survey by Alspach, Heinrich, and Liu [1].

Suppose that f is a k-coloring of a graph G using colors in C and g is a k'-coloring using colors in C' (we can regard f and g as functions $f : E(G) \to C$ and $g : E(G) \to C'$, where $E(G)$ denotes the edge set of G). We say that f and g are *orthogonal* colorings if, for any edges $e, e' \in E(G)$, $(f(e), g(e)) = (f(e'), g(e'))$ implies that $e = e'$ (i.e., if two edges receive the same colors in one of the two colorings, then they receive different colors in the other coloring). Orthogonal colorings generalize the idea of orthogonal one-factorizations.

Just as we can construct a Room square of side n from a pair of orthogonal one-factorizations of K_{n+1}, and a Howell design H($s, 2n$) from a pair of orthogonal one-factorizations of some s-regular graph on $2n$ vertices, so too can we construct a rectangular array from a pair of orthogonal colorings. If C and C' are the color sets of the two colorings, then the dimensions of this array will be $|C| \times |C'|$. Each edge of the underlying graph G will appear in one cell of the array, and conversely, each cell of the array will either be empty or contain an edge of G. Further, each point occurs in at most one cell of any row or column. In the case where the underlying graph G is r-regular and both color classes have k colors, the resulting array is called a *V-square* and is denoted V($k, r; v$). Note that a V($n, n; n+1$) is equivalent to a Room square of side n, and more generally, a V($s, s; 2n$) is equivalent to a Howell design H($s, 2n$).

As an example, we present a V(4,3;10) in Figure 13.1. The underlying graph is the Petersen graph.

Orthogonal colorings were introduced in [25], and several interesting results are proved there. In the following theorem, Δ denotes the maximum degree

12	38	69	45
79	16	$8X$	23
$5X$	49	27	68
34	$7X$	15	

of a vertex in a graph G, and χ' denotes the edge-chromatic number:

Theorem 13.1 ([25], [1]). *If k_1 and k_2 are integers such that $k_1 \geq \chi'$, $k_2 \geq 2(k_1 + \Delta - 2)$, and $k_1 k_2 \geq v$, then G has a pair of orthogonal colorings using k_1 and k_2 colors, respectively.*

A particularly interesting subset of V-squares are those where every cell is filled, namely, when $vr = 2k^2$. Such V-squares are called *crowded*. Necessary and sufficient conditions for the existence of crowded V-squares were determined in [26].

Theorem 13.2 ([26]). *A crowded V-square $\mathrm{V}(k, r; 2k^2/r)$ exists if and only if one of the following two conditions are satisfied:*

1. *$r = sm^2$, where s is odd and square-free, and $(k, r) = (s(m^2 + mn), sm^2)$ for some $n > 0$.*
2. *$r = 2sm^2$, where s is odd and square-free, and $(k, r) = (s(2m^2 + mn), 2sm^2)$ for some $n > 0$; and $(k, r) \neq (2, 2)$.*

Another generalization of orthogonal one-factorizations of K_n is to start with a multigraph instead of a graph. If we start with the multigraph λK_{n+1}, we obtain a generalization of a Room square called a λ*-square* and denoted $\lambda\mathrm{S}(n, \lambda)$, which satisfies the following properties:

1. Every cell either is empty or contains an unordered pair of symbols from S (where $|S| = n + 1$).
2. Each symbol of S occurs once in each row and column.
3. If two pairs of symbols occur in two cells of any given row (column), then they do not occur in any two cells of any column (row).

A $\lambda\mathrm{S}(n, \lambda)$ will have λn rows and columns, since that quantity is the degree of the multigraph λK_{n+1}. Clearly, a $\lambda\mathrm{S}(n, 1)$ is equivalent to a Room square of side n.

There has recently been some interest in the construction of one-factorizations of λK_{n+1} having the property that it is impossible to partition the set of one-factors into one-factorizations of $\lambda_1 K_{n+1}$ and $\lambda_2 K_{n+1}$, where $\lambda_1 + \lambda_2 = \lambda$. Such one-factorizations are called *indecomposable*, and they have been studied in [46] and [24]. Obviously, if we place λ copies of a Room square of side n on the diagonal of a square array of side λn, we obtain a $\lambda\mathrm{S}(n, \lambda)$. If we do this, however, then both the row and column one-factorizations have the property that they can be partitioned into λ one-factorizations of K_{n+1}, and hence are highly decomposable.

