The Stipulation Polynomial of a Uniquely List-colorable Graph

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Abstract

Let G be a simple graph and let $S = (S_1, \ldots, S_n)$ be a set of lists of colors at the vertices of G. G is said to be S list-colorable if there exists a proper coloring of G such that each vertex i takes its color from S_i. Alon and Tarsi [1] have shown that G is S list-colorable if and only if its graph polynomial

$$f_G(\underline{x}) := \prod_{i \sim j} (x_i - x_j)$$

does not lie in the ideal I generated by the annihilator polynomials $g_i(\underline{x})$ of the colors available at the vertices.

We consider the case where G is uniquely list-colorable and determine the irreducible factors of the remainder polynomial (or stipulation polynomial*) $\bar{f}_G = f_G \bmod I$. We establish a bijection between the factors of $\bar{f}_G$ and the edges of G.*

1 Introduction

Let G be a simple graph with vertices $V = \{1, 2, \ldots, n\}$ and edges E. When vertices i and j are joined by an edge, we write $i \sim j$. Let $\mathcal{C} = \{c_1, c_2, \ldots c_q\}$ be a set of indeterminates called *colors* and for each i, $1 \leq i \leq n$, let $S_i \subseteq \mathcal{C}$ be a list of colors available at vertex i. The *S-list coloring* question asks whether there exists a proper (vertex-) coloring $\varphi : V \to \mathcal{C}$ of G such that, for each $i \in V$, $\varphi(i) \in S_i$. Such a proper coloring is called an *S-legal coloring* and G is said to be *S list-colorable*, where $S := \{S_1, \ldots, S_n\}$.

List colorings have arisen in several contexts, mainly in situations where one wishes to determine whether or not a given partial coloring of a graph may be completed to a proper coloring. The actual definition of a list coloring first appeared in Erdös, Rubin, and

Taylor [5]. Their motivation was the so-called "Dinitz problem", which is a special type of list coloring problem (see Janssen [7] and Cipra [3]). A recent paper [6] by Häggvist and Chetwynd gives a history of the general problem (as well as some new results) as does the paper [8] by Kahn. There has been renewed interest in list colorings due to some powerful results obtained recently by N. Alon and M. Tarsi [1]. Indeed, their work has been the motivating factor in the present paper.

Let $\underline{x} = (x_1, x_2, \ldots, x_n)$ be a vector of indeterminates, one for each vertex of G. Define the *edge difference polynomial* of G

$$f_G(\underline{x}) := \prod_{\substack{i \sim j \\ i<j}} (x_i - x_j).$$

It is easy to see that G is list-colorable (with the given lists $S_1, \ldots, S_n$) if and only if there exists a vector $\underline{c} = (c(1), c(2), \ldots, c(n))$ in $S_1 \times S_2 \times \cdots \times S_n$ satisfying $f_G(\underline{c}) \neq 0$.

Now define the *annihilator polynomial* $g_i(\underline{x})$ at vertex i to be the polynomial which is zero precisely when $x_i \in S_i$:

$$g_i(\underline{x}) := \prod_{c \in S_i} (x_i - c).$$

Define I to be the ideal generated by $\{g_i(\underline{x}) : 1 \leq i \leq n\}$ in $\mathbb{Z}[\underline{x}]$. For any proper list coloring $\underline{c}$ of G, we have $f_G(\underline{c}) \neq 0$ while $g_i(\underline{c}) = 0$ for $1 \leq i \leq n$. Extending this, Alon and Tarsi prove

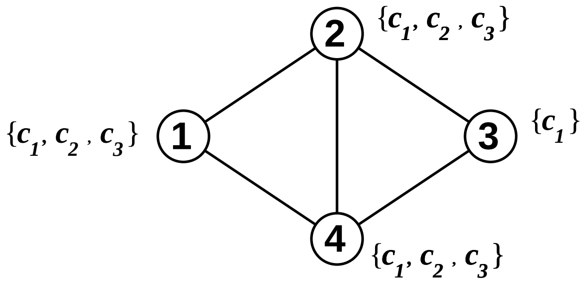
Theorem 1.1 (Alon and Tarsi, [1, Prop. 2.7]) *Either G is S list-colorable or $f_G \in I$, not both.* $\square$

This theorem as it stands is difficult to use. Perhaps the more important part of their paper from the practical viewpoint is the following theorem. (An Eulerian subgraph of a directed graph is a subgraph in which every vertex has in-degree equal to its out-degree.)

