Combinatorics in Undergraduate Courses

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Abstract

A plenary session was held at the 17th British Combinatorial Conference to discuss Combinatorics in Undergraduate Courses. This paper discusses the issues raised by the panel and during subsequent discussions by the participants.

1 Introduction

At the 17th British Combinatorics Conference held in 1999 at The University of Kent at Canterbury, a plenary session was held to discuss "Combinatorics in Undergraduate courses". This was the idea of Professor Donald

Preece, who gathered together a panel of four, namely Ian Anderson, Rosemary Bailey, Jeff Dinitz and Bridget Webb, with Chris Rodger chairing the discussion (John Lamb was originally involved as well, but unfortunately had to withdraw due to illness).

We decided that the best format would be to ask each panelist to discuss a topic on this issue for ten minutes, then after each panelist had spoken, the floor would be open for five minutes for members of the audience to comment on, or to go beyond, the issues raised by the speakers. This resulted in an hour of lively, informative discussions on how undergraduate Discrete Mathematics has developed over the past 40 years, and how we are coping with teaching the variety of subjects that sit under this umbrella.

In this paper, we present some of the points raised by the panelists.

2 Sense of Perspective

Clearly Discrete Mathematics has blossomed over the past 40 years, perhaps largely due to the myriad of applications arising from the prevasive use of computers in all facets of today's society. Frank Harary, one of the audience at the session gave his perspective of this rise, starting with the first graph theory course given by him.

Over this time, not only has Discrete Mathematics been a wonderful source of uses of mathematics in industry and government, but it has also proved itself to be an excellent pedagogical tool for teaching proof techniques and problem solving strategies.

On these topics, Ian Anderson made the following observation.

2.1 Undergraduate courses in Combinatorics

Ian Anderson

Most combinatorialists who were undergraduates in the sixties are self-taught. The separate identity of the subject only began to become clear around that time, with the appearance of the classic books by Ryser [4] and Hall [2], the start of the *Journal of Combinatorial Theory* in 1966 and, in Britain, a conference on Combinatorial Mathematics and its Applications, at Oxford in 1969. Although courses in graph theory had been given by trail blazers such as Frank Harary in the US, it was only around 1970 that combinatorial courses began to find a regular place in British universities.

From slow beginnings, the increase in such undergraduate courses has been remarkable. One way to gauge the growth in Britain is to look at the courses listed in the *British Combinatorial Bulletin.* Let me illustrate with three universities, comparing what is now on offer with what was available twenty years ago.

At Glasgow, where I first gave a combinatorial course in 1971, we had, in 1979, two courses:

Graphs and designs (20 lectures, honours)
Graph Theory (13 lectures, general degree course),

whereas in 1999 this had grown to:

Combinatorics (12 lectures, honours)
Combinatorial designs (25 lectures, honours)
Maths of I.T. (largely coding theory) (25 lectures, honours)
Graphs and networks (22 lectures, non-honours)
Algorithms (taught in Computing Science department).

This is by no means atypical. Here are two other examples, taken from universities in Great Britain.

University A	1979:	Graph Theory (30 lectures)
	1999:	Introduction to discrete mathematics (22)
		Graph Theory (22)
		Combinatorics (22)
		Coding Theory (22)
University B	1979:	Graph Theory (26 lectures)
		Combinatorial optimisation (26)
	1999:	Graph Theory (36)
		Combinatorial optimisation (36)
		Discrete mathematics (36)

I believe it is true to say that in general students enjoy the courses, although they do not find them easy. A slight twist in a counting problem can increase the difficulty greatly. But one of the attractions of the subject is the opportunity it offers for new insights, looking at a problem the 'right' way, and the unexpected connections between apparently different problems. The teacher learns from the insights of his/her students; I have often found that I get a much nicer solution from a student than the one I had been producing for years. I finish with a recent illustration of this.

In studying unordered selections with repetitions allowed, I had established that the number of solutions of $x_1 + \ldots + x_n = k$ in non-negative integers x_i is $\binom{n+k-1}{n-1}$. As a nice application of this I set the following problem:

How many binary sequences with p 0s and q 1s are there, $q \geq p-1$, with no two 0s adjacent?

