Complete Arcs in Steiner Triple Systems

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Abstract

A complete arc in a design is a set of elements which contains no block, and is maximal with respect to this property. The spectrum of sizes of complete arcs in Steiner triple systems is determined without exception here.

1 Definitions

A *Steiner triple system* of order v (briefly STS(v)) is a pair $(X, \mathcal{B})$ where X is a v-element set and $\mathcal{B}$ is a collection of 3-subsets of X (*triples*), such that every pair of X is contained in exactly one triple of $\mathcal{B}$. It is well known that a necessary and sufficient condition for a STS(v) to exist is that $v \equiv 1$ or $3 \pmod 6$.

A subset S of the set X of elements of the STS is *independent*, or an *arc*, if $|B \cap S| \leq 2$ for each $B \in \mathcal{B}$. For a set S of elements, we say that an element x of $X \setminus S$ is *spanned* by a pair $\{s,t\} \in S$ if $\{s,t,x\} \in \mathcal{B}$. A set S is *spanning* if every $x \in X \setminus S$ is spanned by a pair in S. A subset which is both independent and spanning is precisely a *maximal independent set*, or *complete arc*. When $s = |S|$, the term *complete s-arc* is employed.

A subset S of elements is *scattering* if it is independent and, for any two blocks $B, C \in \mathcal{B}$ with $|B \cap S| = |C \cap S| = 2$, we find $B \setminus S \neq C \setminus S$. In other words, all elements spanned by pairs in S are distinct. Now let

$$L(v) = \left\lfloor \tfrac{1}{2}(\sqrt{8v+1} - 1) \right\rfloor$$

$$U(v) = \left\lceil \tfrac{1}{2}(\sqrt{8v+1} - 1) \right\rceil$$

$$T(v) = \begin{cases} \frac{v+1}{2} & \text{if} \quad v \equiv 3,7 \pmod{12} \\ \frac{v-1}{2} & \text{if} \quad v \equiv 1,9 \pmod{12} \end{cases}$$

Sauer and Schönheim [11] establish that $T(v)$ is the largest size of an independent set in an STS(v); Bigelow and Colbourn [1] later generalized this to triple systems of higher index. Colbourn, Dinitz, and Stinson [4] establish that $L(v)$ is the largest size of a scattering set, and that $U(v)$ is the smallest size of a spanning set, in an STS(v).

An independent set of maximum size is necessarily maximal, and hence the result of Sauer and Schönheim gives complete arcs of size $T(v)$. A spanning set of minimum size need not, in general, be independent. However, the technique of Colbourn, Dinitz, and Stinson produces spanning, independent sets – and hence complete arcs – of size $U(v)$. Evidently, the size s of a complete arc in an STS(v) satisfies $U(v) \leq s \leq T(v)$. We have now seen that known results establish that the two extreme values can be realized. In this paper, we establish that the spectrum of sizes of complete arcs in STS(v)s is precisely the interval $[U(v), T(v)]$.

There is an extensive literature on independent sets and complete arcs in triple systems. One example is the study of partitions into complete arcs of minimum cardinality [5, 8]. However, previous work focuses primarily on the question of Erdös and Hajnal [6] asking what the minimum over all STS(v)s of the size of a maximum independent set in the STS. There exist STS(v)s in which the largest independent set has size at most $c\sqrt{v}$, where c is an absolute constant [2, 3, 7, 9]. In addition, there are STS(v)s having no spanning set of size $U(v)$ [4]. Hence it is surely not the case that every STS(v) has complete arcs of each size s satisfying $U(v) \leq s \leq T(v)$; rather, for every such s, we establish that there exists an STS(v) with a complete s-arc.

We use one construction repeatedly, the *singular direct product* for STSs. We outline it here. A *subsystem* of order w in an STS(v) $(V, \mathcal{B})$ is a set $W \subseteq V$ where every triple of $\mathcal{B}$ meets W in 0, 1, or 3 elements. Given an STS(v) with a sub-STS(w), the singular direct product produces an STS($3(v - w) + w$) having subsystems of order v and w (and perhaps others). The construction is as follows. Let $X = V \setminus W$. The STS to be formed has elements $(X \times \{1, 2, 3\}) \cup W$, writing x_i for $(x, i) \in X \times \{1, 2, 3\}$. Take a latin square L of side $v - w$ with rows, columns, and symbols indexed by the elements of X. When $L(a, b) = c$, form a triple $\{a_1, b_2, c_3\}$ (these are *latin square triples*). Now, for $i = 1, 2, 3$, place a copy of the STS(v) on $(X \times \{i\}) \cup W$, so that the subsystem of order w appears on W (these are *i-subsystem triples*). Suppress the second and third copy of the triples on W. The result is an STS($3(v - w) + w$) with a sub-STS(w) on W, and sub-STS(v)s on $(X \times \{i\}) \cup W$ for $i = 1, 2, 3$. In addition, if the STS(v) had other subsystems, these appear again in the STS($3(v - w) + w$). For convenience, we always take $V = \{1, \ldots, v\}$ and $W = \{v - w + 1, \ldots, v\}$.

