Two new infinite families of extremal class-uniformly resolvable designs

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Abstract

In 1991, Lamken, Rees and Vanstone introduced the notion of class-uniformly resolvable designs, CURDs. These are resolvable pairwise balanced designs PBD(v, K, λ) in which given any two resolution classes C and C', for each $k \in K$ the number of blocks of size k in C is equal to the number of blocks of size k in C'. Danzinger and Stevens showed that if a CURD has v points, then $v \leq (3p_3)^2$ and $v \leq (p_2)^2$ where p_i denotes the number of blocks of size i for $i = 2, 3$. They then constructed an infinite class of *extremal* CURDS with $v = (3p_3)^2$ when p_3 is odd and an infinite class with $v = (p_2)^2$ when $p_2 \equiv 2 \pmod 6$. In this note, we construct two new infinite families of extremal CURDs, when $v = (3p_3)^2$ for all $p_3 \geq 1$ and when $v = (p_2)^2$ with $p_2 \equiv 0 \pmod 3$ except possibly when $p_2 = 12$.

1 Introduction

Let K be a set of positive integers. A *pairwise balanced design* PBD$(v, K, 1)$ is a pair $(V, \mathcal{B})$ where $|V| = v$, and B is a collection of subsets of V called blocks. Each subset has size $k \in K$ and each pair of points of V occurs exactly one time in the blocks.

A *group divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

1. G is a partition of a set X (of *points*) into subsets called *groups*,
2. B is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,
3. every pair of points from distinct groups occurs in a unique block.

The *group type* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We use an "exponential" notation to describe types: a GDD has type ${t_1}^{u_1}{t_2}^{u_2} \dots {t_k}^{u_k}$ if there are u_i groups of size t_i

for $1 \leq i \leq k$. A GDD $(X, \mathcal{G}, \mathcal{B})$ will be referred to as a K-GDD of type $t_1{}^{u_1} t_2{}^{u_2} \ldots t_k{}^{u_k}$ if $|B| \in K$ for every block B in $\mathcal{B}$.

A *parallel class* (or *resolution class*) of a PBD is a subset of the blocks that partitions the set of points V. If the set of blocks can be partitioned into parallel classes, then the design is *resolvable*, and we write RPBD$(v, K, 1)$. Given an RPBD$(v, K, 1)$ with $K = \{k_1, \ldots, k_n\}$, if for each k_i there is a corresponding p_i such that $\sum_{i=1}^{n} p_i k_i = v$ and each resolution class contains exactly p_i blocks of size k_i, then the RPBD is a *class-uniformly resolvable design* or CURD. In this case the CURD has partition $k_1^{p_1} k_1^{p_1} \cdots k_n^{p_n}$.

CURDs were first defined and investigated in 1991 by Lamken, Rees and Vanstone [7]. In that paper the authors focus mainly on CURDs with block size 2 and 3 and describe two notable infinite classes of CURDs. In 2001, Danzinger and Stevens [3] continued the investigation by completeing one of the classes from the earlier paper. In addition, they prove the bounds given below in Theorem 1.1 on CURDs which contain only blocks of sizes 2 and 3. Denote a CURD on v points with r resolution classes and p_2 blocks of size 2 and p_3 blocks of size 3 in each resolution class (and no blocks of any other size) as a CURD(v, r, p_2, p_3).

Theorem 1.1 *[3] Assume there exists a CURD(v, r, p_2, p_3). Then*
(a) $v \leq (p_2)^2$ if $p_2 > 1$, and
(b) $v \leq (3p_3)^2$ if $p_3 > 0$.

Motivated by the bounds in Theorem 1.1, a CURD(v, r, p_2, p_3) with $v = (p_2)^2$ or with $v = (3p_3)^2$ is termed an *extremal* CURD. In [3] it was shown that there exists an extremal CURD(v, r, p_2, p_3) with $v = (3p_3)^2$ when p_3 is odd and with $v = (p_2)^2$ when $p_2 \equiv 2 \pmod 6$. In this note we will construct extremal CURDs when $v = (3p_3)^2$ for all $p_3 > 0$ and when $v = (p_2)^2$ for all $p_2 \equiv 0 \pmod 3$ except when $p_2 = 12$.