The 0s have to be separated, so we have a sequence of the following type:

$$x_1 \text{ 1s; } 0;\ x_2 \text{ 1s; } 0;\ x_3 \text{ 1s; } 0; \ldots ; x_p \text{ 1s; } 0;\ x_{p+1} \text{ 1s}$$

where $x_1 + \ldots + x_{p+1} = q$, $x_1 \geq 0$, $x_{p+1} \geq 0$, all other $x_i > 0$. Putting $x_1 = y_1$, $x_{p+1} = y_{p+1}$ and $x_i = 1 + y_i$ otherwise, we want the number of solutions of $y_i + \ldots + y_{p+1} = q - (p-1)$ in non-negative integers y_i. This number is

$$\binom{p+1+q-p+1-1}{p} = \binom{q+1}{p}.$$

Very nice and very clever! But a student has a better idea. Why not place the 1s first, instead of the 0s?

$$\ldots 1 \ldots 1 \ldots 1 \ldots \quad \ldots \quad \ldots 1 \ldots 1 \ldots .$$

There are then $q+1$ "gaps" in which to place p 0s, no two in the same gap. The number of solutions is clearly $\binom{q+1}{p}$. The class were impressed. So was I. And the sense of elegance and satisfaction was shared by all.

3 The Current State

Being such a broad area, it is often quite difficult to decide on which of the many possible topics one should cover given the restrictions on the number of courses one is permitted to teach on Discrete Mathematics, and the number of faculty available to teach the courses. Universities range from having a single introductory course in the subject, to places such as Auburn University and the University of Waterloo where as many as 10 undergraduate courses are taught in Discrete Mathematics. Indeed, students can obtain a degree in Applied Discrete Mathematics at Auburn.

Jeff Dinitz decided to take stock of the current state of affairs by using a questionnaire, the results of which are presented below.

One of the comments from the floor tempered the enthusiasm for lots of interesting Discrete Mathematics classes, reminding one and all that undergraduates need to be exposed to a wide range of mathematics, both classical and modern. Indeed, many results and techniques from these other areas have contributed to progress within Discrete Mathematics. Of course, this is a view with which we heartily concur.

3.1 A Survey of Institutions
Jeff Dinitz

When asked to be part of a panel discussion at the 17th British Combinatorics Conference with the topic being *Combinatorics in Undergraduate Education*, I thought that it might be an informative exercise to do a survey on this topic. I wanted this to be an informal method of gathering the information, so I devised a questionnaire and sent it via email to about 70 people who were on my list of aliases. This probably skewed the results as nearly all of these people are in combinatorics; hence their institutions may offer more in this area. Still it is a start, and I did get interesting information from them.

I put the entire questionnaire as well as all the responses on a web page. The URL is `http://www.emba.uvm.edu/~dinitz/panel.bcc.html`.

3.1.1 The Questionnaire

Below is the questionnaire as sent.

```
>From dinitz@uvm-gen.EMBA.UVM.EDU  Fri Jun 18 09:15:30 1999


Greetings from Jeff Dinitz in Vermont,


  I have been asked to be on a panel discussion at the British
Combinatorial Conference next month; the topic is "Combinatorics in
Undergraduate Courses".  (I know -- I should have told them I had
laryngitis, but I didn't). Being the only representative not from
Britain, I thought it might be interesting if I did a very informal
```

poll of mathematicians at colleges and universities in North America, asking the question, "Tell me about combinatorics in undergraduate classes at your institution". So to make it easy for you and take up as little of your time as possible, I have listed combinatorics classes below that I know are taught at the undergraduate level at some institutions. I'd appreciate it if you would indicate below which of these are taught at your school.

I have only one additional question, "What is the most innovative/interesting/different combinatorics class that is taught at your institution ?"

Please reply to me via e-mail at Jeff.Dinitz@uvm.edu at your earliest convenience. You can forget about this after July 5. Also, please do not e-mail me with any questions or requests for clarification, this is not a scientific survey, just do your best to answer the questions.