This discussion suggests the problem of finding orthogonal indecomposable one-factorizations of λK_{n+1}. The following result was proved by Dinitz in [63]

Theorem 13.3 ([63]). *For all odd $n \geq 11$, there exists a pair of orthogonal simple indecomposable one-factorizations of $2K_{n+1}$.*

Another generalization of a Room square is called a Kirkman square. A *Kirkman square* $\mathrm{KS}_2(n+1,\mu,\lambda)$ is a square array of side $\lambda n/\mu$ that satisfies the following properties:

1. Every cell either is empty or contains an unordered pair of symbols from S (where $|S| = n+1$).
2. Each symbol of S occurs exactly μ times in each row and column.
3. Every unordered pair of symbols occurs in exactly λ cells of the array.

We observe that if condition 3 of the definition of λ-square is deleted, then we get a Kirkman square $\mathrm{KS}_2(n+1,1,\lambda)$.

The spectrum of $\mathrm{KS}_2(n+1,\mu,\lambda)$ has been completely determined in [140], [139], [143], and [264] as follows.

Theorem 13.4 ([140], [139], [143], [264]). *A $KS_2(n+1,\mu,\lambda)$ exists if and only if $\lambda n \equiv 0 \pmod{\mu}$, $\mu(n+1)$ is even, $\lambda > \mu^2/2$, $n+1 \geq 2\lambda/(2\lambda-\mu^2)$, and $(n,\mu,\lambda) \neq (3,1,1)$ or $(5,1,1)$.*

In [136], Lamken proved the existence of $\mathrm{KS}_2(n+1,1,\lambda)$ that cannot be decomposed into two squares $\mathrm{KS}_2(n+1,1,\lambda_1)$ and $\mathrm{KS}_2(n+1,1,\lambda_2)$ (where $\lambda_1+\lambda_2=\lambda$), for all odd $n \geq 3$ and all $\lambda \geq 2$.

14 HOUSES

Of course, a Room square of even side does not exist. The following approximation is called a *house* (note that these were not named after T. G. House). Let n be even, let S be a set of n elements called symbols, and let f be a one-factor of S. A *house* of side n (on symbol set S) is an $n \times n$ array, H, that satisfies the following properties:

1. Every cell of H either is empty or contains an unordered pair of symbols from S.
2. Each symbol of S occurs once in each row and column of H.
3. Every unordered pair of symbols occurs in precisely one cell of H, except for the pairs in f, each of which occurs in precisely two cells of H.
4. Each of the first two rows of H contains the pairs in f.
5. Every column of H contains one pair from f.

In Figure 14.1, we present a house of side 6. This array was first displayed in [228], although it was not called a house there.

In [261], Wallis employed the house of side 6 to prove the existence of

12		34		56	
	12		34		56
35		16			24
46			15	23	
	45		26		13
	36	25		14	

Figure 14.1 A house of side six.

son and Wallis investigated the existence of houses in general and determined their spectrum.

Theorem 14.1 ([242]). *There exists a house of side n for all even $n \geq 6$.*

In [242], some other applications of houses to the construction of Room squares with subsquares are given. One of the constructions for houses given in [242] is a starter-adder type construction. It was shown in [242] that a house of side $2(q+1)$ could be obtained by this method if $q \equiv 1 \pmod 4$ is a prime power. Anderson [11] proves a similar result when $q \equiv 3 \pmod 4$ is a prime power, $q \geq 7$.

A house H of side $2n$ having repeated one-factor f is called *partitionable* if it contains two disjoint $n \times n$ subarrays, each of which is a Howell design $H(n, 2n)$ containing f as a row. Lamken and Vanstone observe in [144] that the existence of a partitionable house of side $2n$ is equivalent to the existence of a maximum empty subarray Room square of side $2n-1$ (Section 9).