2 Extremal CURDs with $v = (3p_3)^2$

First note that in the case when $v = (3p_3)^2$ it necessarily follows that $p_2 = \frac{3p_3(3p_3-1)}{2}$. In [3], Danzinger and Stevens construct a CURD$((3k)^2, 3k(3k-1), \frac{3k(3k-1)}{2}, k)$ for all k odd. In this section we will use a doubling-type construction to prove that these extremal CURDs also exist when k is even and hence that there exists a CURD$((3k)^2, 3k(3k-1), \frac{3k(3k-1)}{2}, k)$ for all $k \geq 1$. We begin with a definition.

A $\{2, 3\}$-GDD of type $(9k^2)^4$ is said to be *3-heavy* if the blocks can be partitioned as follows:

1. $18k^2$ parallel classes with $2k$ blocks of size 3 and $3k(6k-1)$ blocks of size 2 in each class.

2. $18k^2 - 6k$ sets of blocks of size 2 such that each set covers all the points in exactly 2 groups in such a way that every group is missed by the blocks in exactly $9k^2 - 3k$ sets.

The following is our main recursive construction.

Theorem 2.1 *If there exists a 3-heavy* $\{2,3\}$*-GDD of type* $(9k^2)^4$ *and a CURD*$((3k)^2, 3k(3k-1), \frac{3k(3k-1)}{2}, k)$*, then there exist a CURD*$((6k)^2, 6k(6k-1), 3k(6k-1), 2k)$.

Proof: Given a 3-heavy $\{2,3\}$-GDD of type $(9k^2)^4$ by definition there are $18k^2$ parallel classes with $2k$ blocks of size 3 and $3k(6k-1)$ blocks of size 2 in each class, as required.

Now put a CURD$((3k)^2, 3k(3k-1), \frac{3k(3k-1)}{2}, k)$ on the points of each of the groups. We have that each group is missed by exactly $9k^2 - 3k$ of the sets of size 2, and each group contains the blocks of a CURD with exactly this many parallel classes. So for each of the $18k^2 - 6k$ sets of block size 2 add a parallel class from the each of the two groups that it misses to construct a parallel class of blocks containing $9k^2 + 2 \times \frac{3k(3k-1)}{2} = 18k^2 - 3k$ blocks of size 2 and $2k$ blocks of size 3. This completes the construction. ∎

It was shown in [3] that a CURD$((3k)^2, 3k(3k-1), \frac{3k(3k-1)}{2}, k)$ exists for all k odd (including $k = 1$). So in view of the previous theorem CURD$((3k)^2, 3k(3k-1), \frac{3k(3k-1)}{2}, k)$ will exist for all k if it can be shown that 3-heavy $\{2,3\}$-GDD of type $(9k^2)^4$ exist for all k. These 3-heavy GDD will be contructed by an inflation construction given in Lemma 2.2. We need another definition.

A $\{2,3\}$-GDD of type $(9k)^4$ is *3-light* if it satisifies the following conditions:

1. there are $18k$ parallel classes of blocks with 2 blocks of size 3 and $3(6k-1)$ blocks of size 2 in each class, and
2. there are $18k - 6$ sets of block size 2 such that each set covers the points in exactly 2 groups, in such a way that every group is missed by exactly $9k - 3$ sets.

Lemma 2.2 *If there exists a 3-light* $\{2,3\}$*-GDD of type* $(9k)^4$ *and a resolvable transversal design TD*$(3,k)$*, then there exists a 3-heavy* $\{2,3\}$*-GDD of type* $(9k^2)^4$.

Proof: Given a 3-light GDD of type $(9k)^4$, first inflate each point to k points. Then, apply Wilson's construction (see [6]) for resolvable GDDs (using the resolvable TD$(3,k)$ or a resolvable TD$(2,k)$) to construct 3-heavy $\{2,3\}$-GDD of type $(9k^2)^4$. Note that the number of parallel classes in the resulting 3-heavy GDD as well as the sets containing only blocks of size 2 will be k times the number in 3-light GDD, as required . ∎

Theorem 2.3 *There exists a 3-light* $\{2,3\}$*-GDD of type* $(9k)^4$ *for all* $k \geq 1$.

