Complementary Partial Resolution Squares for Steiner Triple Systems

J. H. Dinitz
Dept. of Mathematics and Statistics
University of Vermont
Burlington, VT 05405

E.R. Lamken
Dept. of Mathematics, 253–37
California Institute of Technology
Pasadena, CA 91125

A.C.H. Ling
Dept. of Mathematical Sciences
Michigan Technological University
Houghton, MI 49931–1295

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Abstract

In this paper, we introduce a generalization of frames called partial resolution squares. We are interested in constructing sets of complementary partial resolution squares for Steiner triple systems. Our main result is the existence of six complementary partial resolution squares for Steiner triple systems of order v which can be superimposed in a $v \times v$ array so that the resulting array is also the array formed by the superposition of three mutually orthogonal Latin squares of order v where $v \equiv 1 \pmod 6$, $v \geq 7$, and $v \notin \{55, 115, 145, 205, 235, 265, 319, 355, 415, 493, 649, 697\}$.

1 Introduction.

A balanced incomplete block design ($BIBD$) D is a collection B of subsets (blocks) taken from a finite set V of v elements with the following properties.

(1) Every pair of distinct elements from V is contained in precisely λ blocks of B.

(2) Every block contains exactly k elements.

We denote such a design as a $(v, k, \lambda) - BIBD$.

A $(v, k, \lambda) - BIBD$ D is said to be near resolvable if the blocks of D can be partitioned into classes (resolution classes) $R_1, R_2, \ldots, R_v$ such that for each element x of D there is precisely one class which does not contain x in any of its blocks and each class contains precisely $v - 1$ distinct elements of the design. The classes $R_1, R_2, \ldots, R_v$ form a resolution of D and D is denoted by $NR(v, k, \lambda) - BIBD$. Two necessary conditions for the existence of a $NR(v, k, \lambda) - BIBD$ are $v \equiv 1 \pmod k$ and $\lambda = k - 1$. Results on $NR(v, k, k-1) - BIBD$s can be found in [7] and [2].

Let R and R' be two resolutions of a $NR(v, k, \lambda) - BIBD$. R and R' are said to be orthogonal if $| R_i \cap R'_j | \leq 1$ for all $R_i \in R,\ R'_j \in R'$. (It should be noted that the blocks of the design are considered as being labeled so that if a subset of the elements occurs as a block more than once the blocks are treated as distinct.) If a $NR(v, k, \lambda) - BIBD$ has a pair of orthogonal resolutions, it is called doubly near resolvable and is denoted by $DNR(v, k, \lambda) - BIBD$. We can use a pair of orthogonal resolutions of a $DNR(v, k, \lambda) - BIBD$ to construct a $v \times v$ array. (For convenience, we often refer to this array as a $DNR(v, k, \lambda) - BIBD$.) We index the rows and columns of the array with the pair of orthogonal resolutions R and R'. In the cell labeled (R_i, R'_j), we place $R_i \cap R'_j$. If $R_i \cap R'_j = \emptyset$, the cell is left empty. The rows of the array will contain the resolution classes of the resolution R and the columns will contain the resolution classes of the orthogonal resolution R'. As an example, a $DNR(10, 3, 2) - BIBD$ is displayed in Figure 1.

		8,3,4	6,7,1				9,5,2		
			9,4,0	7,8,2				5,6,3	
8,9,3				5,0,1					6,7,4
6,1,2	9,5,4				7,8,0				
	7,2,3	5,6,0				8,9,1			
		7,9,1					6,8,4		3,2,0
			8,5,2		4,3,1			7,9,0	
				9,6,3		0,4,2			8,5,1
5,7,4					9,6,2		1,0,3		
	6,8,0					5,7,3		2,1,4	

Figure 1
A $(1, 2; 3, 10, 1)$-frame, [5].

If the $DNR(v, k, \lambda) - BIBD$ has the additional property that under an appropriate ordering of the resolution classes R and R', $R_i \cup R'_i$ contains precisely $v - 1$ distinct elements of the design and $R_i \cap R'_i = \emptyset$ for all i, then the array is called a $(1, \lambda; k, v, 1)$-frame. The diagonal of a $(1, \lambda; k, v, 1)$-frame is empty and a unique element of the design can be associated with each cell (i, i). The $DNR(10, 3, 2) - BIBD$ in Figure 1 is a $(1, 2; 3, 10, 1)$-frame. The element associated with cell (i, i) is i for $i = 0, 1, \ldots, 9$.