Thanks for your time. Have a great summer. Maybe I'll see you in England.

Regards,
Jeff

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Name:

Institution:

Mark an X before each class that is taught more-or-less regularly at your institution (anywhere on campus) for at least 2/3 of a term. Mark XX if it is a full year course. Put an R if it is required for a degree.

discrete math (sophomore, typically CS)

combinatorics (roughly 1/2 enumeration, 1/2 graph theory)

combinatorics (broad 1 year course)

enumeration

graph theory

coding theory

cryptography

linear optimization

integer programming

design theory

combinatorial algorithms

information theory

data compression

network flow theory

```
others??

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What is the most innovative/interesting/different combinatorics course
at your institution and what material does it cover?


Thanks,
Jeff
```

3.1.2 Responses to the Questionnaire

I received 38 responses to my solicitation and would like to thank each of those people for taking the time to respond. In Table 1 is a tally of the results. I have placed the classes in order of frequency taught (out of 38 institutions).

It is clear from data in Table 1 that nearly all institutions offer some kind of discrete math class for CS students. What is also not too suprising is that the next most frequently taught class is graph theory. This is followed in frequency by a standard introductory combinatorics class. The next most frequently taught classes (linear optimization, combinatorial algorithms and coding theory) all could be considered as applied combinatorics classes and are taught in roughly 1/4 of the institutions. The remaining classes are taught by a smaller percentage of the schools.

Number of institutions	Course
31	discrete math (sophomore, typically CS)
25	graph theory
22	combinatorics (roughly 1/2 enumeration, 1/2 graph theory)
17	linear optimization
14	combinatorial algorithms
10	coding theory
8	cryptography (and data security)
7	network flow theory
6	integer programming
6	enumeration
4	design theory
3	combinatorics (broad 1 year course)
3	information theory
1	data compression

Others Mentioned:

game theory (1 response)

introduction to proofs in discrete math context (2 responses)

advanced topics (1 response)

Table 1: Questionnaire Response

The second part of my questionnaire asked about the most innovative, interesting and/or different combinatorics course taught at their institution. During my talk at the BCC, I mentioned several of the following responses that I received. (Again let me note that the entire set of responses is available at the URL: `http://www.emba.uvm.edu/~dinitz/panel.bcc.html`.)

Mike Albertson, *Smith College*

Our initial introductory course is the most unusual of any of our courses. It emphasizes beginning proof writing and real applications (e.g. RSA cryptography (we prove the necessary number theory results), minimum weight spanning trees).

Ken Bogart, *Dartmouth College*

Algebraic Combinatorics is a course that starts with counting things like circular arrangements as counting equivalence classes of an equivalence relation and ends with Polya Theory and Mobius Inversion. What makes it unique and fun to teach is that the students work from a list of about 100 exercises (revised each time) with maybe five pages worth of exposition woven in. Most of the time in class is spent working in groups figuring out how to do problems. For problems that are not obvious to everyone someone in a group that has solved it goes to the board with an explanation. They keep a problem notebook and must rewrite solutions until they are acceptable. My experience is that they actually learn Polya theory, for example, and can derive Polya's theorem in an oral exam without advance warning that a derivation will be (very casually) asked for. I purposely ask them in such a way that it is clearly optional to give me a derivation.

Bill Martin, *University of Winnipeg*

Applied Algebra (2nd year): half coding theory, half crypto (as listed above). Sales pitch for modern algebra. So the students enter knowing little more than Z_n. Ideally, they leave willing to learn groups, etc.

One year teaching this course, I got frustrated that I couldn't cover much material, so I gave each of the 17 students a project. They read either a research paper or (more likely) a chapter in an advanced book. Then they wrote a ten page report and gave a 20 minute oral presentation.

They loved it. Topics included: MacWilliams Thm., Assmus-Mattson, El Gamal, MacEliece cryptosystem, factoring, projective planes, compression.