Since the effective application of the singular direct product requires the presence of appropriate subsystems, it is important to limit the size of the subsystems that we require. If $3(v - w) + w$, v, and w are all to be admissible orders for an STS, and we want to write every admissible order in the form $3(v - w) + w$, we note that $w = 0$ works for $v \equiv 3, 9 \pmod{18}$, $w = 1$ works for $v \equiv 1, 7 \pmod{18}$, and $w = 3$ works for $v \equiv 15 \pmod{18}$. But an STS(v) always has subsystems of order 0, 1, and 3 when $v \geq 3$, and hence one can always apply singular direct product with $w \in \{0, 1, 3\}$. Unfortunately, one congruence class remains: $v \equiv 13 \pmod{18}$. In this case, $w = 7$ works; however, in an inductive strategy it is then necessary to ensure that the STSs constructed have sub-STS(7)s.

We therefore prove the following stronger characterization of the spectrum for complete arcs:

Theorem 1.1 *If $x \equiv 1, 3 \pmod{6}$ and $U(x) \leq s \leq T(x)$, there exists an STS(x) with a complete s-arc. When $x = 7$ or $x \geq 15$, this STS(x) has a sub-STS(7) and the sub-STS(7)* *contains at least two elements of the complete arc. When $x \geq 15$, for the complete arc and sub-STS(7) chosen, there is an element of the sub-STS(7) that is spanned by a pair within the complete arc but not within the sub-STS(7).*

The remainder of this paper proves the theorem stated, proceeding inductively on v. We tabulate first the minima and maxima for some small orders.

v	$U(v)$	$T(v)$	v	$U(v)$	$T(v)$	v	$U(v)$	$T(v)$	v	$U(v)$	$T(v)$
3	2	2	7	4	4	9	4	4	13	5	6
15	5	8	19	6	10	21	6	10	25	7	12
27	7	14	31	8	16	33	8	16	37	9	18

The main theorem holds for $x \in \{3, 7, 9, 13\}$, since in these cases there are no sizes strictly between the minimum and the maximum.

2 The middle range

We apply the singular direct product to obtaining different sizes of complete arcs by using both the flexibility in selecting the latin square to be used, and in selecting the ingredient STSs. To produce a number of useful latin squares, we employ the following theorem of Ryser [10]:

Theorem 2.1 *Let L be a partial latin square of order n in which cell (i, j) is filled if and only if $i \leq r$ and $j \leq s$. Then L can be completed to a latin square of order n if and only if $N(x) \geq r + s - n$ for $x = 1, 2, \ldots, n$, where $N(x)$ denotes the number of elements in L that are equal to x.*

Corollary 2.2 *Let n be an integer and $m \leq \frac{n}{2}$. Let σ be an integer satisfying $m \leq \sigma \leq$ min(m^2, n). There exists a latin square of order n in which, in the subarray in rows $1, \ldots, m$ and columns $1, \ldots, m$, the set of distinct symbols that appear is precisely $\{n - \sigma + 1, \ldots, n\}$.*

Proof: Form a latin square of side m on symbols $n - m + 1, \ldots, n$. Now for $i = n - \sigma + 1, \ldots, n - m$, select arbitrarily some symbol occurring two or more times in the square, and replace one of the occurrences by the symbol i. This can be done since $\sigma \leq m^2$, and the resulting partial $m \times m$ square has exactly $\alpha \leq n$ symbols in it. Apply Theorem 2.1, trivially since $r + s - n \leq 0$, to obtain the conclusion. $\square$

Theorem 2.3 *Let $x \equiv 1, 3 \pmod 6$. Let s be an integer satisfying*

(a) $2U(\frac{x}{3}) \leq s \leq \frac{x}{3} - 1 + T(\frac{x}{3})$ *if* $x \equiv 3, 9 \pmod{18}$
(b) $2U(\frac{x+2}{3}) - 1 \leq s \leq \frac{x-1}{2}$ *if* $x \equiv 1, 7 \pmod{18}$
(c) $2U(\frac{x+6}{3}) - 1 \leq s \leq \frac{x-1}{2}$ *if* $x \equiv 15 \pmod{18}$
(d) $2U(\frac{x+14}{3}) - 2 \leq s \leq \frac{x-9}{2}$ *if* $x \equiv 13 \pmod{18}$

Then there is an STS(x) having a complete s-arc, and when $x = 7$ or $x \geq 15$ also having a sub-STS(7) containing at least two elements of the arc.