Theorem 1.2 (Alon and Tarsi, [1, Theorem 1.1]) *Let G and S be given. Suppose there is an orientation D of G such that (i) the out-degree of vertex i in D is less than $|S_i|$ for all i, and (ii) the number of Eulerian subgraphs of D with an even number of edges is not equal to the number of Eulerian subgraphs of D with an odd number of edges. Then G is S list-colorable.* $\square$

While this theorem has been the focus of several recent papers [4, 7] on list coloring, we prefer to work directly with Theorem 1.1. Our strategy is to consider graphs which *can* be list colored. In these cases, we would like to understand the remainder polynomial $\bar{f}_G = f_G \bmod I$. We view the irreducible factors of $\bar{f}_G$ as stipulations on colorability of G. Several examples may help here.

Example 1: Let G be the graph $K_4 - e$ with $V = \{1, 2, 3, 4\}$ and $E = \{12, 23, 34, 14, 24\}$.



Suppose the lists at the vertices are $S_1 = S_2 = S_4 = \{c_1, c_2, c_3\}$ and $S_3 = \{c_1\}$. Then the edge difference polynomial is

$$f_G(\underline{x}) = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_1 - x_4)(x_2 - x_4),$$

and the four annihilator polynomials are

$$g_1(\underline{x}) = (x_1 - c_1)(x_1 - c_2)(x_1 - c_3), \qquad g_2(\underline{x}) = (x_2 - c_1)(x_2 - c_2)(x_2 - c_3),$$

$$g_3(\underline{x}) = (x_3 - c_1), \qquad g_4(\underline{x}) = (x_4 - c_1)(x_4 - c_2)(x_4 - c_3).$$

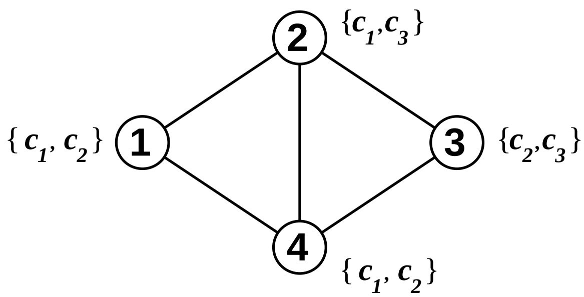
If I is the ideal generated by $g_1(\underline{x}), \ldots, g_4(\underline{x})$ in $\mathbb{Z}[\underline{x}]$, we can compute $\bar{f}_G = f_G \bmod I$ to be

$$\bar{f}_G(\underline{x}) = -(x_1 - c_2)(x_1 - c_3)(x_2 - c_1)(x_4 - c_1)(x_2 - x_4).$$

An evaluation $\varphi : \underline{x} \to \mathcal{C}$ of this polynomial is non-zero if and only if vertex 1 does not get color c_2 or c_3, neither vertex 2 nor 4 get color c_1 and vertices 2 and 4 receive different colors. On the other hand, observe that the evaluation φ is non-zero if and only if $\varphi(x)$ is a proper list-coloring. Note that the condition that a vertex be assigned a color from its list has been "modded out"; i.e., this is to be assumed at this point.

In this way, we feel that the irreducible factors of the polynomial $\bar{f}_G$ correspond to stipulations on the colorings of G. Our task is to decipher the algebraic language in which $\bar{f}_G$ presents these conditions to us. In general, these irreducible factors may be quite unwieldy. We will determine $\bar{f}_G$ completely in the case where G has exactly one S-legal coloring. In this case, we say G is *uniquely list-colorable*. Let us now look at a graph which has a unique S-legal coloring.