15 ORTHOGONAL ONE-FACTORIZATION GRAPHS

If H is a graph and $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are one-factorizations of H, then we can define a graph with vertex set $\mathcal{F}_1, \ldots, \mathcal{F}_n$, in which $\mathcal{F}_i\mathcal{F}_j$ is an edge if $\mathcal{F}_i$ and $\mathcal{F}_j$ are orthogonal one-factorizations. This graph is called an *orthogonal one-factorization graph*, or OOFG, with underlying graph H. If G is any (finite) graph, then we say that G is *realizable* as an OOFG with underlying graph H if there is an OOFG with underlying graph H which is isomorphic to G.

When the underlying graph is the complete bipartitie graph $K_{n,n}$, an OOFG is referred to as an *orthogonal Latin square graph*, or OLSG(n), since the existence of a pair of orthogonal one-factorizations of $K_{n,n}$ is equivalent to the existence of a pair of orthogonal Latin squares of order n. OSLG were investigated by Lindner, Mendelsohn, Mendelsohn, and Wolk in a 1979 paper [158]. In that paper, they proved the following theorem:

Further results on OLSG have been obtained by Bennett [32], Bennett and Mendelsohn [33], and Fu [82], [83]. Given a Latin square L, one can obtain six Latin squares by interchanging the roles of rows, columns, and symbols. These six squares are called the *conjugates* of L. In [33], it is shown for all $n \geq 43$ that the 6-cycle can be realized as an OLSG whose vertices are the six conjugates of some Latin square of order n. In [32], it is shown for all $n \geq 2207$ that K_6 can be realized as an OLSG whose vertices are the six conjugates of some Latin square of order n.

Dinitz investigated the case of OOFG where the underlying graph is the complete graph. He proved a result analogous to Theorem 15.1.

Theorem 15.2 ([60]). *For any finite simple graph G, there exists an integer n_0 such that G is realizable as an OOFG with underlying graph K_n for all $n > n_0$.*

For a finite simple graph G, define the *spectrum* of G to be $\text{Spec}(G) = \{n : G \text{ is realizable as an OOFG with underlying graph } K_n\}$. Clearly, the existence of a Room d-cube of side n implies that $n+1 \in \text{Spec}(K_d)$. From the results of Section 5 on Room d-cubes we have the following:

Theorem 15.3. $\text{Spec}(K_2) = \{n \geq 8, n \textit{ even}\}$; $\text{Spec}(K_3) = \{n \geq 8, n \textit{ even}\}$; $\text{Spec}(K_4) = \{n \geq 10, n \textit{ even}\}$; *and* $\{n \geq 12, n \textit{ even}, n \neq 16\} \subseteq \text{Spec}(K_5) \subseteq \{n \geq 12, n \textit{ even}\}$.

The spectra for some other graphs is also determined in [60]. We can improve the result given in [60] for $\text{Spec}(C_6)$, where C_6 denotes the cycle of length 6.

Theorem 15.4. *If $n \geq 24$ and n is even, then $n \in \text{Spec}(C_6)$.*

Proof. If $n \geq 24$, then there exist a pair of orthogonal one-factorizations of K_n that contains a pair of orthogonal sub-one-factorizations of K_8 (Theorem 4.5). Delete all edges from the sub-one-factorizations of K_8, and call the remaining partial one-factorizations $\mathcal{F}$ and $\mathcal{G}$. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be three orthogonal one-factorizations of K_8. Define $\mathcal{F} \cup \mathcal{A}$, $\mathcal{F} \cup \mathcal{B}$, $\mathcal{F} \cup \mathcal{C}$, $\mathcal{G} \cup \mathcal{A}$, $\mathcal{G} \cup \mathcal{B}$, and $\mathcal{G} \cup \mathcal{C}$ in the obvious way. These six one-factorizations realize C_6. □

We summarize the remaining spectra discussed in [60] in the next theorem. Note that P_i denotes the path with i vertices.

Theorem 15.5 ([60]).