Proof: *Case 1: k odd* Let $V = \mathbb{Z}_{36k}$. The groups are $\{\{4j + i : j \in \mathbb{Z}_{9k}\}$ for $i = 0, 1, 2, 3$. Consider the base blocks

$$\{0, 1, 3\},$$
$$\{i, 18k + 3 - i\} \text{ for } i = 4, 5, \ldots, 9k - 1,$$
$$\{27k, 9k + 3\},\ \{27k + 1, 9k + 2\}, \text{ and } \{2, 18k + 2\}.$$

By adding $18k$ to all but the last block, we get all $36k$ elements, each exactly once. Note that no two blocks have elements with the same difference and that the set of differences that occur in these blocks is $\{1, 2, 3, 5, \ldots 18k-3, 18k-1, 18k, 18k+1, 18k+3, \ldots 36k-5, 36k-3, 36k-2, 36k-1\}$. Develop these blocks by adding $0, 1, \ldots, 18k-1$ to obtain $18k$ disjoint parallel classes where each parallel class has exactly 2 blocks of size 3. Note that all pairs of points with an odd difference (or difference 2 or $18k$) have now appeared together.

If two points have difference $d \equiv 0 \pmod 4$, then they are in the same group and so can not appear in a block together. For each $d \equiv 2 \pmod 4$ with $6 < d \leq 36k-6$ and $d \neq 18k$ construct two sets of blocks $B_{0,d}$ and $B_{1,d}$ where $B_{i,d} = \{\{x, x+d\} | x \equiv i \pmod 4\}$. Note that each set misses exactly two of the groups of the GDD and that the addition of these blocks completes the GDD. Finally, the fact that there are $(9k-3) \times 2 = 18k-6$ of these sets (and clearly every group is missed by exactly $9k-3$ of them) completes the proof.

Case 2: k even Let $k = 2m$ and $V = \mathbb{Z}_{18m} \times \mathbb{Z}_4$. The groups are $\mathbb{Z}_{18m} \times \{i\}$ for $i \in \mathbb{Z}_4$. Consider the base blocks

$$\begin{gathered}\{(0,0),(0,1),(18m-1,2)\} \\ \{(18m-1-i,2),(i,1)\} \text{ for } i = 1, 2, \ldots, 9m-1, \\ \{(9m-i,2),(9m+i,1)\} \text{ for } i = 1, 2, \ldots, 9m-1, \\ \{(9m,1),(9m,3)\}\end{gathered}$$

By adding $(0,2)$ to all but the last block we get all $36k$ elements, each exactly once. Again note that no two blocks have elements with the same difference, hence when these blocks are developed by adding (i,j) where $i \in \mathbb{Z}_{18m}$ and $j = 0, 1$, we obtain $36m = 18k$ disjoint parallel classes where each parallel class has exactly 2 blocks of size 3.

It is easy to check that any two points whose difference is $(d,1)$, for any $d \in \mathbb{Z}_{18m}$, appear in one of the blocks constructed above. Also, points with difference (0,2) and $\pm$(1,2) have appeared together. This leaves exactly those pairs of points whose difference is $(d,2)$ for $18m-3 = 9k-3$ values of d to still be covered by blocks. For each $d \in \mathbb{Z}_{18m}$, $d \neq 0, 1, -1$, define two sets of blocks $\{\{(x,0),(x+d,2)\} \mid x \in \mathbb{Z}_{18m}\}$ and $\{\{(x,1),(x+d,3)\} \mid x \in \mathbb{Z}_{18m}\}$. Each set misses exactly two of the groups of the GDD and by the difference property, the addition of these blocks completes the GDD. Finally, the fact that there are $(9k-3) \times 2 = 18k-6$ of these sets completes the proof. ∎

Since a resolvable TD$(3,k)$ exists for all $k \neq 2, 6$ (see [1]), from Lemma 2.2 and Theorem 2.3 we obtain the following theorem.

Lemma 2.4 *For every $k \neq 2, 6$, there exists a 3-heavy GDD of type $(9k^2)^4$.*

We now fill in both of the missing cases from the previous lemma.