Nabil Shalaby, *Memorial University of Newfoundland*

The 4th year combinatorial design course which is an elective but sometime we get as many as 30 students enrolled. The text I used for the last two years is Ian Anderson's : *Combinatorial Designs: Construction Methods* [1]. The material covered is: Basics of finite fields, BIBDs, MOLS, Room squares, cyclic STSs, Kirkman triple systems and applications.

I think what makes this subject interesting to the students are the assignments and projects that involve constructing these objects (using the difference method). The students appreciate and understand these objects and thus it is easier for them to prove more results about these objects.

Jeannette Janssen, *Dalhousie University*

Game Theory. The course starts with the classical economics stuff, but also has the kind of stuff covered in Berlekamp et al: Winning Ways.

3.1.3 At the University of Vermont

The response by Mike Albertson in the previous section struck a chord with me. Mike notes that they have a class where they basically teach how to do proofs via actual applications in discrete mathematics (and elsewhere). We also have a class like this at the University of Vermont (Math 52). This class is required for all mathematics majors and it does indeed introduce them to some of the constructs and thought processes needed to do upper level mathematics. We have had very good success with this class. Graduating seniors often point to this class as one of the best math classes they had while here. It introduces them to some nice (and modern) applications of mathematics while helping them gain in mathematical maturity.

The syllabus is below. It was developed by Jim Burgmeier and Larry Kost in my department and has been taught (in some form or another) since the fall of 1994. There is currently no text for this class; it is taught from a set of notes that Burgmeier and Kost have prepared.

Math 52 (Fundamentals of Mathematics) at the University of Vermont

An introduction to upper level mathematics (including proofs) through topics in contemporary mathematics.

Syllabus:

1. **Integers:** Euclidean algorithm, base representation, modular arithmetic
2. **Logic:** truth tables, DeMorgan's laws, converse and inverse
3. **Functions and Relations:** sets, reflexive, symmetric, transitive, equivalence relations, partial order, one-to-one and onto
4. **Modular Arithmetic - again:** Euclidean algorithm - extended, inverses, Euler ϕ-function
5. **Public Key Cryptography:** knapsack, RSA, discrete logs
6. **Matrices**
7. **Complex Numbers**
8. **Fractals and Chaos:** Cantor set, Sierpinski gasket, Koch curve, Mandelbrot set, Julia set
9. **Error-Correcting Codes:** sphere-packing bounds, error detection and correction

3.1.4 At Hinesburg Middle School

Even though it is a little off-topic, I can't resist the opportunity to discuss the teaching of combinatorics and discrete math at the middle school level (ages 13-14). In seventh grade, my son Mike (and two other students) took a class in algebra at their school, Hinesburg Middle School. Typically this class was taken by 8th graders, but these three were indeed quite ready for the class and did very well. There was no thought as to what the students would take in the next year (the year before they started high school). There was nothing more offered at their middle school and the high school was inconvenient for them to attend (and I was not in favor of having these young students attend the high school).

I opened my big mouth and told the people at the middle school what I'd do with these bright and eager math students: I would teach them topics in elementary upper level math with an emphasis on discrete math. Of course all my experience is with teaching college students, but I felt that these students could indeed do well in a class modeled after a class we teach at the University of Vermont titled Math 17. This is a class that introduces topics in mathematics to students who are in the arts and sciences and have no intention of taking a subsequent class in math. The middle school took me up on my proposal and the next thing I knew (in 1996) I was teaching Discrete Math to three 8th graders. I used the book that we were using at the university [C.D. Miller, V.E. Heeren, E.J. Hornsby, *Mathematical Ideas* (6th edition), Harper Collins, 1990) but I had to augment it to put it at a little higher level mathematically.

The class was very successful. It was so successful that the school board has adopted it as an integral part of the curriculum and has even hired a *real* teacher (Ann Dutton) who has now taught this class for the past two years (my daughter had it this year from Mrs. Dutton). Even more rewarding is the fact that of the four small towns that comprise the school district, now three of them are teaching discrete math to talented 8th graders (the other towns felt that they needed to keep up with our town). So my grass roots effort to change the curriculum at this level is paying off. One note: the students, in general, really love this class. It is exciting to be exposed to topics like the uncountability of the reals, Euclid's algorithm, combinations and permutations, Euler's planarity formula, group theory and many other topics when you are only 13 or 14 years old (and they really soaked it up). The parents of these students are in general also extremely supportive. They see their child getting excited about mathematics that is clearly at a level beyond what most of them have ever seen (unless they were math majors), while before this, typically, their child was good at but bored with math class.