1. $\text{Spec}(P_3) = \{n \geq 8, n \textit{ even}\}$.
2. *If $i = 4$ or 5, $n \geq 24$ and n is even, then* $n \in \text{Spec}(P_i)$.
3. $\{n \geq 8, n \textit{ even}, n \neq 12\} \subseteq \text{Spec}(C) \subseteq \{n \geq 8, n \textit{ even}\}$

16 BALANCED TOURNAMENT DESIGNS

Let S be a set of $2n$ elements called *symbols*. A *balanced tournament design* of order n (on symbol set S) is an $n \times (2n-1)$ array, F, that satisfies the following properties:

1. Every cell of F contains an unordered pair of symbols from S.
2. Each symbol of S occurs once in each column of F.
3. Each symbol of S occurs once or twice in each row of F.
4. Every unordered pair of symbols occurs in precisely one cell of F.

In the following, we present a balanced tournament design of order 3.

16	35	23	45	24
25	46	14	13	36
34	12	56	26	15

A balanced tournament design of order n will be denoted BTD(n). Intuitively, a BTD(n) is formed by finding a one-factorization of K_{2n} that is orthogonal to a "near two-factorization" of K_{2n}. Gelling and Odeh introduced balanced tournament designs in 1973 in [90]. Haselgrove and Leech established existence for $n \equiv 0, 1 \pmod 3$ in [108]. The spectrum was completed by Schellenberg, van Rees, and Vanstone [214] in 1977. They proved the following:

Theorem 16.1 ([214]). *There exists a balanced tournament design of order n if and only if n is a positive integer, $n \neq 2$.*

We note that there is a unique BTD(3) up to isomorphism. Corriveau enumerated the nonisomorphic BTD(4)s in [52], [53]; there are precisely 47. He also showed for each of the 396 nonisomorphic one-factorizations of K_{10} that there is a BTD(5) having the given one-factorization as the columns of the BTD.

Since 1977, there has been considerable interest in BTDs satisfying certain extra conditions. Many of these special types of BTDs are discussed in the survey by Lamken and Vanstone [149]. These include factored BTDs [138], factor balanced BTDs, partitioned BTDs, and hamiltonian BTDs. We briefly discuss partitioned BTDs and hamiltonian BTDs now.

A BTD(n) is said to be *partitioned* if the $2n-1$ columns can be partitioned

16	20	31	42	53	64	05
25	36	40	51	62	03	14
34	45	56	60	01	12	23

Figure 16.1 An odd balanced tournament design of order 3.

design H$(n, 2n)$, as is the $n \times n$ array formed by the columns in $A \cup C$. It was shown in [237] that the existence of a partitioned BTD(n) is equivalent to the existence of a MESRS of side $2n-1$.

In any row of a BTD(n), every symbol occurs in two cells, except for two symbols that each occur in only one cell. Hence, it is conceivable that the graph formed by the pairs of symbols occuring in the cells of a given row could be a path of length $2n-1$, that is, a hamiltonian path. A BTD(n) is said to be a *hamiltonian path* BTD(n) if this is true for every row of the array. In [116], Horton proves that a hamiltonian path BTD(n) exists if there is a skew starter in $\mathbf{Z}_n$. Hence, there is a hamiltonian path BTD(n) if n is not divisible by 2, 3, or 5.

We next define a class of designs closely related to BTDs. Let S be a set of $2n+1$ symbols. An *odd balanced tournament design* of order n (on symbol set S), or OBTD(n), is an $n \times (2n+1)$ array, F, that satisfies the following properties:

1. Every cell of F contains an unordered pair of symbols from S.
2. Each symbol of S occurs at most once in each column of F.
3. Each symbol of S occurs twice in each row of F.
4. Every unordered pair of symbols occurs in precisely one cell of F.

It is easy to construct an OBTD(n) for any $n \geq 1$ using a patterned starter in $\mathbf{Z}_{2n+1}$; the method of construction should be evident from the example given in Figure 16.1.

Analogous to hamiltonian path BTDs, we might consider OBTDs in which every row gives rise to a hamiltonian cycle; such an OBTD(n) is called a *Kotzig factorization* of order $2n+1$. These designs were studied first by Colbourn and Mendelsohn [48]. The spectrum was determined by Horton [115], where it was shown that a Kotzig factorization of order $2n+1$ exists if $n \geq 1$.

