Disjoint Difference Families with Block Size 3

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Abstract

A disjoint (v,k,λ) difference family is a difference family with disjoint blocks. We show that disjoint (v,3,1) difference families exist for all $v \equiv 1 \bmod 6$.

1 Introduction

Let G be a group of order v. A family of k-tuples of elements from G is a (v,k,λ) difference family if the collection of orbits of the k-tuples (disregarding repeated k-tuples) under the action of G form a balanced incomplete block design, BIBD(v, k, λ). If the k-tuples are disjoint, call the family a *disjoint* (v,k,λ) *difference family.*

In this paper we show that there exists a disjoint (v,3,1) difference family for all $v \equiv 1 \bmod 6$. A more general conjecture of Novák [3] states that every cyclic STS(v) on $v \equiv 1 \bmod 6$ elements has a disjoint set of starter blocks. Novak's conjecture is almost certainly true. This paper provides some evidence for the conjecture by showing that there exists a disjoint set of starter blocks for *some* cyclic Steiner triple system of every order $v \equiv 1 \bmod 6$.

Disjoint (v,3,1) difference families for $v \equiv 1 \bmod 6$ have been shown to be useful in the construction of generalized court–balanced tournament designs [4] and uniformly resolvable designs [2] and indeed provide a partition of the blocks of a cyclic STS(v) into partial parallel classes of $(v-1)/6$ blocks each.

2 Constructions

Theorem 1 *There exists a disjoint (24k+1,3,1) difference family for* $k \geq 1$.

Begin with a ($24k$+1,3,1) difference family that does not have disjoint blocks. The idea is to translate the blocks so that no two intersect. The initial difference family is given below. It is obtained from the Skolem triple system of order $4k$ found in [1], Theorem 8.3.3. This construction requires that $k \geq 3$.

(1a)	0	1	$12k$	
(1b)	0	$4k-1$	$9k-1$	
(1c)	0	$2k$	$10k-1$	
(1d)	0	$4k$	$10k$	
(1e)	0	$2k+2r-1$	$7k+r-1$	$1 \le r \le k-1$
(1f)	0	$2k+2r$	$11k+r-1$	$1 \le r \le k-1$
(1g)	0	$2r+1$	$10k+r$	$1 \le r \le k-1$
(1h)	0	$2r$	$6k+r$	$1 \le r \le k-1$

From the difference family above we obtain the following disjoint difference family by linear translation of the triples.

(1i)	$7k-1$	$7k$	$19k-1$	if $k \equiv 0 \bmod 2$
(1j)	$7k$	$7k+1$	$19k$	if $k \equiv 1 \bmod 2$
(1k)	$2k+2$	$6k+1$	$11k+1$	
(1ℓ)	0	$2k$	$10k-1$	
(1m)	$6k$	$10k$	$16k$	
(1n)	$2r$	$2k+4r-1$	$7k+3r-1$	
(1o)	$2r+1$	$2k+4r+1$	$11k+3r$	
(1p)	$17k+4r$	$17k+2r-1$	$7k+3r$	
(1q)	$21k+r+1+s_k$	$21k+3r+1+s_k$	$3k+2r+s_k$	if $r \not\equiv 0 \bmod 3$
(1r)	$13k+r-\ell_r+5$	$13k+3r-\ell_r+5$	$19k+2r-\ell_r+5$	if $r \equiv 0 \bmod 3$ and $r \neq k-1$
(1s)	$10k+13$	$12k+1$	$17k+2$	if $k \equiv 1 \bmod 3$

where $1 \le r \le k-1$, $s_k = k \bmod 2$, and ℓ_r is defined as follows.
For $k \equiv 0 \bmod 3$: if $r \neq 3r' - \ell_{r'}$ for all $r' < r$ then $\ell_r = 0$; otherwise $\ell_r = 4$.
For $k \equiv 1 \bmod 3$: if $r \neq 3r' - \ell_{r'}$ for all $r' < r$ then $\ell_r = 0$; otherwise $\ell_r = 2$.
For $k \equiv 2 \bmod 3$: if $r \neq 3r' - \ell_{r'} + 2$ for all $r' < r$ then $\ell_r = 2$; otherwise $\ell_r = 4$.

To aid the reader, in the table below we will explicitly give the translation from the first difference family to the second one.