I attach the syllabus for this class below. It is taught for an entire year. Clearly the topics can be done at many levels. My next hope is to change the culture of the high school and have this same syllabus adopted for seniors (to do at a deeper level).

Discrete Mathematics for 8th Grade Students

Syllabus:

1. **Sets:** Set operations, Venn diagrams, infinite sets, countability of rationals
2. **Logic:** statements, truth tables, mathematical induction
3. **Number Theory:** prime and composite numbers, GCD and LCM, Euclid's Algorithm, Fibonacci sequences
4. **Real Numbers:** integers, decimals, irrational numbers

5. **Algebraic Systems:** modular arithmetic, axioms of groups, permutation groups and group actions
6. **Graphs and Functions:** graphing lines and parabolas, slope, systems of equations, use of computer graphing program, linear inequalities, linear programming
7. **Proofs:** rigorous two column proofs of basic theorems of set theory and real numbers
8. **Geometry:** Pythagorean theorem and Pythagorean triples, perimeter and area
9. **Counting:** counting by systematic listing, fundamental counting principles, permutations, combinations, counting with Venn diagrams
10. **Probability:** probability problems thought of as two counting problems
11. **Algebra:** review 7th grade algebra and an extension to Algebra II
12. **Graph Theory:** graph models, isomorphism, edge counting, planar graphs, coloring graphs
13. **Design Theory:** starters and round-robin tournaments, Steiner triple systems, Latin squares

4 Teaching Issues

The development of Discrete Mathematics has been driven by its many applications in industry, government and society. This gives us an excellent opportunity to attract students who think that teaching is the only job for a mathematician! Discrete Mathematics also has the advantage that, although reaching sophisticated levels, it does not depend upon mounds of mathematical knowledge and so can be taught to students with less technical backgrounds.

Bridget Webb describes below how The Open University has designed a course that successfully attracts students with a wide background, both in terms of their knowledge and their stage of life.

4.1 Teaching Combinatorics to 700 students per year Bridget S. Webb

The Open University presents the half-credit (30 point) course MT365 *Graphs, Networks and Design* annually to over seven hundred students. This is a problems-based course, presented in a down-to-earth manner, and is intended for a wide audience. MT365 replaced an earlier course, TM361, of the same name which was presented from 1981 to 1994 and which taught combinatorics to some four thousand students in total. TM361 evolved from separate plans in the Pure Mathematics Department and the Design Group of the Technology Faculty to produce combinatorics-based courses. The resulting joint venture also produced the present course MT365, and an inter-departmental team continues with the upkeep and presentation of this course. Another Open University course that is partly combinatorial in nature is M336 *Groups and Geometry*, which covers orbit counting and Polya's Theorem, tiling, wallpapering and patterns, and Abelian groups, and is presented annually to over two hundred students.