Finally, we should also mention that several applications of BTDs and OBTDs to the construction of resolvable $(v, 3, 2)$-BIBDs are given in [150]. BTDs and OBTDs having resolutions that are orthogonal (for a suitable definition of orthogonal) can be used to produce $(v, 3, 2)$-BIBDs having orthogonal resolutions (for more information, see [133], [146], [148], and [151]). More-
OBTDs having a special type of orthogonal resolutions are equivalent to

TABLE 17.1 Other generalizations of Room squares

Generalization	Reference
Generalized Room square of the first kind	[202]
Generalized Room square of the second kind	[111]
Generalized Howell design	[204]
Multidimensional Room design	[203]
Generalized Euler square	[205]
Kirkman square and Kirkman cube	[246]
Strong Kirkman cube	[245]
Uniform multidimensional generalized Room design	[205]
Room rectangle	[130]
Steiner tableau	[131]
Generalized balanced tournament design	[135]
Weak Room square and weak Room cube	[94]

17 OTHER GENERALIZATIONS

Various other generalizations of Room squares have been studied in the literature. Most of these generalizations involve arrays where the cells contain t-subsets, $t > 2$. We do not discuss any of these generalizations in this survey, but we do give a list of the various types of arrays and references in Table 17.1.

18 OPEN PROBLEMS

In this section, we present a list of open problems, indicating in each case the relevant section.

1. (Section 2.1) Show that any abelian group of odd order admits a strong starter, with the exceptions of the groups $\mathbf{Z}_3$, $\mathbf{Z}_5$, $\mathbf{Z}_9$, and $\mathbf{Z}_3 \times \mathbf{Z}_3$.
2. (Section 2.1) Prove that there exists a skew starter in $\mathbf{Z}_n$ for all n such that $\gcd(n,6) = 1$ (note that such a starter does not exist if $n \equiv 0 \pmod 3$, by Theorem 2.10).
3. (Section 2.1) Prove that there exists a strong frame starter in $\mathbf{Z}_{2n} \backslash \{0, n\}$ for all $n \equiv 0, 1 \pmod 4$ (note that such a starter does not exist if $n \equiv 2, 3 \pmod 4$, by Theorem 2.14).
4. (Section 3) Prove that there exists a frame of type t^5 for all $t > 1$.
5. (Section 3) Prove that there exists a frame of type t^4 for all even $t > 2$.
6. (Section 4) Prove that if $s \geq 7$ is odd, and $n \geq 3s + 2$ is odd, then there exists a Room square of side n containing a subsquare of side s. In par-

7. (Section 4) Prove that a partial Room square of side s can be embedded in a Room square of side at most $f(s)$, where f is some fixed polynomial.

8. (Section 5) Prove or disprove that $\nu(n) \leq (n-1)/2$ for all odd n.

9. (Section 5) Prove or disprove that $\nu(11) = 5$ and that $\nu(15) = 4$.

10. (Section 5) Improve the lower bounds on $\nu(n)$, $n < 100$.

11. (Section 5) If $q = 2^{\alpha}t + 1$ is a prime power, where $t > 1$ is odd, prove that $\nu(q) > t$ (i.e., improve Theorem 2.5).

12. (Section 6) Prove that $\mathrm{NR}(n) \geq c_1 n^{c_2 n^2}$ for positive constants c_1 and c_2.

13. (Section 6) Prove an exponential lower bound on $\mathrm{IR}_3(n)$.

14. (Section 6) Find more examples of strong starters that are not equivalent.

15. (Section 7) Determine necessary and sufficient conditions for the existence of a skew frame of type t^u $(t > 1)$. In the case $t = 2$, prove that there exists a skew frame of type 2^u for all $u \geq 5$. In particular, prove that a skew frame of type 2^6 exists.

16. (Section 8) Prove that there exists a perfect Room square of side q, for every odd prime power $q \geq 11$. In particular, find a perfect Room square of side 13.

17. (Section 8) Find new classes of uniform Room squares.

18. (Section 8) Improve the lower bounds on $\nu_P(n)$, $n < 100$.

