On Maximal Partial Costas Latin Squares

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January 28, 2013

Abstract

A Costas array of order n is an $n \times n$ permutation matrix with the property that all of the $n(n-1)/2$ line segments between pairs of 1's differ in length or in slope. A Costas latin square of order n is an $n \times n$ latin square where for each symbol k, with $1 \le k \le n$, the cells containing k determine a Costas array. The existence of a Costas latin square of side n is equivalent to the existence of n mutually disjoint Costas arrays. In 2012, Dinitz, Östergård and Stinson enumerated all Costas latin squares of side $n \le 27$. In this brief note, a sequel to that paper, we extend this search to sides $n = 28$ and 29. In addition we determine the sizes of *maximal* sets of disjoint Costas latin squares of side n for $n \le 29$.

Keywords: Costas array, Costas property, Latin Square

1 Introduction and definitions

A Costas array of order n is an $n \times n$ permutation matrix with the property that all of the $n(n-1)/2$ line segments between pairs of 1's differ in length or in slope. The condition of having unique displacement vectors between all pairs of 1's is called the *Costas property*. In 1984, J.P. Costas introduced Costas arrays in the context of SONAR detection; these arrays were used by the U.S. Navy for many years (see [1]). For useful references for Costas arrays see [3, 4, 9, 11, 12].

Example 1.1 *All Costas arrays of order 4 (zeros omitted).*

1			
	1		
			1
		1	

$(1,2,4,3)$

			1
		1	
1			
	1		

$(3,4,2,1)$

	1		
1			
		1	
			1

$(2,1,3,4)$

		1	
			1
	1		
1			

$(4,3,1,2)$

1			
		1	
			1
	1		

$(1,4,2,3)$

			1
1			
		1	
	1		

$(2,4,3,1)$

		1	
1			
	1		
			1

$(2,3,1,4)$

		1	
	1		
			1
1			

$(4,2,1,3)$

1			
			1
	1		
		1	

$(1,3,4,2)$

			1
	1		
1			
		1	

$(3,2,4,1)$

	1		
		1	
1			
			1

$(3,1,2,4)$

	1		
			1
		1	
1			

$(4,1,3,2)$

For notational convenience, Costas arrays are often presented using a certain one-line notation. Given a Costas array of order n, let $\pi(i) = j$ whenever the array contains a 1 in cell (i, j). A Costas array of order n can be presented as the permutation $\pi = (\pi(1), \pi(2), \cdots, \pi(n-1), \pi(n))$. We call this the *permutation representation* of a Costas array. Using this notation, the Costas arrays in the first row of Example 1.1 have permutation representations $(1,2,4,3)$, $(4,3,1,2)$, $(2,1,3,4)$ and $(3,4,2,1)$, respectively.

A *Costas latin square* of order n, denoted CLS(n), is a latin square of order n such that for each symbol i, $1 \leq i \leq n$, a Costas array results if a 1 is placed in the cells containing symbol i. Clearly a CLS(n) is equivalent to n disjoint Costas arrays of order n. Costas latin squares were first defined and studied by Etzion [7] and are only known to exist when $n = p - 1$ for all primes p and for $n = 8$. Using the complete listing of all Costas arrays of orders $n \leq 27$ [11] Dinitz, Östergård and Stinson [2] enumerated all Costas latin squares of orders $n \leq 27$. Since that time all Costas arrays of orders 28 and 29 have also been classified (see [5, 6]) and are available for download at [11]. In this paper we will discuss sets of disjoint Costas arrays for those two orders.

We also search for *maximal* sets of disjoint Costas arrays of order n for all $n \leq 29$. These can be superimposed to construct partial latin squares. Define a k-maximal Costas partial latin square of order n, denoted mcpls(n,k), as an $n \times n$ matrix containing the numbers 1 to k satisfying:

1. $k \leq n$,
2. each number 1 to k appears exactly once in each row and each column,
3. each number 1 to k displays the Costas property, and
4. a Costas array of order n cannot be made from the remaining blank cells.

Clearly the existence of a maximal set of k disjoint Costas arrays is equivalent to a mcpls(n,k). Notice also that a mcpls(n,n) is a Costas latin square of side n.