Lemma 2.5 *There exists a 3-heavy GDD of type $(9k^2)^4$ when $k = 2$ and when $k = 6$.*

Proof: We first construct a 3-heavy GDD of type $(36)^4$ (i.e. type $(9k^2)^4$ when $k = 2$). Let the elements of this design be the elements of $\mathbb{Z}_{144}$ and the groups of the design be the 4 translates of $G = \{0, 4, .., 140\}$. Begin with the following base blocks: $\{0, 70, 71\}, \{2, 67, 69\}, \{34, 68\}$,

$\{73, 35\}$, and $\{\{3+i, 66-i\}, 0 \leq i \leq 30\}$. Now, take these base blocks, and add 72 to construct the first parallel class of blocks. Note that this class has 4 blocks of size 3, and 66 blocks of size 2, as required. Translating this parallel class by adding $i = 0, 1, \ldots, 71$ generates the 72 disjoint parallel classes in the GDD. Now, the unused differences from the base blocks are ± 69 and all $d \equiv 2 \pmod 4$ except $d = \pm 2, \pm 34, \pm 38$, and ± 66 (all values modulo 144). For each of the unused differences $d \equiv 2 \pmod 4$ construct two sets of blocks of size 2 (36 in each set) with all blocks having difference d. All the pairs in each set will be of the form $\{x, x+d\}$ with $x \in G$ and $x \in G+1$, respectively. This constructs a total of 56 sets of blocks. From difference 69, construct four sets of blocks of size 2 with all blocks having difference d. All the pairs in each set will be of the form $\{x, x+69\}$ with $x \in G, G+1, G+2$ and $G+3$, respectively. We have constructed the required 60 sets of blocks of size 2, completing the proof.

To construct a 3-heavy GDD of type $(9k^2)^4$ when $k = 6$, we first construct a suitable GDD of type $(9 \cdot 12)^4 = (108)^4$. Consider $V = \mathbb{Z}_{432}$ with groups $\{i, i+4, \ldots, i+428\}$ for $i = 0, 1, 2, 3$. We first construct a set of 108 parallel classes, each with four blocks of size 3 and 210 blocks of size 2. Consider the collection of blocks $\{0, 9, 10\}, \{1, 3, 6\}, \{2+216, 8\}, \{4, 11\}, \{5+216, 7\}, \{12+i, 215-i\}$ for $i = 0, 1, 2, \ldots, 96$, and $\{109+216+i, 118-i\}$ for $i = 0, 1, 2, 3, 4$. It is easy to verify that all differences are distinct and not a multiple of 4 and that when 216 is added to each block that all elements in V are contained in exactly one block. We have thus constructed a parallel class with 4 blocks of size 3 and 210 blocks of size 2. Translating this parallel class by adding $i = 0, 1, \ldots, 107$ generates the 108 disjoint parallel classes. As before, the sets of block size 2 are constructed from the differences missing from the parallel class. Now, applying Wilson's Fundamental Construction (see [6]) with weight 3 gives the desired 3-heavy GDD of type $(9 \cdot 6^2)^4$. ∎

We can now state our main result regarding this extremal class of CURDs.

Theorem 2.6 *For all $n > 1$ there exists a CURD$((3n)^2, 3n(3n-1), \frac{3n(3n-1)}{2}, n)$, i.e. there exists an extremal CURD with $v = (3p_3)^2$ for all values of $p_3 \geq 1$.*

Proof: Let $n = 2^t \cdot k$ with $k \geq 1$ odd. From [3] there exists a CURD$((3k)^2, 3k(3k-1), \frac{3k(3k-1)}{2}, k)$ and from Lemmas 2.4 and 2.5 there is a 3-heavy $\{2,3\}$-GDD of type $(9k^2)^4$. Apply Theorem 2.1 t times to construct a CURD$((3n)^2, 3n(3n-1), \frac{3n(3n-1)}{2}, n)$. ∎

3 Extremal Curds with $v = p_2^2$

The second extremal family from Theorem 1.1 is the family CURD$(n^2, \frac{n(n+1)}{2}, n, \frac{n(n-2)}{3})$, i.e. when $v = p_2^2$. In [3] Danziger and Stevens prove that the basic necessary condition is $n \equiv 0, 2 \pmod 3$ and they show that such CURDs exists whenever $n \equiv 2 \pmod 6$. In this section, we construct an extremal CURD$(n^2, \frac{n(n+1)}{2}, n, \frac{n(n-2)}{3})$ for all $n \equiv 0 \pmod 3$. We divide the proof into two cases, $n \equiv 0 \pmod 6$ and $n \equiv 3 \pmod 6$.