Let $\mathcal{F}$ be a set of t $(1, \lambda; k, v, 1)$-frames, $\mathcal{F} = \{F^1, F^2, \ldots, F^t\}$. Let F be the superposition of $F^1, F^2, \ldots,$ and F^t , $F = F^1 \circ F^2 \circ \ldots \circ F^t$. If the main diagonal of F is empty and each cell of F contains at most one block of size k, then $\mathcal{F}$ is called a set of t complementary $(1, \lambda; k, v, 1)$-frames.

For more general definitions and results on frames and sets of t complementary frames, we refer to [14, 9] and [6]. Frames and sets of complementary frames have been used extensively in constructions for doubly near resolvable and doubly resolvable balanced incomplete block designs, [9, 10, 11, 12, 15, 6]. Recently, a connection between sets of mutually orthogonal Latin squares and sets of complementary frames has been established in [13]. In particular, sets of k complementary frames can be constructed with the additional property that the blocks can be ordered to provide a set of mutually orthogonal Latin squares, [13]. These designs are useful in recursive constructions and some applications are described in [13]. In this paper, we are interested in an analogue of this result which uses s-partial resolution squares.

An s-partial parallel class P of a $(v, k, \lambda) - BIBD$ D is a set of s pairwise disjoint blocks. An s-partial resolution of D is a partition of the blocks of D

into s-partial parallel classes. The number of s-partial parallel classes in an s-partial resolution of D is b/s where b is the number of blocks in D. Note that if $s = (v-1)/k$ and $\lambda = k-1$ then the number of partial parallel classes is v and the design is a $NR(v, k, k-1) - BIBD$.

Two s-partial resolutions R and R' of D are said to be orthogonal if $| R_i \cap R'_j | \leq 1$ for all $R_i \in R,\ R'_j \in R'$. (It should be noted that the blocks of the design are considered as being labeled so that if a subset of the elements occurs as a block more than once the blocks are treated as distinct.) As in the case of resolutions, we can use a pair of orthogonal s-partial resolutions of a design to construct a square array. We index the rows and columns of the array with the pair of orthogonal s-partial resolutions R and R'. In the cell labeled (R_i, R'_j), we place $R_i \cap R'_j$. If $R_i \cap R'_j = \emptyset$, the cell is left empty. The resulting array is an s-partial resolution square for the design D. The rows of the array will contain the s-partial parallel classes of the resolution R and the columns will contain the s-partial parallel classes of the orthogonal resolution R'. Note that if $s = (v-1)/k$ and $\lambda = k-1$, then the s-partial resolution square is a $DNR(v, k, k-1) - BIBD$. In Figure 2, we display a 2-partial resolution square for a $(13, 3, 1) - BIBD$. The symbols are $\{0, 1, \ldots, 9, a, b, c\}$.

In this paper, we will deal exclusively with partial resolution squares for $(v, 3, 1) - BIBDs$ or Steiner triple systems of order v, $STS(v)$. We now restrict our definitions and results to this case, however we note that the ideas clearly extend to more general designs.

We first recall that the number of blocks in a $STS(v)$ is $v(v-1)/6$. So the number of s-partial parallel classes in an s-partial resolution is $v(v-1)/6s$. We are interested in the case when $s = (v-1)/6$. So $v \equiv 1 \pmod 6$ and there are v s-partial resolution classes each containing $(v-1)/6$ blocks. A pair of orthogonal $(v-1)/6$-partial resolutions of a $STS(v)$ can be used to construct a $v \times v$ array. For notational convenience, we call a $(v-1)/6$-partial resolution square for a $STS(v)$ a $PRsq(v)$. Suppose the $STS(v)$ is written on the symbols $\{0, 1, 2, \ldots, v-1\}$. The $PRsq(v)$ is called normalized if the symbol i does not occur in row i or column i for $i = 0, 1, 2, \ldots, v-1$ and the main diagonal of the square is empty. The $PRsq(13)$ in Figure 2 is normalized.

