On Hamiltonian Paths with Prescribed Edge Lengths in the Complete Graph

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Abstract

Marco Buratti has conjectured that given p a prime and a multiset S containing $p-1$ non-zero elements from Z_p, there exists a Hamiltonian path in K_p where the multiset of edge lengths is S. In this paper we completely solve this conjecture when S contains at most two distinct values.

1 Introduction

Given a graph G, a *Hamiltonian Path* in G is a path that visits every vertex exactly once. In a given graph, a Hamiltonian path may or may not exist, however, it is well known in that complete graphs Hamiltonian paths always exist. Let p be a prime and let K_p be the complete graph on p vertices. When the vertices of K_p are labeled with the elements of the cyclic group Z_p one can define the *length* of an edge xy to be $x-y$ or $y-x$, whichever is less than $\frac{p-1}{2}$. Extending this definition slightly we say that an edge with length $k \leq \frac{p-1}{2}$ also has length $p-k$ (there will be no ambiguity when this is used in context). The reader is referred to [4] for further definitions and graph theoretic background and to [1] for an extensive survey of Hamiltonian graphs. In 2007, Marco Buratti made the following conjecture.

Conjecture 1 *(Buratti) Given p a prime and a multiset S containing $p-1$ non-zero elements from Z_p, there exists a Hamiltonian path H in K_p where the multiset of edge lengths in H is S.*

From the above, it is clear that Buratti's conjecture is equivalent to the following: Given p a prime and a multiset S containing $p-1$ non-zero elements between 1 and $\frac{p-1}{2}$, there exists a Hamiltonian path H in K_p where the multiset of edge lengths in H is S. We first consider the two extremal cases, namely when $|S| = 1$ and when $|S| = p - 1$

When $|S| = 1$ all edges have the same length k. The path $0, k, 2k, 3k, \ldots, (p-1)k$ (with all terms reduced modulo p) is a Hamiltonian path with all edge lengths k. Note that since k and p are relatively prime, all vertices in this path are distinct.

Now assume that $|S| = p-1$ and hence $S = \{1, 2, \ldots, p-1\}$. The following is a Hamiltonian path with the prescribed lengths: $0, 1, -1, 2, -2, 3, -3, \ldots, \frac{p-1}{2}, -\frac{p-1}{2}$ (again all terms are reduced modulo p). It is easy to check that the sequence of edge lengths starts with 1 and increases by 1 for each edge added, giving an edge of length d for all non-zero elements $d \in Z_p$. Note that the edge lengths of this Hamiltonian path can also be given as $S = \{1, 1, 2, 2, \ldots, \frac{p-1}{2}, \frac{p-1}{2}\}$. This is essentially the well-known *Walecki Construction* (see [3] for a recent survey of this construction).

The main result of this paper is that when S consists of exactly two different values a and b, that there is always a Hamiltonian path in K_p where the edges all have length either a or b.

2 Paths with Edge Lengths a and b

The case of exactly two edge lengths is greatly simplified by the following two reductions.

Lemma 2 *Let p be a prime. The existence of a Hamiltonian path in K_p where all edges have length a or b is equivalent to the existence of a Hamiltonian path where all lengths are 1 and $a^{-1}b$.*

Proof: Since Z_p is a field, multiplicative inverses exist for all elements of $Z_p \setminus \{0\}$. List the vertices of the Hamiltonian path with lengths a and b in the order in which they were visited, $x_0, x_1, \ldots, x_{p-1}$. Since it is a Hamiltonian path, every vertex appears exactly once. Multiply each vertex by a^{-1}. The resulting sequence is $a^{-1}x_0, a^{-1}x_1, \ldots, a^{-1}x_{p-1}$. Since a^{-1} is a unit in Z_p, each element of Z_p appears once in the new sequence, so it is still a Hamiltonian path. Furthermore, the lengths of each path are now either $a^{-1} * a = 1$ and $a^{-1} * b$. ∎

Clearly it doesn't matter which length was chosen to be labeled a and which was chosen to be labeled b. Hence, one can always label the length that occurs more often as a and the length that occurs less often as b. This reduces Buratti's entire problem with two edge lengths to showing that there is a Hamiltonian path with edge lengths 1 and k for all k where number of occurrences of edges of length 1 is not less than the number occurrences of edges of length k. We record this in the following proposition.