Example 2: Suppose G is again $K_4 - e$ as in the previous example. Let $S_1 := \{c_1, c_2\}$, $S_2 := \{c_1, c_3\}$, $S_3 := \{c_2, c_3\}$, and $S_4 := \{c_1, c_2\}$.



Then

$$f_G(\underline{x}) = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_1 - x_4)(x_2 - x_4),$$

$$g_1(\underline{x}) = (x_1 - c_1)(x_1 - c_2), \qquad g_2(\underline{x}) = (x_2 - c_1)(x_2 - c_3),$$

$$g_3(\underline{x}) = (x_3 - c_2)(x_3 - c_3), \qquad g_4(\underline{x}) = (x_4 - c_1)(x_4 - c_2),$$

and

$$\bar{f}_G(\underline{x}) = (x_1 - c_1)(x_2 - c_1)(x_3 - c_3)(x_4 - c_2)(c_2 - c_3).$$

We describe again how we will "read" this polynomial. It says that vertex 1 cannot be colored c_1, vertex 2 cannot be colored c_1, vertex 3 cannot be colored c_3, and vertex 4 cannot be colored c_2. Thus it gives us the unique list coloring. While the factor $c_2 - c_3$ seems to give us no new information, we will see later that "inessential" factors of this sort arise often in the case where G is uniquely colorable.

Most of the detailed calculations seen here were performed with the aid of the MAPLE computer algebra system.

2 Algebraic Preliminaries

The following lemma was used in the proof of Theorem 1.1.

Lemma 2.1 (Alon and Tarsi, [1, Lemma 2.1]) *Let $P = P(\underline{x})$ be a polynomial in n variables over the ring of integers $\mathbb{Z}$. Suppose that for $1 \le i \le n$, the degree of P as a polynomial in x_i is at most d_i and let $S_i \subset \mathbb{Z}$ be a set of $d_i + 1$ distinct integers. If $P(\underline{x}) = 0$ for all n-tuples $\underline{x} \in S_1 \times \cdots \times S_n$ then $P \equiv 0$.* $\square$

Each of the polynomials $g_i(\underline{x})$ is univariate and monic. Thus, using the division algorithm for any fixed $1 \le i \le n$, there is a natural way to write any polynomial $P(\underline{x}) \in \mathbb{Z}[\underline{x}]$ as

$P(\underline{x}) = q(\underline{x})g_i(\underline{x}) + r(\underline{x})$ where $q(\underline{x}), r(\underline{x}) \in \mathbb{Z}[\underline{x}]$ and $\deg_{x_i}(r) < \deg_{x_i} g_i$. We can perform this reduction repeatedly on $f_G(\underline{x})$ to obtain

$$f_G(\underline{x}) = h_1(\underline{x})g_1(\underline{x}) + h_2(\underline{x})g_2(\underline{x}) + \cdots h_n(\underline{x})g_n(\underline{x}) + \bar{f}_G(\underline{x}).$$

While the polynomials h_i are not unique (they may depend on the ordering of the vertices), the remainder polynomial $\bar{f}_G$ is. For using Lemma 2.1, one can show that $\bar{f}_G$ is the unique polynomial satisfying $\deg_{x_i} \bar{f}_G < \deg_{x_i} g_i$ which is also congruent to f_G modulo the ideal I. For the remainder of this paper, we will refer to $\bar{f}_G = f_G \bmod I$ as the *stipulation polynomial* of the pair (G, S) or, simply the stipulation polynomial of G.

Note that f_G is a homogeneous polynomial of degree $m = |E|$ in the variables $x_1, x_2, \ldots, x_n$. Treating the colors $c_1, c_2, \ldots, c_q$ as indeterminates, each of the annihilator polynomials g_i can be treated as a homogeneous polynomial in the variables $x_1, \ldots, x_n$, and $c_1, \ldots, c_q$. We can obtain any remainder $P(\underline{x}, \underline{c}) \bmod g_i(\underline{x}, \underline{c})$ by repeatedly replacing $x_i^{|S_i|}$ by a homogeneous polynomial in $x_i, c_1, \ldots, c_q$ of degree $|S_i|$ having strictly lower degree in the variable x_i. Thus, when all such substitutions have been made, and we arrive at $\bar{f}_G$, we still have a homogeneous polynomial of degree m, now in the variables $x_1, \ldots, x_n, c_1, \ldots, c_q$. Moreover, every irreducible factor of $\bar{f}_G$ is a homogeneous polynomial in these variables. This proves