We define a *class-disjoint* PBD$(n, \{2,3\})$ is a regular PBD$(n, \{2,3\})$ with exactly n blocks of size 2 that can be partitioned into n classes of pairwise disjoint blocks, each class containing exactly one block of size 2. Note that in a class-disjoint PBD($n, \{2,3\}$) it follows that each point is in $\frac{n+1}{2}$ blocks and hence each point is *not in* $\frac{n-1}{2}$ of the classes.

Theorem 3.1 *For all $n \equiv 3 \pmod 6$, there exists a class-disjoint PBD($n, \{2,3\}$).*

Proof: From [5] there exists a cyclic Steiner triple system on n points with the property that the base blocks are all disjoint. Let B be the set of base blocks in such a system. One of the base blocks is the *short block*, namely $\{0, \frac{n}{3}, \frac{2n}{3}\}$. To construct the first class, replace this short block with $\{0, \frac{n}{3}\}$ and attach the remaining blocks from B. Note that the blocks in this class are pairwise disjoint. The n classes are constructed by developing this first class modulo n. ∎

One other ingredient is needed for our general construction. A *modified group-divisible design* with block size 3 (3-MGDD) of *type u^v*, is a set $U \times V$ with $|U| = u$ and $|V| = v$, partitioned into *first groups* $\{\{u\} \times V : u \in U\}$, and into *second groups* $\{\{U \times \{v\} : v \in V\}$, and equipped with a collection $\mathcal{B}$ of blocks of size 3, so that every pair of elements appears either in a first or second group together, or in exactly one block in $\mathcal{B}$, but not both. A 3-MGDD is resolvable (3-RMGDD) if the blocks can be partitioned into parallel classes each class containing each point exactly once. The following recent theorem of Wang, Tang and Danziger [9] details the existence of 3-RMGDD.

Theorem 3.2 *[9] There exists a 3-RMGDD of type u^v if and only if $u \geq 3,\ v \geq 3,\ uv \equiv 0 \pmod 3$ and $(u-1)(v-1) \equiv 0 \pmod 2$ except when $(u, v) = (3, 6)$ or $(6, 3)$.*

Theorem 3.3 *There exists a CURD($n^2, \frac{n(n+1)}{2}, n, \frac{n(n-2)}{3}$) whenever $n \equiv 3 \pmod 6$.*

Proof: Let $V = \mathbb{Z}_n \times \mathbb{Z}_n$. We construct the blocks of the CURD in two stages, first we will construct n classes consisting of blocks which contain points in the same "row" or the same "column" of V, then we give the classes of blocks containing the other pairs of points.

Since $n \equiv 3 \pmod 6$, from Theorem 3.1 above, there exists a class-disjoint PBD($n, \{2,3\}$) $\mathcal{D}$ with n classes $\{B_0, B_1, \ldots, B_{n-1}\}$. Also, it is well known that when $n \equiv 3 \pmod 6$ there is a Kirkman triple system K of order n (with $\frac{n-1}{2}$ parallel classes). For each $i = 0, 1, \ldots n-1$ and each block $\{x, y, z\}$ (or $\{a, b\}$) in class B_i construct the set of blocks $\{\{(x, j), (y, j), (z, j)\} | 0 \leq j \leq n-1\}$ (or $\{\{(a, j), (b, j)\} | 0 \leq j \leq n-1\}$). Further, for each w not appearing in any of the blocks in B_i ($i = 0, 1, \ldots n-1$) place a parallel class from K on the points $\{w\} \times \mathbb{Z}_n$ in such a way that for each of the $\frac{n-1}{2}$ times that a class from $\mathcal{D}$ misses w, a *different* parallel class from K is used. We have constructed n classes (one for each B_i) with n blocks of size 2 and $\frac{n(n-2)}{3}$ blocks all of size 3. Note that any two points in the same row or column of V have now appeared together in exactly one block.