									2 6 5		1 3 9	
										3 7 6		2 4 a
3 5 b											4 8 7	
	4 6 c											5 9 8
6 a 9		5 7 0										
	7 b a		6 8 1									
		8 c b		7 9 2								
			9 0 c		8 a 3							
				a 1 0		9 b 4						
					b 2 1		a c 5					
						c 3 2		b 0 6				
							0 4 3		c 1 7			
								1 5 4		0 2 8		

Figure 2
A normalized $PRsq(13)$.

The analogue of a set of t complementary frames is a set of t complementary normalized $PRsq(v)$. As in the case of frames, we want to place several partial resolution squares in a single array. Let S_i denote a normalized $PRsq(v)$ for $i = 1, 2, \ldots, t$, and let $\mathcal{S} = \{S_1, S_2, \ldots, S_t\}$. Let S be the superposition of $S_1, S_2, \ldots$, and S_t, $S = S_1 \circ S_2 \circ \ldots \circ S_t$. S is a $v \times v$ array and the main diagonal of S is empty. If each cell of S contains at most one block, then $\mathcal{S}$ is called a set of t complementary $PRsq(v)$. It is easy to see that $1 \leq t \leq 6$.

A set of t complementary (normalized) $PRsq(v)$ is called balanced if each element of $V - \{i\}$ occurs m times in row i and each element of $V - \{i\}$ occurs m times in column i of S. (We note that the array S constructed from a set of $2m$ balanced complementary $PRsq(v)$ is an $(m, 2m; 3, v, 1)$-frame and contains the blocks of a $(v, 3, 2m) - BIBD$, [14, 9].)

In the remainder of this paper, we will restrict our attention to sets of six balanced complementary $PRsq(v)$. For notational convenience, we denote a set of six balanced complementary $PRsq(v)$ and the resulting array by $BCPRsq(v)$. In this case, the array is completely filled except for the diagonal and element i does not occur in row i or column i for all i. (The underlying design is a $(v, 3, 6) - BIBD$ which can be decomposed into six $(v, 3, 1) - BIBDs$.) We need one further definition before describing our results.

Suppose that it is possible to order the symbols in the filled cells of a $BCPRsq(v)$ in such a way that a square consisting of the ith element of each cell ($i = 1, 2, 3$) is a latin square L_i (when an idempotent main diagonal is added), and suppose that L_1, L_2, and L_3 form a set of three mutually orthogonal latin squares, then the $BCPRsq(v)$ is called a mutually orthogonal Latin square ordered $BCPRsq(v)$ or a MOLS-ordered $BCPRsq(v)$. (Definitions and results on mutually orthogonal Latin squares can be found in [1] or the texts [3] or [4].)

An example of a MOLS-ordered $BCPRsq$ appears in Figure 3. This square, a $BCPRsq(7)$, was constructed by superimposing 6 complementary normalized $PRsq(7)$s. The filled cells of the ith of these squares can be seen below in the i^{th} (extended) diagonal. Note that all off-diagonal cells are filled, that the i^{th} row (column) contains every symbol except i exactly 3 times (once as the first symbol, once as the second and once as the third), that the three squares comprised only of either the 1^{st}, the 2^{nd} or the 3^{rd} symbol in each cell are three idempotent MOLS(7) (when the main diagonal is filled in appropriately), and that the set of all filled cells is a $(7, 3, 6) - BIBD$.

	653	536	412	365	241	124
235		064	640	523	406	352
463	346		105	051	634	510
621	504	450		216	162	045
156	032	615	561		320	203
314	260	143	026	602		431
542	425	301	254	130	013	

Figure 3
A MOLS-ordered $BCPRsq(7)$.

A cyclic Steiner triple system of order v contains exactly $(v-1)/6 = s$ base blocks. If the base blocks are disjoint, then clearly a $(v-1)/6$-partial resolution can be constructed cyclically from these base blocks. The set $\{a_1, \ldots a_s\}$ is an adder for a set of base blocks $\{B_1, \ldots B_s\}$ if $\{B_1 + a_1, \ldots B_s + a_s\}$ is also a set of disjoint blocks of the design. Note that the cyclic translates of the disjoint base blocks and the cyclic translates of the base blocks plus adders are a pair of orthogonal partial resolutions and yield a $PRsq(v)$. Also note that when a $PRsq(v)$ is formed in this manner the s-transversal of filled cells can be decomposed into s disjoint transversals each consisting of the translates of a starter block. The $PRsq(13)$ in Figure 2 was constructed in this manner with the set of base blocks $\{\{2,6,5\},\{1,3,9\}\}$ and the adder $\{4,2\}$.