Proposition 3 *Let $1 < k \leq \frac{p-1}{2}$ and suppose that there exists a Hamiltonian path in K_p containing u edges of length 1 and v edges of length k for every u and v with $u + v = p - 1$ and $u \geq v$. Then, given any a and b with $1 \leq a, b \leq p - 1$ and any n_a and n_b with $n_a + n_b = p - 1$ there exists a Hamiltonian path H in K_p with exactly n_a edges of length a and n_b edges of length b .*

3 Constructions

For the remainder of the paper we will be constructing Hamiltonian paths in K_p ($p \geq 5$ a prime) with all edges of length 1 or k with $1 < k \leq \frac{p-1}{2}$. Let n denote the number of edges of length k in the Hamiltonian path and hence there are $p - 1 - n$ edges of length 1 in the path. From Proposition 3 we can assume that $1 \leq n \leq \frac{p-1}{2}$. Using the division algorithm, we write $n = qk + r$ where $0 \leq r < k$. We require four fairly similar constructions to cover all the cases. We consider the cases when k is even and when k is odd with r even, r odd, and $q = 1$.

3.1 k even

Lemma 4 *If $n = qk + r$ with k even, then there is a Hamiltonian path in K_p with n edges of length k and $(p - 1) - n$ edges of length 1.*

Proof: First consider the case where $n = qk + r$ with k even, r even. We begin by placing all the edges of length k and construct a path in the following manner:

Start at 0 and go upwards in increments of k until the vertex $(q+1)k$ is reached. This picks up all of the vertices congruent to 0 modulo k between 0 and $(q + 1)k$. Add one to get to $(q + 1)k + 1$, then travel backwards in increments of k. This picks up all of the vertices congruent to 1 modulo k. At 1, add one to get to the vertex 2 and then travel to the vertices congruent to 2 modulo k by way of paths of length k. Continue in this pattern until you have reached the vertex $r - 1$. At this point, add 1 to get to the vertex r and travel upwards in increments of k. Instead of going to $(q + 1)k + r$, end at $qk + r$. Add 1 and travel downwards to $r + 1$. Pick up the remaining congruence classes (modulo k) in this manner. Since k is even, the last congruence class will be obtained by traveling downward and ending at $k - 1$. This part of the Hamiltonian path containing the edges of length k is summarized in the list below:

$$
\begin{array}{l}
0, k, 2k, \ldots, (q+1)k, \\
(q+1)k+1, qk+1, \ldots, 1, \\
2, k+2, 2k+2, \ldots, (q+1)k+2, \\
(q+1)k+3, qk+3, \ldots, 3, \\
\ddots \\
r-2, k+(r-2), \ldots, (q+1)k+(r-2), \\
(q+1)k+(r-1), qk+(r-1), \ldots, r-1, \\
r, k+r, 2k+r, \ldots, qk+r, \\
qk+r+1, \ldots, r+1, \\
r+2, \ldots, qk+(r+2), \\
\ddots \\
qk+(k-1), \ldots, k-1
\end{array}
$$

Note that if $r = 0$, it is necessary to start at the row beginning with r and not the row listed above as beginning with 0.

This path exhibits certain properties. First, it is easy to see that every vertex between 0 and $(q+1)k+(r-1)$ has been visited. Furthermore, the degree of every vertex in that range is 2 with the exceptions of 0 and $k-1$. Second, the path contains exactly n edges of length k. This follows since the first r congruence classes have $q+1$ edges of length k going forward and the last $k-r$ congruence classes contain exactly q edges going forward giving a total of $r(q+1)+(k-r)q = qk+r = n$ edges of length k. Lastly, the vertex 0 has an open end, so the additional edges of length 1 can be added in a counterclockwise manner starting at 0 until reaching the vertex $(q+1)k+r$. The addition of these edges of length 1 yields a Hamiltonian path.