Example 1.2 *A Costas latin square of order 12 and a* 10*-maximal Costas latin square of order 21.*

1	11	4	5	3	7	12	2	9	8	10	6
2	9	8	10	6	12	11	4	5	3	7	1
3	7	12	11	9	8	10	6	1	2	4	5
4	1	3	7	5	2	9	12	10	6	8	11
5	3	7	1	11	9	8	10	6	12	2	4
6	5	11	4	12	3	7	8	2	9	1	10
7	8	2	9	1	10	6	5	11	4	12	3
8	10	6	12	2	4	5	3	7	1	11	9
9	12	10	6	8	11	4	1	3	7	5	2
10	6	1	2	4	5	3	7	12	11	9	8
11	4	5	3	7	1	2	9	8	10	6	12
12	2	9	8	10	6	1	11	4	5	3	7

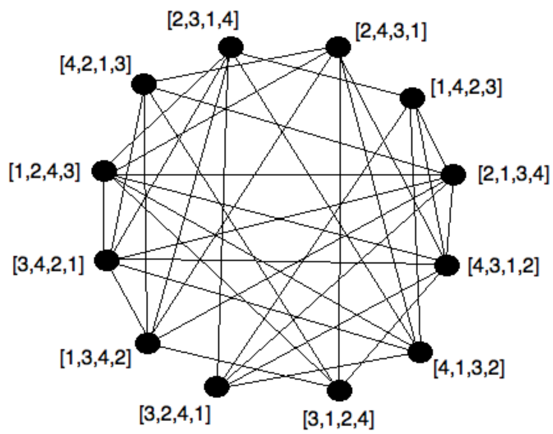
cls(12)

	5	6	8	2	10			9	4	3						1	7			
			4	9	2		7		8		10	3	1				5			6
		4					2	8	1	5				9	3			10	6	7
					5	7	1	6			4	10			8	3		2	9	
1	7					3		10		6				2		4	9	5	8	
2	8		7				5		9			4	6	3		10				1
	4			5	3	9	6	1	2					8					7	10
		3						5	10	7					6	8	2	1	4	9
3		9			6	10	4			1			7	5			8			2
			9	10		8			6		2	5	4		1			7		3
	2	1	3	6	7										10	9	4	8	5	
4		10			8		3	2	5		9			1		7	6			
5			1			2	10			8			3	7	9			6		4
6	3	8	5	1	9					10	7	2						4		
7	10					1					5	8	9	6	4	2			3	
8				7		4	9	3			6		2				10		1	5
	1	2	6	3		5				9		7		4					10	8
	6	5		4	1			7	3			9	8	10	2					
10	9	7			4	6				2	8	1	5					3		
9			2				8	4	7		1		10		5	6	3			
			10	8						4	3	6			7	5	1	9	2	

mcpls(21,10)

For Costas arrays of order n, the *disjointness graph of order* n, denoted G_n, is a graph representing all Costas arrays of order n where each vertex represents a Costas array and two vertices are adjacent if the Costas arrays those vertices represent are disjoint. Note the number of vertices in G_n equals the number of distinct Costas arrays of order n. We exhibit G_4 in Example 1.3 below.

Example 1.3 *G_4 constructed from the Costas arrays in Example 1.1.*



In the next section of the paper we discuss the methodology used for searching for maximal Costas partial latin squares and in the final section we present and discuss our results.

2 Finding Costas latin squares and maximal Costas latin squares

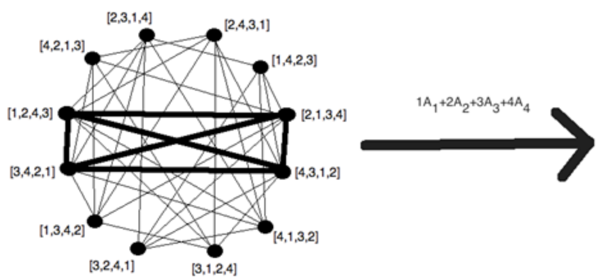
To continue the work from [2] we performed computer searchs for Costas latin squares and maximal Costas partial latin squares of orders 1 to 29. Checking if two Costas arrays are disjoint is only a matter of comparing the list representations of those Costas arrays element wise; if at each index in the lists, the elements are different, then the two Costas arrays are disjoint. Given a set of k mutually disjoint Costas arrays of order n, $A_1, A_2, \ldots, A_k$, we use the following equation:

$$\sum_{i=1}^{k} i \cdot A_i \tag{1}$$

to yield an $n \times n$ matrix where each number 1 to k appears exactly once in each row and each column, and each number has the Costas property. If a Costas array cannot be made from the remaining blank cells after performing equation (1), then we have found a k-maximal Costas latin square of order n and equivalently the set of Costas arrays is a maximal set of disjoint Costas arrays. If $k = n$, then we have found a Costas latin square of order n.