In the next section, we describe a direct construction for $BCPRsq(v)$ which uses this type of structure. This construction is used together with recursive constructions in section 3 to prove our main result on MOLS-ordered $BCPRsq(v)$ for $v \equiv 1 \pmod 6$.

2 Direct Construction.

In this section we give a direct construction using finite fields. Our general construction is the following.

Lemma 2.1 *Let g be a generator of the multiplicative group of the field F_q where $q = 6t+1$ is a prime power. Let ind(s) be the index of the element $s \in F_q$ (i.e. if $k = ind(s)$, then $g^k = s$). If there exists an x such that the following conditions are satisfied:*

1. $|ind(g^{2t}+g^x) - ind(1+g^x)| \geq t$
2. $|ind(g^{2t}+g^x) - ind(g^{4t}+g^x)| \geq t$
3. $|ind(g^{4t}+g^x) - ind(1+g^x)| \geq t$

then there exists a MOLS-ordered $BCPRsq(q)$.

Proof: From the Bose construction, it is well known that the starter blocks $T = \{\{g^i, g^{2t+i}, g^{4t+i}\}, 0 \leq i \leq t-1\}$ can be used to generate an STS(q). Clearly, $g^{jt}T = T_j$ for $0 \leq j \leq 5$ yields six sets of starter blocks of (isomorphic) STS(q)s. It is our intention to pack these 6 STSs in a square of size q by q. For our purposes, it is convenient to order each block in these triple systems. In particular, we order each block in T as $\{(g^i, g^{2t+i}, g^{4t+i}), 0 \leq i \leq t-1\}$, and then we extend this ordering to each T_j in the obvious way. Note that for all $v-1$ ordered starter blocks, the blocks are uniquely identified by the first element in the ordered block. Define $Q_k = (g^k, g^{k+2t}, g^{k+4t})$ for each $k \in Z_{6t}$. We let the corresponding adder for Q_k be g^{k+x}. The q by q square R is indexed by the elements in F_q. In the first row of the square,

Q_k is placed in column $-g^{k+x}$ for all $k \in Z_{q-1}$. The remaining square is constructed by developing the first row using F_q (i.e. if $R(a,b) = (u,v,w)$, then cell $R(a+h, b+h) = (u+h, v+h, w+h)$ for all $h \in F_q$).

We must check that this construction satisfies the conditions for a MOLS-ordered $BCPRsq(q)$. First we must check that the square R is well-defined, i.e. that there is exactly one triple in every off-diagonal cell. This is true since the adders are all distinct and consist of every nonzero element of F_q. Hence the square R does indeed contain 6 distinct STS(q) in the union of all the cells.

Next we check the row and column balance. Row balance is obvious since each row is a translate of the first row and it is easy to see that each non-zero element of F_q occurs exactly three times in that row. Further note that each symbol occurs exactly once in each position in the ordered triples of the first row and hence of each row. Now we check the columns. The first column contains all of the ordered triples $Q_k + g^{k+x} = (g^k, g^{k+2t}, g^{k+4t}) + g^{k+x}$ for all $k \in Z_{6t}$. We see that the set of first symbols occurring in the triples of the first column are $\{g^k + g^{k+x} | k \in Z_{6t}\} = \{g^k(1+g^x) | k \in Z_{6t}\}$. Clearly this consists of every non-zero element of F_q exactly once. In a similar manner we see that the set of elements in the second positions and the set of elements in the third position in the first column also contain each symbol of F_q exactly once. Hence by construction, each column of R has this property.

Next we show that the three latin squares formed from the first, second and third symbols in each cell are indeed orthogonal (latin was proved in the paragraph above). Consider the first two: call them L_1 and L_2. In the first row of the superposition of L_1 and L_2 are the pairs (g^k, g^{k+2t}) for all $k \in Z_{6t}$. The (ordered) differences between the first element and the second element is $g^{k+2t} - g^k = g^k(g^{2t} - 1)$. Clearly as k goes through all the elements of Z_{6t}, every difference occurs exactly once. Hence when this first row is developed in the additive group of the field, every pair will occur exactly once (except the pairs (i,i) which are assumed to be on the diagonal). Hence L_1 and L_2 are mutually orthogonal. That L_1 and L_3, and L_2 and L_3 are orthogonal can be shown similarly.