Proof: **Case (a):** $x \equiv 3, 9 \pmod{18}$. Set $v = x/3$ and $w = 0$. When $s \geq 3U(v)$, write $s = 2q+\beta$ for $\beta \in \{q-1, q, q+1\}$; otherwise, write $q = U(v)$ and $s = 2q+\beta$. Now $q < v/2$, so apply Corollary 2.2 with $m = q$ and $\sigma = v-\beta$ to form a latin square L of side v. Use L to form the latin square triples in the singular direct product. For $i = 1, 2$, the i-subsystem triples are formed by an STS(v) having a complete q-arc on $\{1, \ldots, q\}$. The 3-subsystem triples are formed by an STS(v) having an independent set on $\{1, \ldots, \beta\}$. The first and second systems exist since $\frac{v-1}{2} \geq q \geq U(v)$. Since $\beta \leq q+1$ when $q \leq \frac{v-3}{2}$, and $\beta \leq T(v)$ when $q = \frac{v-1}{2}$, the third STS(v) with an independent set of size β also exists. The singular direct product produces an STS(x) having a complete s-arc on $(\{1, \ldots, q\} \times \{1, 2\}) \cup (\{1, \ldots, \beta\} \times \{3\})$. Independence is immediate. Every element in the first two subsystems is spanned by the complete q-arcs, while all elements of the third ingredient STS that are not in the arc are spanned by the latin square triples. Inductively, we can select the first STS(v) to have a sub-STS(7) when $v = 7$ or $v \geq 15$. It remains to treat $x \in \{27, 39\}$.

Case (b): $x \equiv 1, 7 \pmod{18}$. Write $v = \frac{x+2}{3}$ and $w = 1$. When $s \geq 3U(v)$, write $s = 2q + \beta$ for $\beta \in \{q-1, q, q+1\}$; otherwise when $q \geq 2U(v)$, write $q = U(v)$ and $s = 2q + \beta$. Now proceed as above, but with L of side $v-1$, ensuring that the point v appears in neither of the complete arcs, nor the independent set of the third system. If $s = 2U(v) - 1$, set $q = U(v) - 1$, $\beta = 0$, and include the single point of the subsystem in the complete arc. To ensure the presence of a sub-STS(7), it remains to treat $x \in \{25, 37\}$.

Case (c): $x \equiv 15 \pmod{18}$. If $x = 15$ see Lemma 3.2. Otherwise, write $v = \frac{x+6}{3}$ and $w = 3$. When $s \geq 3(U(v) - 1) + 1$, write $s = 2q + \beta + 1$ for $\beta \in \{q-1, q, q+1\}$; otherwise, write $q = U(v) - 1$ and $s = 2q + \beta + 1$. Apply Corollary 2.2 with $m = q$ and $\sigma = \frac{x-3}{3} - \beta$ to form a latin square L of side $v - 3$, yielding the latin square triples. Choose the first two STS(v)s with complete q-arcs, arranged in such a way that $v - 2$ is a point of the q-arc, but $v - 1$ and v are not; this can always be done since $q \leq \frac{v-1}{2}$. Choose the last STS(v) so that $\{1, \ldots, \beta\} \cup \{v-2\}$ is an independent set. The complete s arc is on $(\{1, \ldots, q\} \times \{1, 2\}) \cup (\{1, \ldots, \beta\} \times \{3\}) \cup \{v-2\}$. To ensure the presence of a sub-STS(7) inductively, we must treat the case $x = 33$. Parameters $v = 15$ and $w = 3$ in case (c) also handle $x = 39$, left from case (a).

Case (d): $x \equiv 13 \pmod{18}$. Write $v = \frac{x+14}{3}$ and $w = 7$. Although we have required that the systems constructed have sub-STS(7)s, we have exercised no control over the size of the intersection of various complete arcs with the subsystem. As a result, we obtain a weaker result here. We may suppose inductively that an STS(v) exists having a complete t-arc for $U(v) \leq t \leq T(v)$; further suppose that in the STS(v) with the complete t-arc, we find exactly δ_t of the points of the arc in the sub-STS(7). Evidently, $2 \leq \delta_t \leq 4$. We write $s = 2q + \beta - \delta_q$ so that $q \geq U(v)$, and $0 \leq \beta \leq T(v) - 4$. When this can be done, choose latin square triples as before, then place STS(v)s having a complete q-arc on elements $\{1, \ldots, q - \delta_q\} \cup \{v-6, \ldots, v-7+\delta_q\}$, and place finally the third STS with an independent set on $\{1, \ldots, \beta\} \cup \{v-6, v-5, v-4, v-3\}$. If we follow the prescription for the singular direct product blindly, this third STS would require an independent set meeting a sub-STS(7) in precisely four points, and our induction hypothesis does not support this. Instead, we therefore omit the sub-STS(7)s when placing the first two STS(v)s, and place the third STS(v) on $(\{1, \ldots, v-7\} \times \{3\}) \cup \{v-6, \ldots, v\}$ so that its independent set of size $\beta + 4$ appears on the desired points. This may, of course, fail to place an STS(7) on the points $\{v-6,\ldots,v\}$. Nevertheless, provided the STS(v) employed indeed has a sub-STS(7), it remains in the STS(x) and continues to satisfy the induction hypothesis.

Since the $\{\delta_q\}$'s are unknown, one must be careful to see what values can be handled. For each value of q, by selecting $0 \le \beta \le T(\frac{x+14}{3}) - 4$, we handle an interval of values, and δ_q contributes only some small uncertainty about which interval is covered. However, the interval for q and the interval for $q+1$ must meet (or overlap) if $T(\frac{x+14}{3}) - 4 \ge 3$, which always holds since $x > 13$.