Notice that a set of k mutualy disjoint Costas arrays of order n will be represented as a k-clique in G_n, and a maximal set of k mutually disjoint Costas arrays of order n will be represented by a maximal k-clique in G_n. Hence, we downloaded a database of all Costas arrays for each order 1 to 29 from [11] and built the disjointness graph, G_n, for each order n from 1 to 29. We then used the clique-finding C-routines *Cliquer* [10] to search the disjointness graphs for maximal cliques, each of which represents a maximal Costas latin square. For instance, from Example 1.3 we see that $A_1 = [1,2,4,3], A_2 = [3,4,2,1], A_3 = [2,1,3,4]$, and $A_4 = [4,3,1,2]$ form a maximal 4-clique in G_4 and hence form a CLS(4) after using Equation 1. This is demonstrated in the example below.

Example 2.1 *Constructing a CLS(4) from* G_4



1	2	4	3
2	1	3	4
3	4	2	1
4	3	1	2

The search was performed on one node of the Vermont Advanced Computing Core. This node contains two quad-core processors running at 2.8 Ghz. We used this machine for over two months to find the values in the Table 3.1 below.

3 Results and Observations

As noted above, in this paper we complete the enumeration of Costas latin squares of side 28 and 29. There are precisely 1,371,168 different Costas latin squares of order 28 and there are none of order 29. The fact that there exists at least one Costas latin square of order 28 was given in [2], however the precise number was not determined in that paper.

The following table contains the a summary of our search. For each $n < 29$ we give the following data:

1. $|V(G_n)|$, the number of vertices in G_n – this is the number of distinct Costas arrays of order n.

2. $|E(G_n)|$, the number of edges in G_n – this is the number of pairs of disjoint Costas arrays of order n.

3. Maximum – the maximum clique size. This is the size of the largest set of disjoint Costas arrays of order n.

4. Maximal – the size(s) of maximal sets of disjoint Costas arrays of order n.

Table 3.1 *Results from the computer search.*

n	$\|V(G_n)\|$	$\|E(G_n)\|$	Maximum	Maximal
1	1	0	1	1
2	1	1	2	2
3	4	2	2	2
4	12	34	4	4
5	40	274	4	3,4
6	116	2508	6	6
7	200	6960	6	5,6
8	444	37228	8	6,8
9	760	103298	8	6,7,8
10	2160	847198	10	6,7,8,10
11	4368	3388642	10	7,8,9,10
12	7852	11142006	12	7,8,10,11,12
13	12828	29478750	11-12	7,8,10,11
14	17252	53281396	12-13	8,9,10
15	19612	68683920	11-14	9,10,11
16	21104	79791926	16	8,10,11,16
17	18276	59541992	12-16	8,9,10,11
18	15096	40454286	18	9,10,11,18
19	10240	18556592	12	9,10,11,12
20	6464	7384300	12	8,9,10,11,12
21	3536	2196898	11	8,9,10,11
22	2052	766072	22	22
23	872	129848	9	6,7,8,9
24	200	7288	8	5,7,8
25	88	1032	5	5
26	56	6239	6	4,6
27	204	7784	8	6,8
28	712	128744	28	28
29	164	3704	6	5,6

The following are some comments and observations concerning the results presented in Table 3.1.

- We give the precise value for the maximum clique sizes for G_{19} and G_{20} (both found to be 12) and for G_{29}. We also find *lower bounds* for the size of the maximum cliques in G_{13}, G_{14}, G_{15} and G_{17}. These are new results. All other values for the maximum clique size agree with the prior results reported in [2].

- All values given for the sizes of maximal cliques are new to this paper.