Finally, we must check that each column of R contains 6 partial parallel classes. We will show that the translates of each partial parallel class in the first row form a partial parallel class of the first column (i.e. no symbol occurs twice in the translates). Consider the partial parallel class $T_j = \{\{g^{i+jt}, g^{(2+j)t+i}, g^{(4+j)t+i}\}, 0 \le i \le t-1\}$ for $0 \le j \le 5$. We have previously established that all the symbols occurring the first position are distinct, next we show that no symbol in the first position of a translate can equal a symbol in the second position of a translate. Assume equality, then for some s, k with $|s-k| < t$ we must have $g^{s+jt} + g^{s+jt+x} = g^{k+(2+j)t} + g^{k+jt+x}$. Hence, $g^{s+jt}(1+g^x) = g^{k+jt}(g^{2t}+g^x)$, which implies $g^s(1+g^x) = g^k(g^{2t}+g^x)$. Now the first condition in the hypothesis of this lemma, namely that $|\mathrm{ind}(g^{2t} + g^x) - \mathrm{ind}(1+g^x)| \geq t$ gives the required contradiction. In a similar manner, the other two conditions of the hypothesis insure that the first and second and the second and third symbols in the translates are distinct. Hence the first column consists of six partial parallel classes. Now, by construction we have that every column of R contains 6 partial parallel classes, completing the proof. □

We illustrate this construction for $v = 13$: the element $x = 1$ satisfies the hypothesis of Lemma 2.1 and $g = 2$ is a generator of F_{13}. Table 1 gives six sets of starter blocks as well as the column in the first row where each block is placed.

	starter blocks	column
$T =$	$(g^0, g^4, g^8) = (1, 3, 9)$	$-2 = -g^{0+1}$
	$(g^1, g^5, g^9) = (2, 6, 5)$	$-4 = -g^{1+1}$
$g^2T =$	$(g^2, g^6, g^{10}) = (4, 12, 10)$	$-8 = -g^{2+1}$
	$(g^3, g^7, g^{11}) = (8, 11, 7)$	$-3 = -g^{3+1}$
$g^4T =$	$(g^4, g^8, g^0) = (3, 9, 1)$	$-6 = -g^{4+1}$
	$(g^5, g^9, g^1) = (6, 5, 2)$	$-12 = -g^{5+1}$
$g^6T =$	$(g^6, g^{10}, g^2) = (12, 10, 4)$	$-11 = -g^{6+1}$
	$(g^7, g^{11}, g^3) = (11, 7, 8)$	$-9 = -g^{7+1}$
$g^8T =$	$(g^8, g^0, g^4) = (9, 1, 3)$	$-5 = -g^{8+1}$
	$(g^9, g^1, g^5) = (5, 2, 6)$	$-10 = -g^{9+1}$
$g^{10}T =$	$(g^{10}, g^2, g^6) = (10, 4, 12)$	$-7 = -g^{10+1}$
	$(g^{11}, g^3, g^7) = (7, 8, 11)$	$-1 = -g^{11+1}$

Table 1
Starter blocks and adders for a $BCPRsq(13)$.

Note that the $BCPRsq(13)$ in Figure 2 is constructed from T in Table 1. The square in Figure 4 is a $BCPRsq(13)$ constructed using the starter blocks and adders listed in Table 1. (For typesetting purposes we substituted the symbol a for 10, b for 11 and c for 12).

	652	ca4	526	b78	4ca	a4c	391	913	265	8b7	139	78b
89c		763	0b5	637	c89	50b	b50	4a2	a24	376	9c8	24a
35b	9a0		874	1c6	748	09a	61c	c61	5b3	b35	487	a09
b1a	46c	ab1		985	207	859	1ab	720	072	6c4	c46	598
6a9	c2b	570	bc2		a96	318	96a	2bc	831	183	705	057
168	7ba	03c	681	c03		ba7	429	a7b	3c0	942	294	816
927	279	8cb	140	792	014		cb8	53a	b8c	401	a53	3a5
4b6	a38	38a	90c	251	8a3	125		0c9	64b	c90	512	b64
c75	5c7	b49	49b	a10	362	9b4	236		10a	75c	0a1	623
734	086	608	c5a	5ac	b21	473	ac5	347		21b	860	1b2
2c3	845	197	719	06b	6b0	c32	584	b06	458		32c	971
a82	304	956	2a8	82a	17c	7c1	043	695	c17	569		430
541	b93	415	a67	3b9	93b	280	802	154	7a6	028	67a	

Figure 4
A MOLS-ordered $BCPRsq(13)$.