Finally, we must handle the cases when $v \in \{25, 27, 33, 37\}$. We simply outline the strategy here. To produce an STS(v) having a complete s-arc, we employed Stinson's hill-climbing algorithm with two modifications. First, we forced the selection of an STS(7) on the points $\{0, 1, 2, s, s+1, s+2, s+3\}$. Then we required that whenever a triple is added, its largest element is at least s. This ensures that the elements $\{0, \ldots, s-1\}$ form an arc, and then a simple verification is made that the arc is complete. Solutions for $7 \le s \le 12$ when $v = 25$, $8 \le s \le 13$ when $v = 27$, $8 \le s \le 16$ when $v = 33$, and $9 \le s \le 18$ when $v = 33$, were found in this way (and are available from the authors). For $v = 25$, the system

```
abm acn bco aop bnp cmp mno adx aet afw ags ahy aiv ajq aku alr bdv ber
bfx bgu bht biq bjw bks bly cds cev cfr cgt chu cix cjy ckq clw deq dfm
dgw dhp dir djt dky dlo dnu efo egx ehw eim eju ekp els eny fgn fhq fit
fjs fkv flu fpy ghr giy gjp gkm glv goq his hjo hkn hlx hmv ijn iko ilp
iuw jkr jlm jvx klt kwx lnq mqw mrt msy mux nrw nsv ntx ory osx otu ovw
pqu prx psw ptv qrv qst qxy rsu twy uvy
```

has complete arcs

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abcfkty, abchkrvx, abcepsuxy, abehnoquvx, abcdefghijkl
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A second STS(25),

```
abl acm bcn ano bmo clo lmn adx aet afu ags ahp aiq ajr aky avw bdu bev
bft bgr bhw bix bjq bks bpy cdq cew cfy cgu chv cip cjt ckr csx deo dfs
dgy dhn dit djm dkl dpw drv efm egp ehq eil ejs ekx enr euy fgq fhl fir
fjv fkn fow fpx ght giv gjn gkm glw gox him hjo hku hrx hsy ijy iko inu
isw jkp jlx juw kqv ktw lpr lqs ltu lvy mpt mqy mrw msv mux npq nst nvx
nwy ops oqu ory otv puv qrt qwx rsu txy
```

has complete arcs

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abehmru, abdhmrty, abcgopqtw, abcdfgkopv, abcdefghijk
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An STS(27) and some of its complete arcs are:

```
abn aco bcp apq boq cnq nop adA aes afu agz ahy aiw ajr akt alv amx bdx
bez bfr bgA bht biv bjy bkw bls bmu cdv cex cfy cgs chr cit cjw cku clA
cmz deq dfn dgu dho dis djp dkz dlr dmt dwy efv egn ehp eiA ejt eky elo
emw eru fgt fhs fip fjz fkA flx fmq fow ghw gio gjv gkq gly gmr gpx hix
hju hkn hlz hmA hqv ijn ikr ilu imy iqz jks jlq jmo jxA klp kmv kox lmn
ltw mps nry nst nuz nvA nwx orA osz otv ouy prv pty puA pwz qrt qsu qwA
qxy rsw rxz svx syA tux tzA uvw vyz
abdglopt, abfopvxyA, abdefgmopy, abdfgkopsvy, abcdefghijklm
```

An STS(33) and some of its complete arcs are:

```
abq acr bcs ast brt cqt qrs adD aew afy agC ahv aiG ajz akE alB amu anA
aoF apx bdE bex bfG bgA bhD biF bjv bkw blu bmC bny boz bpB cdu ceB cfF
cgx chz civ cjA ckC clw cmy cnE coD cpG dev dfx dgq dhB diw djs dkr dlt
dmG dnz doy dpF dAC eft egr ehs eiC ejD ekA elE emF enu eoq epz eyG fgw
fhq fiD fjE fkz flv fms fnB foA fpu frC ghy giu gjF gkv glG gmz gnD goB
gpt gsE hiE hjx hkF hlA hmw hnG hor hpC htu ijB ikt ilr imA ins iox ipq
iyz jkG jlq jmr jnC jot jpy juw kly kmB knq kou kps kxD lmx lnF los lpD
lzC mnt moE mpv mqD nov npr nwx opw oCG pAE quv qwG qxz qyE qAB qCF ruz
rvG rwF rxB ryA rDE suB svF swA sxC syD szG tvx twy tzE tAD tBC tFG uxy
uAF uCE uDG vwE vyC vzA vBD wzB wCD xAG xEF yBF zDF BEG
abcdefzC, abgprsuDF, abefjkrsBF, abcfhiouwBC, abchowxyABCE, abcdefghijklmnop
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An STS(37) and some of its complete arcs are:

```
abs act bcu auv btv csv stu adJ aeK afy agw ahB aiH ajA akE alC amx anF
aoD apI aqG arz bdy bex bfD bgE bhJ biF bjC bkH blA bmG bnz boI bpK bqB
brw cdE ceI cfF cgH chD ciC cjK ckx clJ cmy cnB coz cpA cqw crG deu dfB
dgt dhK div djI dkG dlH dmw dnD dox dps dqF drC dzA efC egB eht eiA ejv
ekD els emz enH eoy epw eqE erJ eFG fgz fhE fix fjt fkI flw fmJ fns fou
fpH fqA frK fvG ghs giI gjG gkA glK gmC gny goF gpx gqv gru gDJ hiu hjy
hkw hlv hmI hnA hoC hpG hqx hrH hzF ijs ikz ilD imK inG iow ipy iqt irB
iEJ jkF jlB jmD jnu joH jpE jqJ jrx jwz klt kms knC koK kpJ kqy krv kuB
lmF lnx loG lpz lqI lry luE mnv moB mpu mqH mrA mtE noE npt nqK nrI nwJ
opv oqs ort oAJ pqC prF pBD qrD quz rsE swH sxG syz sAC sBF sDI sJK twF
txA tyG tzK tBH tCD tIJ uwx uyH uAD uCK uFI uGJ vwA vxJ vyC vzD vBE vFK
vHI wyE wBC wDG wIK xyK xzH xBI xCF xDE yAI yBJ yDF zBG zCJ zEI ABK AEF
AGH CEH CGI DHK EGK FHJ
abjopqruy, abcegivyDG, abcejryBDEH, abegjltyzFHI, abcmrBCDEFIJK,
abcdefghijklmnopqr
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Except when $(v, s) =$ (27,12), (33,13), (33,14), (33,15), (37,14), (37,15), (37,16), or (37,17), these examples treat all cases with $v \in \{25, 27, 33, 37\}$. We leave the remaining cases for s as easy exercises for the reader. □

3 At and near the maximum

Theorem 2.3 misses a narrow but growing interval of values near the minimum, and only a few values near the maximum. We treat the missed values near the maximum first. The maximum value itself, $T(v)$, has been settled by Sauer and Schönheim [11], but we must repeat it in order to ensure the presence of the required sub-STS(7).

We employ here another standard construction, the $v \to 2v + 1$ construction. A 1-factor in a graph is a regular, spanning subgraph of degree 1 (i.e., a perfect matching). A 1-factorization of a graph G is a set $\mathcal{F} = \{F_1, \ldots, F_r\}$ where each F_i is a 1-factor (here G must be an r-regular graph). Now if an STS(v) on elements V exists, and a one-factorization of the complete graph of order $v+1$ on a disjoint set X of elements exists, one produces an STS($2v+1$) on $V \cup X$ as follows. Include all triples in the STS(v). Then let $V = \{z_1, \ldots, z_v\}$, and let the 1-factorization be $\mathcal{F} = \{F_1, \ldots, F_v\}$. Whenever $\{a, b\} \in F_i$, add the triple $\{z_i, a, b\}$. The result is easily seen to be an STS($2v+1$), but more is true: X forms an independent set of size $v+1$.

Typically the STS($2v+1$) constructed has many more complete arcs. We examine this next. A set D of vertices in a graph is *independent* if it induces a void subgraph, and it is *dominating* if every vertex not in D has at least one neighbour in D. Consider a 1-factorization $\mathcal{F} = \{F_1, \ldots, F_v\}$ of K_{v+1}, and let $G_r = \bigcup_{i=1}^{r} F_i$. Now choose an STS($v$) for which $\{z_1, \ldots, z_r\}$ is a complete arc, and choose a 1-factorization $\mathcal{F}$ of K_{v+1} so that D is an independent dominating set of size s in G_r. Then applying the $v \to 2v+1$ construction, it is easily verified that $\{z_1, \ldots, z_r\} \cup D$ is a complete $(r+s)$-arc in the STS($2v+1$).

It is easy to produce 1-factorizations of K_{v+1}, v odd, in which the first $\frac{v-1}{2}$ 1-factors form a bipartite graph with an independent, dominating set of size $\frac{v+1}{2}$ — this is just a class of the bipartition. In fact, the same independent, dominating set is present in G_r for every r satisfying $1 \le r \le \frac{v-1}{2}$. Hence we conclude:

Theorem 3.1 *If an STS(v) having a complete s-arc exists, then an STS($2v+1$) having a complete ($s + \frac{v+1}{2}$)-arc also exists. If the STS(v) has a sub-STS(7) meeting the conditions of Theorem 1.1, so also does the STS($2v+1$).*

Inductively, Theorem 3.1 fails to produce the required sub-STS(7) within the sub-STS(v) when $2v+1 \in \{19, 27\}$. However, in both cases, one can choose the 1-factorization to have a sub-1-factorization of order 4. Then attaching the three 1-factors of the subfactorization to three elements of a triple in the sub-STS(v) yields the required sub-STS(7).

The structure of the independent dominating sets can also be exploited:

Lemma 3.2 *There is an STS(15) having a complete s-arc in which δ_s arc elements lie in a sub-STS(7) when $(s, \delta_s) \in \{(8,4), (7,4), (6,4), (5,3)\}$.*

Proof: For (8,4), apply Theorem 3.1. For (7,4), consider the 1-factorization:

1,4	1,5	1,6	1,7	1,8	1,2	1,3
2,8	2,4	2,5	2,6	2,3	3,6	2,7
3,7	3,8	3,4	3,5	4,6	4,7	4,5
5,6	6,7	7,8	4,8	5,7	5,8	6,8

Each column forms a 1-factor, and the first four 1-factors yield a graph with {1,2,3} as an independent dominating set. Employ these in the doubling construction using an STS(7) having a complete 4-arc, attaching the first four 1-factors to the arc elements. The 4-arc, together with elements {1,2,3} from the 1-factorization, give the complete 7-arc.