Lemma 2.2 *If G is a graph with n vertices and $\cup S_i = \{c_1, \ldots, c_q\}$, then the stipulation polynomial $\bar{f}_G$ of the pair (G, S) is a homogeneous polynomial of degree $m = |E(G)|$ in the variables $x_1, \ldots, x_n, c_1, \ldots, c_q$. Moreover, every proper factor of $\bar{f}_G$ is homogeneous (of lesser degree) in these variables.* $\square$

We shall call an irreducible factor of $\bar{f}_G$ a *k-th order stipulation* if it is homogeneous of degree k. We shall see below that first order stipulations are relatively easy to interpret but those of larger order can be quite difficult. In the case where G is uniquely list-colorable, we will observe in Section 3 that all stipulations are first-order.

3 The stipulation polynomial

In this section, we completely determine the stipulation polynomial $\bar{f}_G$ of a graph G when G is uniquely list-colorable.

Our first result is essentially obtained by Lagrange interpolation. Let graph G be given with a set $S = \{S_1, \ldots, S_n\}$ of lists of colors at the vertices of G. For a given vector $\underline{c} \in S_1 \times \cdots \times S_n$, $\underline{c} = (c(1), \ldots, c(n))$ say, define $\hat{g}_{i,c}(\underline{x}) = g_i(\underline{x})/(x_i - c(i))$ for $i = 1, \ldots, n$.

Theorem 3.1 *With G, S given, we have*

$$\bar{f}_G(\underline{x}) = \sum_{\underline{c} \in S_1 \times \cdots \times S_n} \frac{\prod_{i=1}^n \hat{g}_{i,c}(\underline{x})}{\prod_{i=1}^n \hat{g}_{i,c}(\underline{c})} f_G(\underline{c}). \tag{1}$$

Proof: It is clear that this polynomial has degree less than $|S_i|$ in variable x_i for each i. Moreover, for any $\underline{s} \in S_1 \times \cdots \times S_n$, we have

$$\bar{f}_G(\underline{s}) = f_G(\underline{s}) = \frac{\prod_{i=1}^n \hat{g}_{i,s}(\underline{s})}{\prod_{i=1}^n \hat{g}_{i,s}(\underline{s})} f_G(\underline{s}) = \sum_{\underline{c} \in S_1 \times \cdots \times S_n} \frac{\prod_{i=1}^n \hat{g}_{i,c}(\underline{s})}{\prod_{i=1}^n \hat{g}_{i,c}(\underline{c})} f_G(\underline{c}).$$

So the polynomial given takes on the same values as $\bar{f}_G$ on the grid $S_1 \times \cdots \times S_n$. Using Lemma 2.1, we conclude that they are equal. $\square$

Now, in the case where G is uniquely S list-colorable, we have $f_G(\underline{c}) = 0$ for all but one point $\underline{c}$ on the grid $S_1 \times \cdots \times S_n$. Thus, we immediately get

Corollary 3.2 *Assume G and S are given and that G has a unique S-legal coloring $\underline{c}$. Then*

$$\bar{f}_G(\underline{x}) = \frac{\prod_{i=1}^n \hat{g}_{i,c}(\underline{x})}{\prod_{i=1}^n \hat{g}_{i,c}(\underline{c})} f_G(\underline{c}).$$

Theorem 3.1 also gives us some information about linear factors in the general case. Let i be a vertex of G and let c be a color. We say that color c is *forbidden* at vertex i if $c \in S_i$ and there is no S-legal coloring of G which assigns color c to vertex i. Let $F_i \subseteq S_i$ be the set of colors which are forbidden at vertex i. Then we have