From Theorem 3.2, there exist a 3-RMGDD on the points $V = \mathbb{Z}_n \times \mathbb{Z}_{n+1}$. Now, truncate the RMGDD by deleting the last column, $\mathbb{Z}_n \times n$. Note that now some blocks have size 2, in fact, every parallel class of the truncated RMGDD now has exactly n blocks of size 2 and $\frac{n(n-2)}{3}$ blocks of size 3. We now see that no pairs covered by this truncated MGDD are in the same row or column of V and that if two points are in different rows or columns of V, then they are in a block of the truncated RMGDD. This completes the proof.

∎

We now consider the existence of CURD($n^2, \frac{n(n+1)}{2}, n, \frac{n(n-2)}{3}$) in the case where $n \equiv 0 \pmod 6$. The proof is fairly similar to the one for $n \equiv 3 \pmod 6$, except that since there is no resolvable STS on n points we are forced to use resolvable group divisible designs and frames. As before we need a few definitions.

Let $(X, \mathcal{G}, \mathcal{A})$ be a 3-GDD of type 2^{3n} with points $X = \{0, 1, \ldots 6n-1\}$ and groups $\mathcal{G} = \{\{i, i+3n\} \mid 0 \leq i \leq 3n-1\}$. We say that $(X, \mathcal{G}, \mathcal{A})$ is *class disjoint* if the blocks can be partitioned into $6n$ classes B_i, $0 \leq i \leq 6n-1$, such that for each i the blocks in class B_i are disjoint and miss the group $\{i, i+3n\}$ ($i+3n$ is modulo $6n$). Note that these classes are not required to be full parallel classes. In addition, since the replication number of a 3-GDD of type 2^{3n} is $3n-1$ and since there are $6n$ classes, we see that each point $x \in X$ *does not* appear in precisely $6n - (3n-1) = 3n+1$ of the classes.

A *3-frame of type g^n* is a 3-GDD $(X, \mathcal{G}, \mathcal{A})$ of type g^n with the additional property that the blocks can be partitioned into partial parallel classes (called *frame parallel classes*) so that each frame parallel class partitions the points of $X \setminus G_i$ for some $G_i \in \mathcal{G}$.

Lemma 3.4 *For every $n \geq 3$ there exists a class disjoint 3-GDD of type 2^{3n}.*

Proof: We begin with a 3-frame of type 6^n which was shown to exist in [8] if $n \geq 4$ Let the points of this frame be $X = \{0, 1, \ldots 6n-1\}$ and the groups be $G_i = \{0+i, n+i, 2n+i, 3n+i, 4n+i, 5n+i\}$ for $0 \leq i \leq n-1$. We partition these groups further into the groups $\Gamma_i = \{\{i, i+3n\} \mid 0 \leq i \leq 3n-1\}$ to construct the class disjoint 3-GDD. Remember, we need to construct the classes of blocks B_i, $0 \leq i \leq 6n-1$, such that for each i the blocks in class B_i are disjoint and miss the group Γ_i.

It is easy to show that for each group G_i, $0 \leq i \leq n-1$, there are three frame parallel classes missing G_i. Call these classes B_i, B_{n+i} and B_{2n+i}. Notice that we have just defined $3n$ of the classes of the class disjoint 3-GDD (B_i for $0 \leq i \leq 3n-1$) and that any pair of points that are in different groups in the frame have appeared together in some block in one of these classes. Also, clearly B_i misses group Γ_i for $0 \leq i \leq 3n-1$.

For each i with $0 \leq i \leq n-1$ on the points of G_i place a 3-GDD of type 2^3 with groups Γ_i, Γ_{n+i} and Γ_{2n+i}. This GDD has 4 blocks b_{i1}, b_{i2}, b_{i3} and b_{i4}. We define the remaining clases in the class disjoint GDD as follows. For each $0 \leq i \leq n-1$ let $B_{3n+i} = \{b_{(i+1)1}, b_{(i+2)2}\}$, $B_{4n+i} = \{b_{(i+1)3}\}$, and $B_{5n+i} = \{b_{(i+1)4}\}$ where all subscripts of the $b_{i,j}$ are taken modulo n. It is now straightforward to check that we have constructed a class disjoint 3-GDD on the points $X = \{0, 1, \ldots 6n-1\}$ with groups $\Gamma_i = \{\{i, i+3n\} \mid 0 \leq i \leq 3n-1\}$ and $6n$ classes of blocks B_i, $0 \leq i \leq 6n-1$ when $n \geq 4$