Lemma 2.2 *For every prime power* $q \equiv 1 \pmod 6$, $7 \leq q \leq 5077$ *and for* $q = 5779, 5827, 8053$ *and 11827, there exists an* x *satisfying conditions 1,2, and 3 of Lemma 2.1.*

Proof: The values for x are given in Appendix 1. For $q = p^n$ with $n > 1$, a table giving the irreducible polynomial and a generator for GF(q) is in Appendix 2. □

¿From the two lemmas above we have the following theorem.

Theorem 2.3 *For every prime power* q *with* $q \equiv 1 \pmod 6$, $7 \leq q \leq 5077$ *and for* $q = 5779, 5827, 8053$ *and 11827, there exists a MOLS-ordered* $BCPRsq(q)$.

3 Recursive Constructions and Spectrum.

In this section we describe some recursive constructions which when combined with the results from the previous section will give us the spectrum of MOLS-ordered $BCPRsq$s. We begin with a PBD-closure result. (For definitions and results on PBD-closure, we refer to [17].)

Lemma 3.1 *If there exists a pairwise balanced design of order v with block sizes from the set K (a PBD(v, K)) and if there exists a MOLS-ordered $BCPRsq(k)$ for all $k \in K$, then there exists a MOLS-ordered $BCPRsq(v)$. Equivalently, the set $S = \{v \mid$ there exists a MOLS-ordered $BCPRsq(v)\}$ is PBD-closed.*

Proof: This construction is the standard PBD construction for sets of mutually orthogonal Latin squares and block designs. Since a MOLS-ordered $BCPRsq(k)$ consists of three superimposed idempotent MOLS(k), it follows that the resulting $v \times v$ array, S, is also MOLS-ordered. It also follows that in S all off-diagonal cells are filled and that it is row and column balanced. It remains to show only that S is the array formed by the superposition of six complementary $PRsq(v)$.

For every block of size k ($k \in K$) of the pairwise balanced design there exists a $BCPRsq(k)$, and each of these arrays consists of a set of six complementary $PRsq(k)$s. We use the blocks of the ith partial resolution square for each of the blocks in the PBD to construct a $PRsq(v)$, S_i, for $i = 1, 2, \ldots, 6$. So it is straightforward to check that S is the superposition of $S_1, S_2, \ldots, S_6$. □

We now use a PBD closure result from Mullin and Stinson [16] and Greig [8]. Let $Q_1 = \{q : q \equiv 1 \pmod 6, q \leq 5077,\ q \text{ a prime power }\}$, let $P = \{55, 115, 145, 205, 235, 265, 319, 355, 391, 415, 445, 451, 493, 649, 667, 685, 697, 745, 781, 799, 805, 1315\}$ and let $E = \{5779, 5827, 8053, 11827\}$.

Theorem 3.2 *If $v \geq 7$ and $v \notin P$, then there exists a PBD$(v, Q_1 \cup E)$.*

Proof: The proof is implicitly contained in [16]. We will give an outline of the proof here but we refer to [16] for the details.

Lemma 4.1 of [16] (with some additional PBD's found in [8]) shows that if $n \equiv 1 \pmod 6$, $n \leq 5071$, and $n \notin P$, then there exists a PBD(n, Q_1).

In text following the proof of Lemma 5.1 in [16], the authors assert that for every $n \equiv 1 \pmod 6$ between 5077 and 46357 (except $n \in E$), there exists a PBD(n, Q_1) . Finally, Lemma 5.1 in [16] proves by induction that if there exists a PBD$(n, Q_1 \cup E)$ for all $1927 \leq n \leq 46357$ with $n \equiv 1 \pmod 6$, then there exists a PBD$(n, Q_1 \cup E)$ for all $n \equiv 1 \pmod 6$ with $n \geq 1927$. The result follows. □

¿From Theorems 2.3 and 3.2 and Lemma 3.1 we have the following.