In the case where k is even and r is odd, the same construction above can be used with one slight modification. Let $r = s+1$ and use s in the construction. The path contains $qk+r-1$ edges of length k and ends at $k-1$. Add the edge $(k-1, -1)$. This time the open vertex is at -1, so the extra edges of length 1 needed to make a Hamiltonian path are added this time in a counterclockwise manner starting at -1.

The last property that needs to be checked is that this construction fits in K_p for all p. In the case of r even, the largest vertex visited is $(q+1)k+r-1$. In the case of r being odd, one more vertex was used, so if the vertices are relabeled so the open end is at 0, the largest vertex visited is $(q+1)k+r$ (we must show this value is less than p). Recall that by Proposition 3, $n = qk+r \leq \frac{p-1}{2}$ and $k \leq \frac{p-1}{2}$. It follows that, $(q+1)k+r = (qk+r)+k \leq \frac{p-1}{2}+\frac{p-1}{2} = p-1$. Therefore, this construction works for all p. ■

Figures 1 and 2 show an example of the construction when k is even. Figure 1 shows a path where k is even and r is even. In this example, 10 edges of length 4 are placed in Z_{23}. In this case, $n = qk + r = 2 * 4 + 2$. Figure 2 shows how the case where k is even and r is even can be extended to r being odd by adding the edge $(k-1, -1)$. For this case we demonstrate 11 edges of length 4 in Z_{23}.

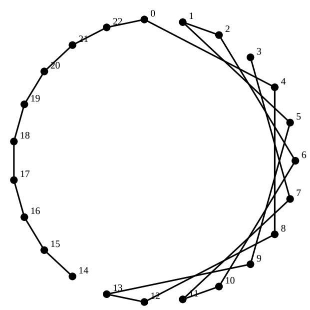

Figure 1: A Hamiltonian path with $p = 23, n = 10, k = 4$

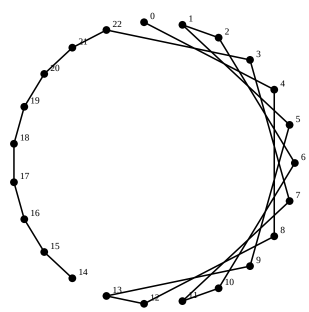

Figure 2: A Hamiltonian path with $p = 23, n = 11, k = 4$

3.2 k odd

It is necessary to break this case into three parts. Lemma 5 covers the case when k is odd and r is even. Lemmas 6 and 7 cover the case when k is odd and r is odd with Lemma 6 providing a construction for $q \geq 2$ and Lemma 7 providing the construction for $q = 1$.

Lemma 5 *If $n = qk+r$ with k odd and r even, then there is a Hamiltonian path in K_p with n edges of length k and $(p-1)-n$ edges of length 1.*

Proof: In the case of k odd and r even, we use essentially the same pattern that was used in the k even case with even remainder. However note that instead of ending the path at the vertex $k-1$, the path now ends at $qk+k-1$. This change is noted in the summary of the pattern below.

$$
\begin{array}{l}
0, k, 2k, \ldots, (q+1)k, \\
(q+1)k+1, qk+1, \ldots, 1, \\
2, k+2, 2k+2, \ldots, (q+1)k+2, \\
(q+1)k+3, qk+3, \ldots, 3, \\
\ddots \\
r-2, k+(r-2), \ldots, (q+1)k+(r-2), \\
(q+1)k+(r-1), qk+(r-1), \ldots, r-1, \\
r, k+r, 2k+r, \ldots, qk+r, \\
qk+r+1, \ldots, r+1, \\
r+2, \ldots, qk+(r+2), \\
\ddots \\
k-1, \ldots, qk+(k-1)
\end{array}
$$

Once again, this path visits the first $(q+1)k+(r-1)$ vertices and contains exactly $n = qk+r$ edges of length k, since again the first r congruence classes have $q+1$ edges of length k going forward and the last $k-r$ congruence classes contain exactly q edges going forward. To extend this path to a Hamiltonian path, the remaining edges of length 1 are be added in a counterclockwise manner starting at the open end, i.e. at the vertex 0.