We now construct a class disjoint 3-GDD of type 2^9 directly. Let the points be $\mathbb{Z}_{16} \cup \{\infty_1, \infty_2\}$ and the groups be $\{\{0+i, 8+i\} \mid 0 \leq i \leq 7\} \cup \{\infty_1, \infty_2\}$. Now let $A = \{\{0, 1, 6\}, \{3, 7, 10\}\}$ and $B = \{\{\infty_1, 0, 2\}, \{\infty_2, 6, 8\}\}$. It is easy to check that the set of blocks $\{A + i \mid 0 \leq i \leq 15\} \cup \{B + i \mid i = 0, 1, 4, 5, 8, 9, 12, 13\}$ is indeed a class disjoint 3-GDD of type 2^9. ∎

Theorem 3.5 *There exists a CURD($m^2, \frac{m(m+1)}{2}, m, \frac{m(m-2)}{3}$) whenever $m \equiv 0 \pmod 6$ except possibly when $m = 12$.*

Proof: Let $m = 6n$ and let $V = \mathbb{Z}_{6n} \times \mathbb{Z}_{6n}$. We again construct the blocks of the CURD in two stages, first constructing $6n$ classes consisting of blocks which contain points in the same "row" or the same "column" of V, then giving the classes of blocks containing the other pairs of points.

We will be placing the blocks of a 3-GDD of type 2^{3n} on the rows and columns of V so we define the groups in row i as the set of points $\{\{(i,j),(i,j+3n)\} \mid 0 \le j \le 3n-1\}$. Define the groups in the columns similarly.

From Theorem 3.4 above there is a class disjoint 3-GDD of type 2^{3n}, $(X, \mathcal{G}, \mathcal{A})$, with $6n$ classes $\{B_0, B_1, \ldots, B_{6n-1}\}$ as long as $n \ge 3$. Also, from [11] there exists a resolvable 3-GDD $\mathcal{D}$ of type 2^{3n}; this GDD necessarily has $3n-1$ parallel classes.

Now let $0 \le i \le 6n-1$. For each block $\{x,y,z\}$ in class B_i of the class disjoint 3-GDD construct the set of blocks $\{\{(x,j),(y,j),(z,j)\} \mid 0 \le j \le 6n-1\}$ on V. Further, for each w not appearing in any of the blocks in B_i, except $w = i$ or $w \equiv i + 3n \pmod{6n}$, place a parallel class from $\mathcal{D}$ on the points $\{w\} \times \mathbb{Z}_{6n}$ in such a way that for each of the $3n-1$ times that a class from the class disjoint 3-GDD misses w, a *different* parallel class from $\mathcal{D}$ is used. Finally, from the definition we have that the blocks in class B_i miss the points i and $i + 3n \pmod{6n}$. If $0 \le i \le 3n-1$ add the following set of (vertical) blocks of size 2 to B_i: $\{\{(i,j),(i+3n,j)\} \mid 0 \le j \le 6n-1\}$. If $3n \le i \le 6n-1$ add the following set of (horizontal) blocks of size 2 to B_i: $\{\{(k,i-3n),(k,i)\} \mid 0 \le k \le 6n-1\}$. Notice that each of the $6n$ classes contructed on V has precisely $6n$ blocks of size 2 and all other blocks of size 3 and that each is indeed a parallel class of V. Also, any two points in the same row or column of V have now appeared together in exactly one block.

As in the case of Theorem 3.3, adding the blocks of a 3-RMGDD of type $(6n)^{6n+1}$ which has been truncated by one column gives the blocks covering all points *not* in the same row or the same column. Also, each resolution class now has precisely $6n$ blocks of size 2 and all the remaining blocks are still of size 3. This completes the proof when $n \ge 3$, i.e when $m \ge 18$. A CURD$(m^2, \frac{m(m+1)}{2}, m, \frac{m(m-2)}{3})$ when $m = 6$ is given in [7] ∎

Combining Theorems 3.3 and 3.5 yields the main result of this section.

Theorem 3.6 *There exists a CURD*$(m^2, \frac{m(m+1)}{2}, m, \frac{m(m-2)}{3})$ *whenever* $m \equiv 0 \pmod{3}$ *except possibly when* $m = 12$.

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