Theorem 3.3 *If $v \geq 7$ with $v \equiv 1 \pmod 6$ and if $v \notin P$, then there exists a MOLS-ordered $BCPRsq(v)$.*

We will now construct MOLS-ordered $BCPRsq$s for some of the values missing from the theorem above. We first note that the singular direct product holds for normalized partial resolution squares and for sets of t complementary partial resolution squares. Since these constructions are straightforward generalizations of the singular direct product constructions for frames and sets of t complementary frames, we omit the proofs and refer to [14, 9]. (The decomposition into six complementary $PRsq$s follows from the decomposition of the base design, a $BCPRsq(u)$, and the subdesigns, $BCPRsq(v)$.) We state the construction for the case of interest to us.

Theorem 3.4 *If there is a $BCPRsq(u)$, a $BCPRsq(v)$, and three mutually orthogonal Latin squares of order $v-1$, then there is a $BCPRsq(u(v-1)+1)$.*

Corollary 3.5 *If there is a MOLS-ordered $BCPRsq(u)$, a MOLS-ordered $BCPRsq(v)$, and three mutually orthogonal Latin squares of order $v-1$, then there is a MOLS-ordered $BCPRsq(u(v-1)+1)$.*

Proof: This follows immediately since the singular direct product also holds for sets of mutually orthogonal Latin squares.□

Lemma 3.6 *For $w \in \{391, 445, 451, 667, 685, 745, 781, 799, 805, 1315\}$ there exists a MOLS-ordered $BCPRsq(w)$*

Proof: Each of these is an application of Corollary 3.5. For each w, Table 2 gives the appropriate value of u and v so that the corollary can be applied. □

w	u	v	w	u	v
391	13	31	445	37	13
451	25	19	667	37	19
685	19	37	745	31	25
781	13	61	799	19	43
805	67	13	1315	73	19

Table 2
Parameters for applications of Corollary 3.5.

¿From Theorem 3.3 and Lemma 3.6 we have the following result.

Theorem 3.7 *For every* $n \equiv 1$ *(mod 6),* $n \geq 7$*, except possibly for* $n \in \{55, 115, 145, 205, 235, 265, 319, 355, 415, 493, 649, 697\}$ *there exists a MOLS-ordered* $BCPRsq(n)$*.*

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Appendix 1

g	x	q	g	x	q	g	x	q	g	x	q	g	x	q
3	0	7	2	1	13	2	1	19	3	2	31	2	2	37
3	0	43	2	1	61	2	2	67	5	5	73	3	4	79
5	0	97	5	0	103	6	0	109	3	0	127	2	1	139
6	0	151	5	2	157	2	2	163	2	1	181	5	4	193
3	0	199	2	3	211	3	12	223	6	6	229	7	6	241
6	0	271	5	0	277	3	0	283	5	2	307	10	1	313
3	0	331	10	2	337	2	1	349	6	1	367	2	3	373
2	3	379	5	0	397	21	1	409	2	2	421	5	4	433
15	7	439	13	0	457	3	7	463	3	0	487	7	5	499
2	1	523	2	1	541	2	8	547	3	3	571	5	1	577
7	3	601	3	3	607	2	1	613	2	1	619	3	2	631
11	1	643	2	1	661	5	1	673	3	6	691	2	3	709
5	1	727	6	0	733	3	0	739	3	0	751	2	2	757
11	0	769	2	2	787	3	9	811	3	0	823	2	3	829
2	7	853	2	6	859	2	1	877	2	1	883	2	1	907
7	3	919	5	1	937	5	0	967	6	1	991	7	5	997
11	4	1009	10	3	1021	5	0	1033	3	1	1039	7	1	1051
3	2	1063	6	1	1069	3	8	1087	5	0	1093	2	10	1117
2	1	1123	11	4	1129	5	2	1153	2	5	1171	11	2	1201
2	2	1213	3	0	1231	2	1	1237	7	2	1249	3	7	1279
2	3	1291	10	0	1297	6	1	1303	13	10	1321	3	0	1327
2	2	1381	13	8	1399	3	4	1423	6	0	1429	3	6	1447
2	1	1453	3	0	1459	6	1	1471	2	1	1483	14	13	1489
2	1	1531	5	15	1543	2	5	1549	3	2	1567	3	0	1579
11	6	1597	7	0	1609	2	1	1621	3	0	1627	11	13	1657
3	2	1663	2	7	1669	2	1	1693	3	0	1699	3	3	1723
2	1	1741	2	4	1747	7	1	1753	6	2	1759	5	0	1777
10	0	1783	6	0	1789	11	3	1801	3	4	1831	2	3	1861
2	7	1867	10	0	1873	6	0	1879	5	2	1933	3	3	1951
2	2	1987	5	0	1993	3	3	1999	3	5	2011	5	3	2017
2	3	2029	2	17	2053	2	1	2083	7	2	2089	5	0	2113
2	6	2131	10	5	2137	3	3	2143	23	0	2161	7	1	2179
5	0	2203	2	1	2221	3	2	2239	7	0	2251	2	1	2269
7	0	2281	19	0	2287	2	1	2293	3	1	2311	7	0	2341