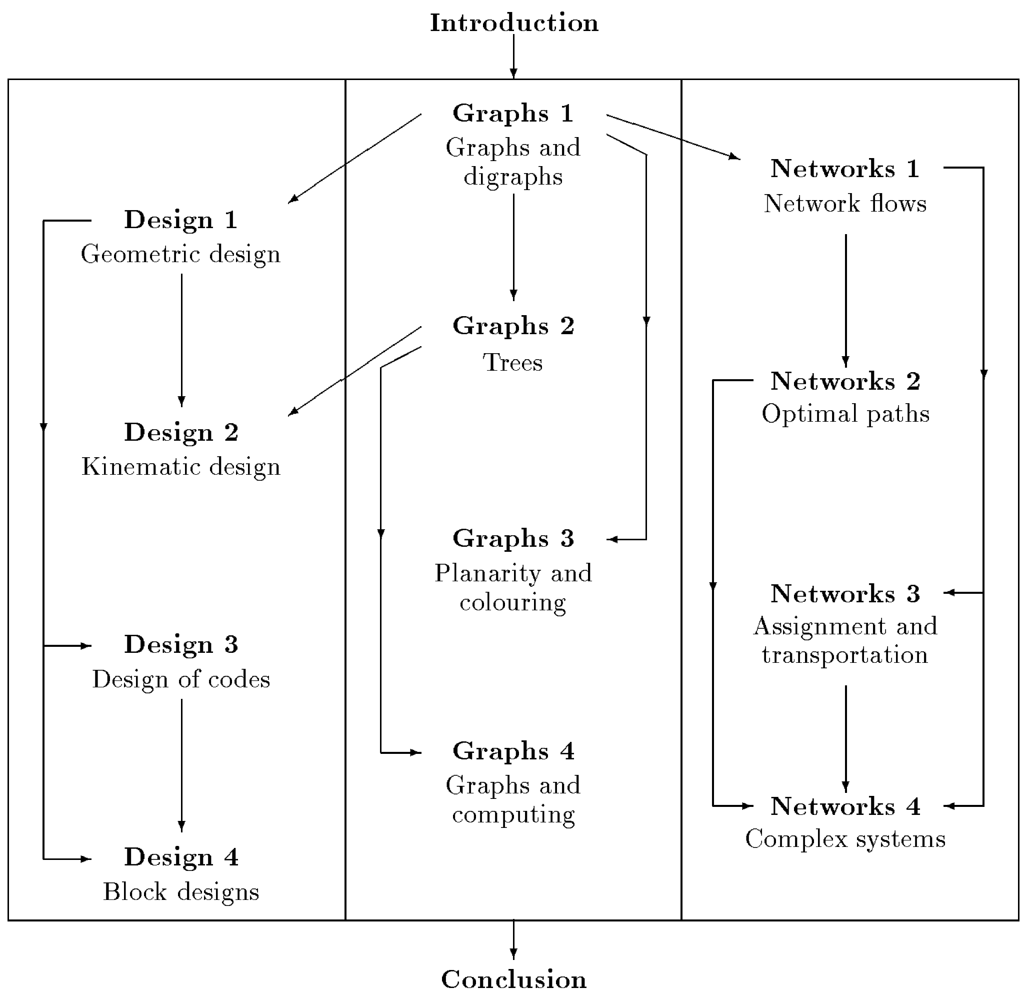
Open University students are adults, mostly in full-time employment, who are studying part-time. They live around the UK and elsewhere. Their backgrounds and experience are diverse—some students studying mathematics courses have no formal mathematics education prior to embarking with the Open University, while others may have had considerable experience. Many students of MT365 use combinatorics in their professional lives.

The course MT365 is intended for highly motivated students who do not necessarily have a strong mathematical background—maturity rather than knowledge is important. The course is popular with mathematicians, technologists, scientists and computer scientists alike. The emphasis is on solving problems and applying algorithms, rather than understanding proofs; it is very much a "doing" course. The development of combinatorics as a subject is integrated with the modelling of practical situations. Mathematicians get to see mathematics in action and technologists and scientists get to see the importance and usefulness of developing a mathematical framework to solve related problems.

The three strands of the course are interlaced: Graphs, Networks (applied graph theory) and Design (encompassing tilings and polyhedra, kinematic design, codes and block designs). The course highlights four different types of problem solving: existence, construction, enumeration, and optimization. Although it is a broadly based course, there is enough depth to give meaning to the topics covered.

MT365 is presented through a variety of different media. There are fourteen printed correspondence texts that provide the bulk of the teaching material; the titles of these texts are given in the following diagram, with the arrows showing the interrelationships between the topics covered. Each text is expected to take between ten and fifteen study hours and includes worked examples as well as problems and exercises with full solutions. The course handbook summarizes the material and may be taken into the final examination. The focus is on understanding and applying the results rather than memorizing information.