For (6,4), choose a 1-factorization in which the first three 1-factors form two disjoint complete graphs on four elements each. Then the first four 1-factors induce a graph with an independent dominating set of size two.

For (5,3), employ the same 1-factorization, noting that the first three 1-factors also induce a graph with an independent dominating set of size two. Suppose without loss of generality that the edge joining the two elements in the independent dominating set appears in the seventh 1-factor. Then choose the STS(7) to be used to have triples $\{x_1, x_2, x_4\}$, $\{x_1, x_3, x_5\}$, and $\{x_2, x_3, x_6\}$, without loss of generality. Then $\{x_1, x_2, x_3\}$ is not a complete arc, but it fails to span only x_7. But x_7 is spanned by the two elements of the independent dominating set. □

Now we turn to cases when $x \equiv 1, 9 \pmod{12}$. When $x \equiv 1, 3, 7, 15 \pmod{18}$, Theorem 2.3 reaches the maximum, $\frac{x-1}{2}$. Indeed when $x \equiv 9 \pmod{18}$ and $x \equiv 9 \pmod{12}$, we find $x \equiv 9 \pmod{36}$ so that $\frac{x}{3} \equiv 3 \pmod{12}$ and again Theorem 2.3 reaches the maximum. Therefore the only remaining case is when $x \equiv 13 \pmod{36}$.

Theorem 3.3 *Let $v = 36t + 13$, $t \geq 1$. When $15t + 5 \leq s \leq 18t + 6$, there exists an STS(v) having a complete s-arc. In addition, it contains a sub-STS(7) meeting the conditions of Theorem 1.1.*

Proof: Form an STS($6t+3$) on elements $(X \times \{0, 1\}) \cup \{\infty\}$ in which $\{\infty, x_0, x_1\}$ is a triple for each $x \in X$. Let $\mathcal{D}$ be the set of triples not containing ∞. On $X \times \{0, 1\} \times \{0, 1, 2\} \times \{0, 1\}$, we form a set of triples, writing the 4-tuple (x, i, a, p) as (x_i, a_p). When $\{x_i, y_j, z_k\} \in \mathcal{D}$, we form the 36 triples obtained by taking $\{(x_i, a_p), (y_j, b_q), (z_k, c_r)\}$ whenever $a + b + c \equiv 0 \pmod 3$ and $p + q + r \equiv 1 \pmod 2$, and of course $\{p, q, r\} \subseteq \{0, 1\}$ and $\{a, b, c\} \subseteq \{0, 1, 2\}$. Call the resulting set of triples $\mathcal{B}$. Now no triple contains more than two elements of $X \times \{0, 1\} \times \{0, 1, 2\} \times \{0\}$, since in each triple chosen we required that $p + q + r \equiv 1 \pmod 2$. However, for every point of $X \times \{0, 1\} \times \{0, 1, 2\} \times \{1\}$, there are two points in $X \times \{0, 1\} \times \{0, 1\} \times \{0\}$ with which it forms a triple of $\mathcal{B}$. To verify this, consider a point (z_k, c_1). Find a triple of $\mathcal{D}$, say $\{x_i, y_j, z_k\}$, that contains z_k. If $c = 2$, select $a = 0$ and $b = 1$; otherwise select $a = b = c$. In either case, choose $p = q = 0$. Then $\{(x_i, a_p), (y_j, b_q), (z_k, c_1)\}$ is indeed in $\mathcal{B}$, and $\{(x_i, a_p), (y_j, b_q)\}$ appear in the required set.

Let $\widehat{s} = s - (15t + 5)$. For any $\widehat{s}$ different selections of $x \in X$, place on $\{x\} \times \{0, 1\} \times \{0, 1, 2\} \times \{0, 1\}$ together with a new point ∞ an STS(13) with a complete 5-arc on

$$\{(x_i, a_p) : (i, a, p) \in \{0, 1\} \times \{0, 1\} \times \{0\} \text{ or } (i, a, p) = (1, 2, 0)\}.$$

For the remaining $3t + 1 - \widehat{s}$ choices of $x \in X$, on $\{x\} \times \{0, 1\} \times \{0, 1, 2\} \times \{0, 1\}$ together with the new point ∞, place an STS(13) with a complete 6-arc on $\{x\} \times \{0, 1\} \times \{0, 1, 2\} \times \{0\}$. In both cases, we ensure (without loss of generality) that $\{\infty, (x_i, 0_0), (x_i, 0_1)\}$ is a triple for $i = 0, 1$.