Corollary 3.3 *If F_i is the set of colors forbidden at vertex i, then for each i,*

$$\gcd(\bar{f}_G, g_i) = \prod_{c \in F_i} (x_i - c).$$

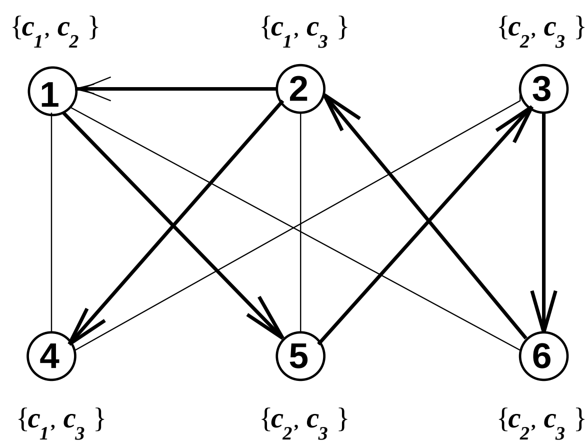
Proof: Clearly, if $x_i - c$ divides $\bar{f}_G(\underline{x})$, then any coloring which uses color c at vertex i corresponds to a vector $\underline{c}$ for which $\bar{f}_G(\underline{c})$ is zero. Thus $f_G(\underline{c}) = 0$ as well and so the coloring is not proper. Conversely, if no proper list-coloring of G uses color c at vertex i, then the factor $x_i - c$ divides every non-zero summand on the right-hand side of Equation 1. So it also divides the left-hand side. $\square$

Of course it is clear that, when G has many S-legal colorings, the factors of $\bar{f}_G$ can become quite complex. We are interested in finding and interpreting factors of small degree. In the next section, we report our progress in this direction.

4 Essential sets of edges

We have shown that, when G is uniquely list-colorable, its stipulation polynomial factors into $m = |E(G)|$ linear factors, each either of the form $(x_i - c_j)$ or $(c(i) - c(j))$. We call factors of the former type *essential* and factors of the latter type *inessential*. The original polynomial f_G has one linear factor for each edge of G. Thus, the numerology of this situation suggests that there could be a set $\mathcal{E}$ of edges of G which correspond to the essential factors with the remaining edges playing the part of the inessential factors. When G is uniquely list-colorable, define a subset $\mathcal{E} \subseteq E(G)$ to be an *essential set* if the members of $\mathcal{E}$ can be oriented so that, for each vertex i and for each forbidden color c at vertex i, there is a unique directed edge $k \to i$ in this set $\mathcal{E}$ such that $c(k) = c$. A naive interpretation here is that when vertex k is colored with color c, this forbids color c at vertex i and this contributes the factor $(x_i - c)$ to $\bar{f}$.

Example 3: The following graph is uniquely list-colorable. An essential set is indicated in bold.



Here

$$\bar{f} = \pm(x_1 - c_1)(x_2 - c_3)(x_3 - c_3)(x_4 - c_1)(x_5 - c_2)(x_6 - c_2)(c_1 - c_3)(c_2 - c_3)^3.$$

In this section, we will show that, whenever G is uniquely list-colorable, such an essential set exists. However, (i) this set of edges is not always unique, and (ii) the presence of an essential set does *not* prove that the coloring is unique.

Theorem 4.1 *Let G be a uniquely list-colorable graph. Then $E(G)$ contains an essential set.*

Proof: Let G be as given with lists $S = (S_1, S_2, \ldots, S_n)$. Let $c(i)$ be the forced color at vertex i and define $F_i = S_i - c(i)$. Construct a bipartite graph B with bipartition (X, E) where

$$X = \{(i, c) : 1 \leq i \leq n, \quad c \in F_i\}$$

and $E = E(G)$. A node $(i, c) \in X$ is joined to node $e \in E$ in B precisely when e is incident to i in G and the other endpoint of e is forced to take on color c. It is not difficult to see that an essential set in G is equivalent to a matching in B which saturates X. (See Example 4.)