Finally, it is necessary to check that this construction will fit into K_p. Once again, the highest vertex visited is $(q+1)k+r-1$. The conditions that $n = qk+r \leq \frac{p-1}{2}$ and $k \leq \frac{p-1}{2}$ still hold by Proposition 3. It follows that $(q+1)k+r-1 = (qk+r)+k-1 \leq \frac{p-1}{2} + \frac{p-1}{2} - 1 = p-2$, completing the proof. ■

Figure 3 shows the construction in Lemma 5 for $p = 29$, $k = 5$, and $n = 12 = 2 * 5 + 2$.

Lemma 6 *If $n = qk+r$ with k odd, r odd and $q \geq 2$, then a Hamiltonian path with n edges of length k and $(p-1)-n$ edges of length 1 exists.*

Proof: Similar to what was done with the case where k is even, the construction for k odd and r even can be extended to k odd and r odd by adding one edge. As before, we do the construction for k odd and r even

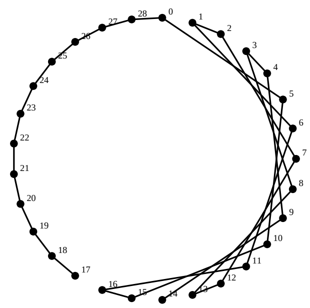

Figure 3: A Hamiltonian path with $p = 29, n = 12, k = 5$

using $s = r - 1$ as the remainder. This time the path ends at the vertex $qk + k - 1$. Add in the edge $(qk + k - 1, qk + 2k - 1)$. The vertices between $(q + 1)k + r - 1$ and $(q + 2)k - 1$ can be reached by paths of length 1 by starting at the open end of the path at $qk + 2k - 1$ and subtracting 1. The vertices between $qk + 2k - 1$ and 0 can be reached by paths of length 1 where 0 is the starting point and the edges proceed counterclockwise.

In this case, it is necessary to show that $qk + 2k - 1 \leq p - 1$. By Proposition 3, $qk + r \leq \frac{p-1}{2}$. Since $qk \leq qk + r$, it follows that $qk \leq \frac{p-1}{2}$. Now using the hypothesis that $q \geq 2$ we get that $qk + 2k - 1 \leq qk + qk - 1 \leq \frac{p-1}{2} + \frac{p-1}{2} - 1 \leq p - 1$, completing the proof. ■

Figure 4 shows the construction from Lemma 6 for $p = 29$, $n = 13$, and $k = 5$. Note that the constructed path is the same as in Figure 3 with the addition of the edge $(14, 19)$ and the subtraction of the edge $(19, 20)$.

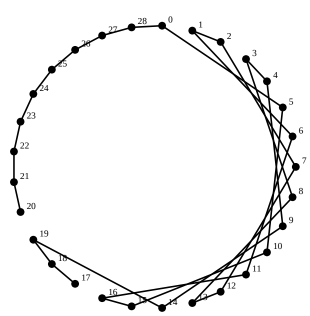

Figure 4: A Hamiltonian path with $p = 29, n = 13, k = 5$

In the case where $q = 1$, Lemma 6 will not work when $2k$ is greater than $\frac{p-1}{2}$. Our final lemma gives a construction for the $q = 1$ case.

Lemma 7 *If $n = qk + r$ with k odd, r odd and $q = 1$, then a Hamiltonian path with n edges of length k and $(p-1) - n$ edges of length 1 exists.*

Proof: Consider the following construction: Place an edge of length k from 0 to k. Instead of going forward 1 (as in the previous constructions), subtract 1 to get to the vertex $k-1$. At this point go forward k to the vertex $2k-1$. Add in two edges of length 1 by going forwards to $2k$ and $2k+1$. From here, the pattern is similar to previous constructions. Travel backwards and pick up the class of numbers congruent to 1 modulo k. Add 1 and travel forwards by lengths of k, which adds the vertices congruent to 2 modulo k to the path. Continue in this pattern until the vertex r is reached by traveling backwards from $2k+r$. This gives $2k+r$ as the largest vertex visited thus far. At this point use the following pattern: add 1, add k, add 1, subtract k. This continues until the vertex $k-2$ is reached by traveling backwards from $2k-2$. This path is summarized in the following list:

$$
\begin{array}{l}
0, k, \\
k-1, 2k-1, 2k, \\
2k+1, k+1, 1, \\
2, k+2, 2k+2, \\
\ddots \\
r-1, k+r-1, 2k+r-1, \\
2k+r, k+r, r, \\
r+1, k+r+1, \\
k+r+2, r+2, \\
\ddots \\
r-3, k+(r-3)k+(r-2), r-2
\end{array}
$$