The set of all blocks forms an STS(v) with $v = 36t + 13$. The unions of the complete arcs in the STS(13)s so placed is a complete s-arc as required. When $\{x_i, y_j, z_k\} \in \mathcal{D}$, we find a sub-STS(7) satisfying the conditions of Theorem 1.1, on $(\{(x, i), (y, j), (z, k)\} \times \{0\} \times \{0, 1\}) \cup \{\infty\}$. □

4 At and near the minimum

Theorem 2.3 treats all values from approximately $\frac{2\sqrt{2}}{\sqrt{3}}\sqrt{x}$ upwards, whereas $U(x)$ is approximately $\sqrt{2}\sqrt{x}$. To continue with the singular direct product, we must then abandon the assumption that within the first two STS(v)s the elements included form a complete arc of the subsystem. In this case, the machinery developed in [4] can be used.

Colbourn, Dinitz, and Stinson [4] prove that

Theorem 4.1 *For every* $v \equiv 1, 3 \pmod 6$*, there is an STS(v) with a scattering set of size* $L(v)$*. Moreover, when* $v = 31$ *or* $v \geq 43$*, there is an STS(v) with a scattering set of size* $L(v)$ *in which exactly two points of the scattering set are in a sub-STS(7), and at least four other points of the sub-STS(7) are spanned by the scattering set.*

They also employ the singular direct product, but the latin squares required have somewhat different restrictions. A latin square of side r is (a, b)-*scattered* if

1. The $a \times a$ subarray indexed by rows $1, \ldots, a$ and columns $1, \ldots, a$ contains a^2 distinct symbols, none of which are in the set $\{1, \ldots, b\}$.
2. In rows $1, \ldots, a$, the symbols $1, \ldots, b$ appear in ab distinct columns.
3. In columns $1, \ldots, a$, the symbols $1, \ldots, b$ appear in ab distinct rows.

Lemma 4.2 *[4] If* $b \leq a$ *and* $r \geq max(a^2+b, ab+a+b)$*, then an* (a, b)*-scattered latin square of side* r *exists.*

We follow the prescription by Colbourn, Dinitz, and Stinson [4] closely.

Theorem 4.3 *Let* $x = 3r + w$*, with* $w = 0$ *when* $x \equiv 3, 9 \pmod{18}$*,* $w = 1$ *if* $x \equiv 1, 7 \pmod{18}$*,* $w = 3$ *when* $x \equiv 15 \pmod{18}$*, and* $w = 7$ *when* $x \equiv 13 \pmod{18}$*. Let* $e = w$ *when* $w \in \{0, 1\}$*,* $e = 2$ *when* $w = 7$*, and* $e = U(x) \bmod 3$ *when* $w = 3$*. Let* a_1, a_2, a_3 *satisfy* $\lceil \frac{U(x)-e}{3} \rceil \geq a_1 \geq a_2 \geq a_3 \geq \lfloor \frac{U(x)-e}{3} \rfloor$ *and* $a_1 + a_2 + a_3 + e = U(x)$*. Then whenever* $U(x) \leq s \leq a_1 + a_2 + e + (r - a_1 a_2)$*, there is an STS(x) having a complete s-arc which contains an STS(7) meeting the conditions of Theorem 1.1.*

Proof: We form the STS(x) on $(\{1, \ldots, r\} \times \{1, 2, 3\}) \cup \{r+1, \ldots, r+w\}$, writing x_i for (x, i). First we form an (a_1, a_3)-scattered latin square of side r, which is shown to exist in [4]. For concreteness, suppose that the latin square L constructed has symbols $\{r - a_1 a_2 + 1, \ldots, r\}$ in the leading $a_1 \times a_2$ subarray, that it has symbols $1, \ldots, a_3$ in rows $1, \ldots, a_1$ appearing in columns $r - a_1 a_3 + 1, \ldots, r$, and that it has symbols $1, \ldots, a_3$ in columns $1, \ldots, a_2$ appearing in rows $r - a_2 a_3 + 1, \ldots, r$. Our objective is to form a complete arc that contains (at least) the points $P = \{r+1, \ldots, r+e\} \cup \bigcup_{i=1}^{3} (\{1, \ldots, a_i\} \times \{i\})$. If we use L to form latin square triples, then immediately we find that no element x_i can be added to P while keeping the set independent if $x > a_j a_k$ and $\{i, j, k\} = \{1, 2, 3\}$. One can, however, for fixed $i \in \{1, 2, 3\}$, add all elements $\{x_i : a_i < x \leq r - a_j a_k, \{i, j, k\} = \{1, 2, 3\}\}$ to P while retaining independence among the latin square triples. But one cannot, in general, add elements with different subscripts and retain independence.