- Some searches did not finish. Our computer resources included a maximum wall-time of 77 days before the computer nodes needed to be restarted. We did not complete those searches which required more time than the 77 days allowed. Unfortunately, if the search does not complete, we do not get any indication of how close the program was to termination.

- Note that for order 14 a maximal clique of size 10 was found, yet the size of the maximum clique is either size 12 or 13. This indicates that the program found a clique of size 12, however it ran out of time before determining whether or not this clique was maximal. The value of 13 is from [2]. Similarly for $n = 17$, a maximal clique of size 11 was found and a clique of size 12 was also found, but the program ran out of time before determining if the clique of size 12 was maximal. Again, in [2] it is given that that the size of the maximum clique is at most 16.

- Note that when $n = 2, 3, 4, 6, 22, 25$ and 28 there are no maximal cliques that are not maximum. Hence for these orders, *any* set of disjoint Costas arrays of order n can be extended to a set of maximum size. We find this to be very surprising.

- There are cases where the four rotations of a Costas array are mutually disjoint. We term such a Costas array a *super Costas array*. We searched for sets of disjoint super Costas arrays in the sense that if C and D are super Costas arrays, then every rotation of C is disjoint from D. If n disjoint super Costas arrays are found then we have $4n$ disjoint Costas arrays. One particularly nice example of this is given in Example 3.1 below. It is a MCPLS(20,12) composed of the rotations of 3 disjoint super Costas arrays.

- It is interesting to note that some of the maximal partial Costas latin squares are very sparse. We found MCPLS(26,4), MCPLS(29,5), MCPLS(25,5) and MCPLS(27,6) which are only 15.34%, 17.24% , 20% and 22.22% filled, respectively. A MCPLS(26,4) is given in Example 3.2 below.

- The undergraduate honors thesis of the first author is accessible online at [8]. In that thesis one can find an example of a maximal Costas latin square as well as a Costas latin square for each known order. Other examples found in that thesis include a super Costas array of each possible order and a frequency map of the filled cells in the set of all Costas arrays of each size n with $3 \leq n \leq 29$.

Example 3.1 *A MCPLS(20,12) composed of the rotations of 3 disjoint super Costas arrays.*

	9	8	10	3	5	4							2	7	1	12	6	11	
1	6	9	4	7				12			10				5	2	11	8	3
2		11	12		6		7		1	3		5		8		10	9		4
	1	10		5		8	11		4	2		9	6		7		12	3	
	8	7		1	4		12	9			11	10		2	3		5	6	
3			11		12		5	2	6	8	4	7		10		9			1
4			7	9		1		8	10	12	6		3		11	5			2
	11				2	10	3	7	8	6	5	1	12	4				9	
5		12		2		6		3	11	9	1		8		4		10		7
6	3		5		10	2	9					11	4	12		7		1	8
7	10		8		3	11	4					2	9	1		6		12	5
8		1		11		7		10	2	4	12		5		9		3		6
	2				11	3	10	6	5	7	8	12	1	9				4	
9			6	4		12		5	3	1	7		10		2	8			11
10			2		1		8	11	7	5	9	6		3		4			12
	5	6		12	9		1	4			2	3		11	10		8	7	
	12	3		8		5	2		9	11		4	7		6		1	10	
11		2	1		7		6		12	10		8		5		3	4		9
12	7	4	9	6				1			3				8	11	2	5	10
	4	5	3	10	8	9							11	6	12	1	7	2	

Example 3.2 *A MCPLS(26,4). This square has only 15.34% of cells filled.*

			4										2							1			3		
						2								4			1			3					
					1				4								2	3							
					3			2	1															4	
							1					4						2			3				
			3							1						4							2		
						4				3	2										1				
1					2							3			4										
	4	2										1										3			
						1		4							2		3								
		4				3	2												1						
			2								3				1					4					
	2						3						4										1		
2											4								3						1
										4				2		1									3
	1						4									3					2				
										2	1				3		4								
				1					3										2			4			
		3		4					2													1			
		1		2	4								3												
	3		1													2		4							
				3														1			4	2			
								3															4	1	2
3												2		1					4						
													1							2				3	4
4								1						3										2	

4 Acknowledgements

The authors acknowledge the Vermont Advanced Computing Core which is supported by NASA (NNX 06AC88G), at the University of Vermont for providing High Performance Computing resources that have contributed to the research results reported within this paper.

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