This path indeed contains the correct number of edges of length k since every vertex between 0 and $k+r$ has an edge of length k traveling forward with the exception of vertex k. Also, it is easy to see that there are no isolated vertices between 0 and $k+r$. Lastly, the end at 0 is open, which enables us to pick up the vertices between $2k+r$ and 0 with edges of length 1.

The last thing to check is that the construction fits in K_p. By Proposition 3, $n = k + r \leq \frac{p-1}{2}$. Also $k \leq \frac{p-1}{2}$. Therefore, $2k + r = k + k + r \leq \frac{p-1}{2} + \frac{p-1}{2} = p - 1$, completing the proof. ■

Figure 5 shows the construction from Lemma 7 for $p = 23$, $k = 7$, and $n = 10 = 1 * 7 + 3$.

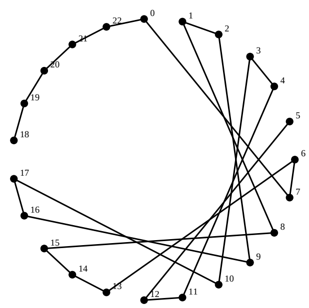

Figure 5: A Hamiltonian path with $p = 23, n = 10, k = 7$

4 Conclusion

We combine the constructions of the previous section to get the following theorem.

Theorem 8 *Let p a prime, $k \leq \frac{p-1}{2}$, and $n \leq \frac{p-1}{2}$, then there exists a Hamiltonian path in K_p containing n edges of length k and $(p-1)-n$ edges of length 1.*

Proof: This theorem follows directly from Lemmas 4, 5, 6, and 7. ■

From the reductions of Section 2 and Theorem 8 we now have a solution to Buratti's Conjecture for the case of two distinct edge lengths.

Theorem 9 *Given p a prime, $1 \leq n \leq p-1$ and any nonzero lengths a, b with $\{a, b\} \in \mathbb{Z}_p$, there exists a Hamiltonian path in K_p containing n edges of length a and $(p-1)-n$ edges of length b.*

Proof: By Theorem 8, there exists a Hamiltonian path in Z_p containing n edges of length k and $(p-1)-n$ edges of length 1. The existence of this path in conjunction with Proposition 3 give the existence of the desired path. ■

We note here that Theorem 4 was proven indenendently by Horak and Rosa in [2]. The interested reader is referred to that paper for further discussion of Buratti's problem.

The following corollary deals with the case of Hamiltonian paths in complete graphs of nonprime order.

Corollary 10 *Given $s \in \mathbb{N}$ and $n \leq \frac{s-1}{2}$, there exists a Hamiltonian path in K_s containing n edges of length k and $(s-1)-n$ edges of length 1.*

Proof: Examining the proofs of Lemmas 4, 5, 6, and 7, it can be observed that the condition of s being a prime was not used. Therefore, the constructions work for any $s \in \mathbb{N}$. $\blacksquare$

Our final corollary deals with Hamiltonian paths with edges of length a and b in K_s with s not prime.