Computer activities, using especially developed course software, are an important part of the course. These occur throughout the texts and are



Introduction
Graphs 1
Graphs and digraphs
Design 1
Geometric design
Networks 1
Network flows
Graphs 2
Trees
Design 2
Kinematic design
Networks 2
Optimal paths
Graphs 3
Planarity and colouring
Networks 3
Assignment and transportation
Design 3
Design of codes
Graphs 4
Graphs and computing
Networks 4
Complex systems
Design 4
Block designs
Conclusion

designed to help consolidate or develop ideas that occur in the texts. The software comprises many different packages, including: *Map Colouring*; *Maximum Flows*; *Tilings*; *Animals*; *Codes*; *Critical Path*; *Latin Squares* and an on-line version of the handbook. There is a *Graph Database* package containing the 1252 standard graphs with up to 7 vertices, and a *Graph Editor* that can be used to create graphs and add, delete, move, label or colour vertices and edges. The clipboard enables graphs and networks to be exported and imported from one package to another. Most of the algorithms can be run in either *manual* or *automatic* mode.

The seven accompanying television programs show many of the ideas from the course in practical situations (the design of compact disks, offshore gas pipeline networks, etc.) and the audio tapes (used in conjunction with printed *audio-frames*) help to illustrate the step-by-step implementation of many of the algorithms. All students are assigned a local Tutor who marks their Tutor Marked Assignments and gives face-to-face teaching at day schools and/or tutorials.

The course is assessed by means of Tutor Marked Assignments (TMAs), Computer Marked Assignments (CMAs), and a three-hour final examination. The TMAs are marked by the local tutor, using a marking scheme produced by the Course team, and their marking is monitored by the central team. The CMAs are carefully constructed multiple choice questions designed to home in on the detailed parts of the course that the TMAs can't reach! The computer software may be used to answer TMA questions, as long as the printout is annotated in such a way that it is clear to the Tutor how the answers are obtained.

Like most Open University courses, this course was prepared by a Course Team over a period of about three years. The Course Team comprised 22 people, including academics (both mathematicians and technologists), an administrator, a regional staff member, computer software designers, BBC producers (for television and audio-tapes), an academic editor and publishing editors. The course material evolved through several drafts and was commented on by critical readers (internal and external) at each stage. The external assessor was Lowell Beineke (Professor of Mathematics, Indiana University-Purduse University).

The course continues to be maintained by an Examination and Assessment Board which holds an examination meeting annually with an external examiner. The Course Team Chair is Robin Wilson and external examiners for the course and its predecessor TM361 have included Norman Biggs (London), Peter Cameron (Oxford, now London), Douglas Woodall (Nottingham) and Ian Anderson (Glasgow).

5 Conclusion

It is clear that Discrete Mathematics has become an integral part of the course offerings in the current university mathematics curricula. The subject has many things going for it: captivating problems that can be easily grasped, yet require depth to prove; applications that often drive the development of the subject; and a wealth of excellent examples of various proof techniques; to name three.

Perhaps the main challenge that each university faces is to make a selection from the myriad of possible Combinatorics courses that will fit in with the need for students to see various branches of mathematics. It is no longer a question of whether Combinatorics is to be studied, but one of which areas or how much material can be included.

The answer to this question will vary from university to university, but will also vary greatly over time. Currently, university courses in Discrete Mathematics start from scratch, as is necessitated by the background of the students. This however, will definitely change as the subject enters the high school, and even earlier, curricula. In the USA, Discrete Mathematics has been incorporated into the national high school mathematics standards [3], and the National Science Foundation is funding efforts through DIMACS to educate kindergarten to eigth grade teachers in this area (see http://dimacs.rutgers.edu/lp/institutes/k8_am.html).

It is a dynamic, exciting time for all of us involved in curriculum and course development. Hopefully forums such as this will help us all as we try to answer the questions and deal with the issues raised in this paper.

References

[1] I. Anderson, Combinatorial Designs: Construction Methods, Halsted, 1990.

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[4] H. J. Ryser, Combinatorial Mathematics, The Carus Mathematical Monographs 14, MAA, 1963.