Suppose there is no matching in B which saturates X. By Hall's Theorem (see, e.g., [2]), there must exist a set $S \subseteq X$ such that the neighbor set $N(S) = \cup_{s \in S} N(s)$ (where $N(s)$ is the set of neighbors of node s) has strictly smaller cardinality than S. By throwing components away if necessary, we may assume that the subgraph M of B induced by $S \cup N(S)$ is connected.

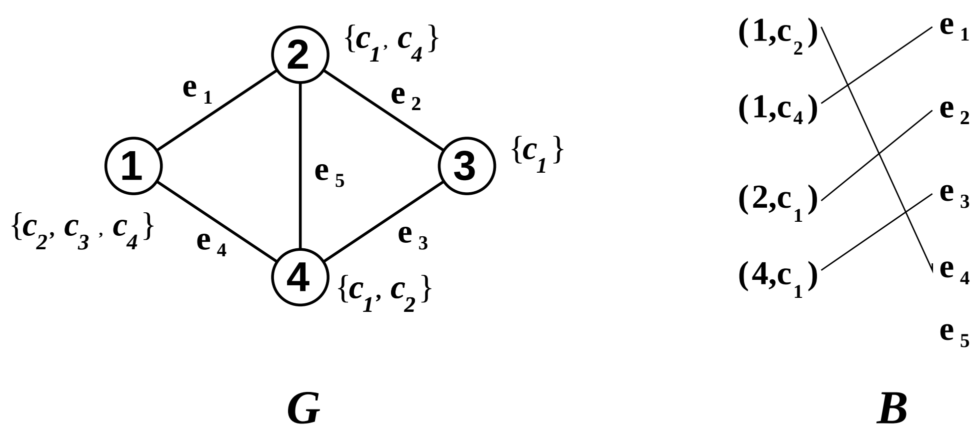
Now in B, every element of E has either zero, one, or two neighbors. We claim that every element of $N(S)$ has two neighbors in S. Let $|N(S)| = k_1 + k_2$ where k_i is the number of vertices whose degree in M is i. The number of edges in M is $k_1 + 2k_2$. Since M is connected, it contains a spanning tree: so

$$k_1 + 2k_2 \geq k_1 + k_2 + |S| - 1 \geq 2k_1 + 2k_2$$

since, under our assumption that there is no matching, $|S| > |N(S)|$. This forces $k_1 = 0$ as desired. Thus $|N(S)| = k_2 = \frac{1}{2}|E(M)| < |S|$ and, since M is connected, M looks like a tree on S in which each edge has been subdivided once.

Let (i, c) be a node in S. Then every node in S at distance 2 from (i, c) in M must be of the form $(j, c(i))$ by construction of B. Extending this, there must be only two colors, c and c' say, which occur as second coordinates of members of S. Let (i, c) be any node of S. Then every neighbor of i in G which is colored c' must be in S since M contains no nodes $e \in E$ of degree one. Therefore, if for each node $(i, c) \in S$ we recolor vertex i of G with color c, we obtain a second S-legal coloring of the graph G. Yet our hypothesis was that G is uniquely list-colorable, So this must be impossible. We conclude that the matching we seek exists. This provides us with our essential set. $\square$

Example 4:



Whenever G is uniquely list-colorable and we have an essential set $\mathcal{E}$ of edges, we can easily read off the stipulation polynomial of G. Given essential set $\mathcal{E}$, we define an edge $e = ij$ to be *inessential* if it does not belong to $\mathcal{E}$. These edges are then directed arbitrarily. We are now in a position to prove our main theorem.