The subsystem triples on $\{1_i, \ldots, r_i, r+1, \ldots, r+w\}$ for $i = 1, 2, 3$ must now be selected. We choose the first system to be an STS($r + w$) with a sub-STS(w), having a scattering set of size $a_1 + e$ with e of the scattering set elements in the sub-STS(w). In general, we further prescribe the number d_1 of elements of the sub-STS(w) spanned by the scattering set. Place the STS with scattering set on $(\{1,\ldots,a_1\}\times\{1\})\cup\{r+1,\ldots,r+e\}$, so that the $\binom{a_1+e}{2}$ triples having two elements in the scattering set have third elements (which are all different) equal to $\{a_1+1,\ldots,a_1+\binom{a_1+e}{2}-d_1\}$ outside of the subsystem, and to $\{r+e+1,\ldots,r+e+d_1\}$ within the subsystem. Evidently, when $a_1+\binom{a_1+e}{2}-d_1 \geq r-a_2a_3$, the 1-subsystem triples, together with the latin square triples, prevent the addition of any element x_1 with $x>a_1$ to P, while retaining independence. We always take $d_1=0$ when $x\equiv 1,3,7,9 \pmod{18}$. We take $d_1=1$ when $x\equiv 15 \pmod{18}$, and we take $d_1=4,3,3$ when $U(x)\equiv 0,1,2 \pmod 3$, respectively. Colbourn, Dinitz, and Stinson [4] verify that the required inequality above is met by these choices of parameters.

The second subsystem is placed similarly, with third elements in the sub-STS(w) from $\{r+e+1,\ldots,r+e+\binom{e}{2}\}\cup\{r+e+d_1+1,\ldots,r+e+d_1+d_2-\binom{e}{2}\}$. When $r+e+d_1+d_2-\binom{e}{2}$ exceeds $r+w$, for the latter set take $\{r+w-d_2+\binom{e}{2}+1,\ldots,r+w\}$. Again the verification requires that we specify d_2. We take $d_2=d_1$ except when $x\equiv 13 \pmod{18}$ and $U(x)\equiv 0 \pmod 3$, in which case we take $d_2=2$. Colbourn, Dinitz, and Stinson [4] verify that $a_2+\binom{a_2+e}{2}-d_2 \geq r-a_1a_3$, so that no element of the form x_2 can be added to P while retaining independence once the 2-subsystem triples are present.

If we were to place the third subsystem also similarly, this would in fact realize the minimum size $U(x)$ of the complete arc. However, we vary the prescription for this last ingredient. The basic fact upon which we rely is that having placed latin square, 1-subsystem, and 2-subsystem triples, the only candidates to add to P while retaining independence are $\{x_3 : a_3 < x \leq r-a_1a_2\}\cup\{r+e+d_1+d_2-\binom{e}{2}+1,\ldots,r+w\}$. Some computation shows that with the specified choices of e, d_1, and d_2 for each choice of w, the latter set (within the subsystem) is empty except when $x\equiv 13,15 \pmod{18}$, in which cases it contains at most one element. This limitation depends in no way on the selection of the third subsystem to place.

So write $\tilde{s}=s-U(x)$, the amount by which the size of the desired complete arc exceeds the size of a minimum complete arc. Form an STS($r+w$) on $\{1,\ldots,r+w\}$ with a sub-STS(w) on $\{r+1,\ldots,r+w\}$ having an independent set on $\{1,\ldots,a_3+\tilde{s}\}\cup\{r+1,\ldots,r+e\}$ by which all elements $\{a_3+\tilde{s}+1,\ldots,r-a_1a_2\}\cup\{r+e+d_1+d_2-\binom{e}{2}+1,\ldots,r+w\}$ are spanned. We have seen that the size of the latter set (within the subsystem) is at most one. When $w\in\{0,1,3\}$, this can be ensured for any STS($r+w$). When $w=7$, the induction hypothesis provides the required system (except when $r+w\leq 13$, which cannot hold).

Indeed we can choose $0\leq\tilde{s}\leq r-a_1a_2-a_3$ as we like, since a_1a_2 comprises approximately two-thirds of the elements. That one can always choose $\tilde{s}$ at the maximum without exceeding the size of an independent set in the STS($r+w$) is an easy but tedious verification (which we completed with a simple MAPLE program). The more serious issue is whether we can choose $\tilde{s}$ small. However, Colbourn, Dinitz, and Stinson [4] verify that when $\tilde{s}=0$ we have $a_3+\binom{a_3+e}{2}-d_3\geq r-a_1a_2$, so the result is always a complete arc. The required sub-STS(7), when present in one of the ingredients, remains in the STS constructed. □

5 Putting the pieces together

The remaining issue is to consider those values that are too large for Theorem 4.3, but too small for Theorem 2.3. Since the largest size treated by Theorem 4.3 grows approximately as $\frac{x}{9}$, while the smallest size treated by Theorem 2.3 grows only as $\sqrt{\frac{8x}{3}}$, it is easy to see that for x sufficiently large, all sizes of complete arcs are treated by one (or both) of these two constructions. However, for small values of x, one must verify that there remains no gap between the two constructions. This verification is lengthy, but easily done.

It is not unexpected that the possible sizes of complete arcs form an interval, and indeed that there is substantial flexibility in forming them. Nevertheless, the techniques used appear to encounter essentially different problems near the minimum, in the middle, and near the maximum sizes. It would be of interest to find single systems that admit many different sizes of complete arcs. Our experience is that most systems have very many small complete arcs but few (if any) of sizes near the maximum. This suggests the problem of determining the maximum, over all Steiner triple systems of a given order, of the size of a smallest complete arc in the system. We expect this number to be close to $U(v)$ on the (admittedly weak) basis of computational evidence.

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