Corollary 11 *Given $s \in \mathbb{N}$ and assume that a and b are nonzero lengths in $\mathbb{Z}_s$ with $\gcd(s, a) = 1$ and $\gcd(s, b) = 1$, then there exists a Hamiltonian path in K_s containing n edges of length a and $(s-1)-n$ edges of length b for all $0 \le n \le s-1$.*

Proof: The case of $n = 0$ or $n = p - 1$ is solved by considering the path consisting of the successive multiples of a (or b) in the cyclic group $\mathbb{Z}_s$. Now without loss of generality assume that $n \le \frac{s-1}{2}$. Since b is a unit in Z_s, the existence of a Hamiltonian path in K_s with n edges of length a and $(s-1)-n$ edges of length b is implied by the existence of a Hamiltonian path in K_s with n edges of length ab^{-1} and $(s-1)-n$ edges of length 1. The result now follows from Corollary 10. $\blacksquare$

A basic computer search conducted using *Mathematica* has shown Buratti's Conjecture is true for $p = 7$. The following is a table that contains every possible multiset S of lengths in Z_7 and a Hamiltonian path in K_7 whose edge lengths correspond to the values in the multi-set S.

S	Hamiltonian Path	S	Hamiltonian Path
$\{1,1,1,1,1,1\}$	$\{0,1,2,3,4,5,6\}$	$\{1,1,3,3,3,3\}$	$\{0,3,2,6,5,1,4\}$
$\{1,1,1,1,1,2\}$	$\{0,1,2,3,4,6,5\}$	$\{1,2,2,2,2,2\}$	$\{0,2,4,5,3,1,6\}$
$\{1,1,1,1,1,3\}$	$\{0,1,2,3,6,5,4\}$	$\{1,2,2,2,2,3\}$	$\{0,2,4,1,6,5,3\}$
$\{1,1,1,1,2,2\}$	$\{0,1,2,3,5,4,6\}$	$\{1,2,2,2,3,3\}$	$\{0,2,5,6,4,1,3\}$
$\{1,1,1,1,2,3\}$	$\{0,1,2,3,6,4,5\}$	$\{1,2,2,3,3,3\}$	$\{0,1,4,2,5,3,6\}$
$\{1,1,1,1,3,3\}$	$\{0,1,2,5,6,3,4\}$	$\{1,2,3,3,3,3\}$	$\{0,3,6,4,1,2,5\}$
$\{1,1,1,2,2,2\}$	$\{0,1,2,4,6,5,3\}$	$\{1,3,3,3,3,3\}$	$\{0,3,4,1,5,2,6\}$
$\{1,1,1,2,2,3\}$	$\{0,1,2,6,4,5,3\}$	$\{2,2,2,2,2,2\}$	$\{0,2,4,6,1,3,5\}$
$\{1,1,1,2,3,3\}$	$\{0,1,2,5,4,6,3\}$	$\{2,2,2,2,2,3\}$	$\{0,2,4,6,1,5,3\}$
$\{1,1,1,3,3,3\}$	$\{0,1,4,5,2,3,6\}$	$\{2,2,2,2,3,3\}$	$\{0,2,4,6,3,1,5\}$
$\{1,1,2,2,2,2\}$	$\{0,1,6,4,2,3,5\}$	$\{2,2,2,3,3,3\}$	$\{0,2,5,3,6,4,1\}$
$\{1,1,2,2,2,3\}$	$\{0,2,3,5,6,1,4\}$	$\{2,2,3,3,3,3\}$	$\{0,3,5,2,6,1,4\}$
$\{1,1,2,2,3,3\}$	$\{0,1,3,6,4,5,2\}$	$\{2,3,3,3,3,3\}$	$\{0,3,6,4,1,5,2\}$
$\{1,1,2,3,3,3\}$	$\{0,3,4,1,6,5,2\}$	$\{3,3,3,3,3,3\}$	$\{0,3,6,2,5,1,4\}$

The cases in K_8 and K_9 were also run on the computer. In the case of K_8, there are 120 possible multi-sets S and 105 of them are realizable as Hamiltonian paths. In the case of K_9, there are 165 multi-sets S with 161 of these having realizable Hamiltonian paths. The only four sets that do not have realizable paths are $\{1,3,3,3,3,3,3,3\}$, $\{2,3,3,3,3,3,3,3\}$, $\{3,3,3,3,3,3,3,3\}$ and $\{3,3,3,3,3,3,3,4\}$. Clearly these paths are not realizable as it is not possible to attain elements of more than 2 of the congruence classes modulo 3. It is interesting to note that these are the only multisets in $\mathbb{Z}_9$ with no realizable path.

References

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