Theorem 4.2 *Let G be a uniquely list-colorable graph with coloring $\underline{c} = \{c(1), \ldots, c(n)\}$ and with essential set $\mathcal{E}$. Let $\mathcal{I} := E(G) \setminus \mathcal{E}$ be the set of inessential edges. Then*

$$\bar{f}_G = \pm \prod_{ij \in \mathcal{E}} (x_j - c(i)) \prod_{ij \in \mathcal{I}} (c(i) - c(j)).$$

Proof: By definition of an essential set, the right hand side is

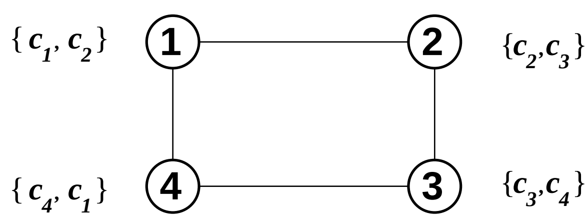
$$\prod_{i=1}^{n} \hat{g}_i(\underline{x}) \prod_{ij\in\mathcal{I}} (c(i) - c(j)).$$

So, using Theorem 3.2, we need only prove that

$$\frac{f_G(\underline{c})}{\prod_{i=1}^{n} \hat{g}_i(\underline{c})} = \prod_{ij\in\mathcal{I}} (c(i) - c(j)).$$

Examining the left-hand side of this equation, this quotient corresponds set-theoretically to deleting an essential set of edges. Thus, the terms of $f_G(\underline{c})$ which remain correspond precisely to the terms that arise from the inessential edges, up to multiplication by -1. □

The presence of an essential set is not sufficient to declare a graph uniquely list-colorable. In the following example, there are two colorings and an essential set for each.
Example 5:



Indeed, uniquely list-colorable graphs are rare. However, even in the case where the graph has many colorings, empirical evidence suggests that the irreducible factors can still be interpreted combinatorially. More precisely, we believe that there will always be a function from the edge set of G to the set of irreducible factors of $\bar{f}_G$ with the property that the pre-image of any irreducible factor induces a connected graph whose stipulation polynomial (perhaps with some colors deleted from the lists) is precisely that factor. A weaker conjecture is that, for each irreducible factor h, the set $\{i : x_i \text{ appears in } h\}$ induces a connected subgraph of G.

5 Further results and examples

The following is an immediate corollary to Theorem 1.2 and Lemma 3.3. The proof is omitted.

Corollary 5.1 *Suppose G has an orientation D satisfying Theorem 1.2 and having outdegree sequence $d = (d_1, \ldots, d_n)$. Then there are at most d_i colors forbidden at vertex i for $1 \leq i \leq n$.* □

Let us say that a color c is *allowable* at vertex i if there exists a proper list coloring φ of G with $\varphi(i) = c$. Suppose that the only stipulations involving x_i are of the form $x_i - c$ where c is a color. Then, for any coloring of G which makes $\bar{f}$ non-zero, we can change the color of vertex i to any color allowable there — leaving all other colors the same — and $\bar{f}$ will remain non-zero. (This is true since the only way this expression can be made zero simply by changing x_i is by choosing a forbidden color there.) This proves

Proposition 5.2 *Suppose the polynomial*

$$\frac{\bar{f}_G(\underline{x})}{\prod_{c \in F_i}(x_i - c)}$$

has degree zero in x_i and let j be a neighbor of vertex i. Then no color can be allowable at i and also allowable at j. □

We now construct, for each positive integer k a graph with a unique coloring having all lists of size k.

In [5], Erdös et al. construct, for each k, a complete bipartite graph E_k which is not k-choosable. The vertices of E_k correspond to two copies of the k-sets of a set of size $2k-1$. These k-sets are chosen as the lists at the $2\binom{2k-1}{k}$ vertices. If k or more colors are utilized on one side of the bipartition, then some vertex on the other side has its entire list forbidden. On the other hand, at least k colors must be used on each side since, for every set of $k-1$ or fewer colors, there is a vertex whose list contains none of these.

To construct a graph with a unique coloring, we proceed as follows. For each j $(1 \leq j \leq 2k-1)$, construct a complete bipartite graph as above and modify just one list. Choose a vertex, v_j, whose list does not contain color c_j and replace some element of that list with color c_j. The bipartite graph is now list-colorable, but color c_j must be used at vertex v_j. For each of these graphs, we choose a fixed list coloring and join a vertex i to those vertices v_j such that color c_j is not the color of vertex i. The resulting graph has a unique coloring. It is not difficult to modify this argument so as to guarantee that the graph obtained is bipartite.

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