g	x	q	g	x	q	g	x	q	g	x	q	g	x	q
3	3	2347	2	7	2371	5	1	2377	5	4	2383	2	5	2389
2	4	2437	2	3	2467	5	0	2473	3	1	2503	17	2	2521
2	2	2539	6	6	2551	2	2	2557	7	0	2593	5	1	2617
3	2	2647	2	3	2659	7	1	2671	2	1	2677	2	4	2683
19	1	2689	2	1	2707	5	0	2713	3	6	2719	3	9	2731
6	1	2749	3	0	2767	6	2	2791	2	5	2797	2	6	2803
5	2	2833	2	1	2851	11	0	2857	5	2	2887	5	9	2917
13	0	2953	10	0	2971	14	11	3001	2	1	3019	2	7	3037
11	8	3049	6	0	3061	2	1	3067	6	1	3079	6	1	3109
7	0	3121	3	0	3163	7	3	3169	7	2	3181	2	1	3187
5	0	3217	6	1	3229	2	4	3253	3	5	3259	3	0	3271
6	3	3301	2	1	3307	10	7	3313	6	0	3319	3	0	3331
5	9	3343	22	5	3361	5	3	3373	3	0	3391	5	0	3433
7	0	3457	3	0	3463	2	1	3469	2	1	3499	7	4	3511
2	7	3517	17	0	3529	7	1	3541	2	1	3547	3	4	3559
2	1	3571	3	3	3583	5	8	3607	2	1	3613	15	12	3631
2	6	3637	2	2	3643	5	3	3673	2	2	3691	5	0	3697
2	10	3709	3	8	3727	2	5	3733	7	0	3739	7	1	3769
5	8	3793	3	1	3823	5	1	3847	2	1	3853	2	1	3877
11	3	3889	2	17	3907	3	8	3919	2	1	3931	3	3	3943
6	5	3967	2	5	4003	2	3	4021	3	9	4027	10	4	4051
5	0	4057	2	1	4093	2	1	4099	12	11	4111	13	4	4129
5	8	4153	3	0	4159	5	0	4177	11	7	4201	2	1	4219
3	0	4231	2	1	4243	2	20	4261	5	1	4273	5	3	4297
3	1	4327	10	7	4339	2	6	4357	2	5	4363	3	1	4423
21	0	4441	3	3	4447	2	2	4483	2	1	4507	7	1	4513
3	2	4519	6	1	4549	11	5	4561	3	2	4567	11	6	4591
5	8	4597	2	1	4603	2	3	4621	3	1	4639	3	3	4651
15	0	4657	3	0	4663	2	6	4723	17	1	4729	3	0	4759
6	0	4783	2	1	4789	7	8	4801	2	1	4813	3	0	4831
11	0	4861	3	1	4903	6	2	4909	2	1	4933	6	1	4951
2	1	4957	11	2	4969	2	6	4987	5	0	4993	3	0	4999
2	9	5011	3	3	5023	2	3	5059	2	3	5779	2	2	5827
2	4	8053	2	7	11827									

Appendix 2

Irreducible Polynomial	Generator	x	q
$x^2 + x + 2$	$2x$	7	25
$x^2 + 8x + 10$	$5x + 4$	1	121
$x^2 + 5x + 9$	$8x + 16$	0	289
$x^2 + 14x + 5$	$20x + 18$	0	529
$x^2 + 23x + 26$	$27x + 2$	1	841
$x^2 + 12x + 1$	$40x + 38$	0	1681
$x^2 + 38x + 10$	$39x + 16$	0	2209
$x^2 + 22x + 34$	$31x + 40$	0	2809
$x^2 + 12x + 53$	$52x + 43$	5	3481
$x^2 + 24x + 12$	$18x + 48$	2	5041