19. (Section 9) Complete the spectrum of maximum empty subarray Room squares.

20. (Section 10) Show that a skew symmetric balanced starter exists in $\mathbf{Z}_n$, for n a Fermat prime.

21. (Section 10) Find BRS(36) and BRS(92).

22. (Section 10) Find more recursive constructions for balanced Room squares.

23. (Section 10) Prove that there is a skew resolution in $\mathrm{AG}(m,2)$ for all even $m \geq 4$.

24. (Section 10) Find examples of $\mathrm{BRS}_{\lambda}(2n)$, $\lambda > 2$.

25. (Section 11) Determine the spectrum of $*$-designs.

26. (Section 11) Prove or disprove that $\nu(s,2n) \leq n-1$.

27. (Section 11) Find an example of an $\mathrm{H}_{n-1}(s,2n)$, where $s \neq n, 2n-1$ or $2n-2$.

29. (Section 12) Find more small examples of OSTS(n) for $n \equiv 3 \pmod 6$. In particular, find an OSTS(21).

30. (Section 12) Find more constructions for sets of t OSTS, $t \geq 3$.

31. (Section 13) Find classes of (simple, indecomposable) orthogonal one-factorizations of λK_n, $\lambda > 2$.

32. (Section 15) Determine Spec(C_n) and Spec(P_n) for more values of n.

We now discuss the progress on the ten problems that were posed by W. D. Wallis in the 1972 monograph on Room squares [266]. Most of these questions are now solved. A short answer to each question is supplied here along with a reference to the relevent sections of this survey where more details are given.

1. *Is there a Room square of side 257? If so is there a skew one?* There are Room squares and skew Room squares for all odd orders $n \geq 7$ (Sections 1, 3, and 7).

2. *Suppose that there is a Room square of side r with a subsquare of side n. Is is necessarily true that $r \geq 3n+2$; however the best result is $r = 4n+1$. Is it possible that $3n+2 \leq r \leq 4n+1$? Is there a stronger bound than $r \geq 3n+2$?* The necessary condition is indeed $r \geq 3n+2$. This condition is necessary and sufficient if $n = 7, 9, 11, 13$, or 15, or if $n \geq 393$ (Section 4).

3. *Is there an abelian group of order $3n$ with a strong starter for some n prime to 3?* A strong starter exists in every cyclic group of odd order n for $11 \leq n \leq 999$. It is still unknown in general which groups admit a strong starter (Section 2.1).

4. *Find infinite families of adders for patterned starters in groups that are not of prime power order.* This is the equivalent to asking for an infinite family of strong starters in groups that are not of prime power order. Direct constructions for strong starters of prime power orders can be used with multiplication theorems to obtain strong starters in groups that are not of prime power order (Section 2.1).

5. *Are there skew Room squares of sides 9, 15, or 21? Are there sides for which skew squares (or embedded squares) do not exist?* Skew Room squares exist for all odd orders $n \geq 7$ (Section 7).

6. *Is there a theorem of the form "if there exist n pairwise orthogonal Latin squares of side r then there is a skew Room square of side r"? Is there such a theorem if "skew" is deleted?* We doubt that there is any direct connection between Room squares and sets of orthogonal Latin squares.

7. *One could ask: "Find out something about isomorphism." More specifically,* the text contains four constructions for Room squares of side $r = 6n + 1$

the number of nonisomorphic Room squares of order n, then it is known that $\mathrm{NR}(n) \to \infty$ as $n \to \infty$ (Section 6).

8. *Is there a Room square that has a complement but is not isomorphic to an embedded square? To a skew square? Are there infinite families of them?* To our knowledge, this problem has not been studied.

9. *Find examples of starters and adders in nonabelian groups.* Not much work has been done on this problem, and as far as we know, there are only two papers on this topic: [103] and [154] (Section 2.1).

10. *What is the maximum number of pairwise orthogonal symmetric Latin squares of order 9?* The maximum number of pairwise orthogonal symmetric Latin squares of order 9 is four. For every odd $n \geq 11$ (except possibly $n = 15$), there are at least five pairwise orthogonal symmetric Latin squares of order n (Section 5).

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