From	To	Add	Comments
(1a)	(1i)	$7k-1$	if $k \equiv 0 \bmod 2$
(1a)	(1j)	$7k$	if $k \equiv 1 \bmod 2$
(1b)	(1k)	$2k+2$	
(1c)	(1ℓ)	0	
(1d)	(1m)	$6k$	
(1e)	(1n)	$2r$	for $1 \le r \le k-1$
(1f)	(1o)	$2r+1$	for $1 \le r \le k-1$
(1g)	(1p)	$17k+4r$ to the triples $\{0, -(2r+1), -(10k+r)\}$	for $1 \le r \le k-1$
(1h)	(1q)	$21k+r+s_k$	for $1 \le r \le k-1$ if $r \not\equiv 0 \bmod 3$
(1h)	(1r)	$13k+r-l_r+5$	for $1 \le r \le k-2$ if $r \equiv 0 \bmod 3$
(1h)	(1s)	$10k+3$	if $r \equiv 0 \bmod 3$

Since the first set given is a difference family and the second is just obtained from the first by translations, then clearly the new set of triples is also a difference family.

It is relatively easy to check that the triples are all disjoint. A note is in order concerning the introduction of the parameter ℓ_r in (1r). ℓ_r ensures that the triples of (1r) will be disjoint from each other. Specifically, ℓ_r is chosen such that $13k + r - \ell_r + 4 \neq 13k + 3r' - \ell_{r'} + 4$ for all $r' < r$. Furthermore, ℓ_r is also chosen so that the triples of (1o) are disjoint from the triples of (1r).

The following small cases complete the proof.

$v = 25$ ($k = 1$): (0,2,9), (4,7,12), (6,10,16), (7,8,19), (12,15,20)

$v = 49$ ($k = 2$): (0,4,19), (2,7,16), (3,9,25), (6,13,23), (8,44,46), (12,20,32), (17,35,37), (21,22,45)

Theorem 2 *There exists a disjoint (24k+7,3,1) difference family for* $k \geq 0$.

We again begin with a (24k+7,3,1) difference family that does not have disjoint blocks and then translate the blocks. The initial difference family is given below. It is obtained from the Skolem triple system of order $4k + 1$ found in [1], Theorem 8.3.3. This construction requires that $k \geq 5$.

(2a)	0	1	$12k+3$	
(2b)	0	$4k-1$	$9k$	
(2c)	0	$4k$	$10k+1$	
(2d)	0	$4k+1$	$12k+1$	
(2e)	0	$2k$	$12k$	
(2f)	0	$2k+2r-1$	$7k+r$	$1 \leq r \leq k-1$
(2g)	0	$2k+2r$	$11k+r$	$1 \leq r \leq k-1$
(2h)	0	$2r+1$	$10k+r+1$	$1 \leq r \leq k-1$
(2i)	0	$2r$	$6k+r+1$	$1 \leq r \leq k-1$

From the difference family above we obtain the following disjoint difference family by linear translation of the triples.

$(2j)$	$7k-2$	$7k-1$	$19k+1$	
$(2k)$	$2k+1$	$6k$	$11k+1$	
(2ℓ)	$6k-1$	$10k-1$	$16k$	
$(2m)$	$6k+1$	$10k+2$	$18k+2$	if $k \equiv 2 \bmod 4$
$(2n)$	$6k+2$	$10k+3$	$18k+3$	if $k \equiv 1 \bmod 4$
$(2o)$	$6k+3$	$10k+4$	$18k+4$	if $k \equiv 0 \bmod 4$
$(2p)$	$6k+4$	$10k+5$	$18k+5$	if $k \equiv 3 \bmod 4$
$(2q)$	$5k+s_k$	$7k+s_k$	$17k+s_k$	
$(2r)$	$2r$	$2k+4r-1$	$7k+3r$	
$(2s)$	$2r+1$	$2k+4r+1$	$11k+3r+1$	
$(2t)$	$17k+4r+2$	$17k+2r+1$	$7k+3r+1$	
$(2u)$	$21k+r+6+s_k$	$21k+3r+6+s_k$	$3k+2r+s_k$	if $r \not\equiv 0 \bmod 3$
$(2v)$	$13k+r-\ell_r+4$	$13k+3r-\ell_r+4$	$19k+2r-\ell_r+5$	if $r \equiv 0 \bmod 3$

where $1 \le r \le k-1$, $s_k = k \bmod 2$, and ℓ_r is defined as follows.
For $k \equiv 0 \bmod 3$: if $r \ne 3r' - \ell_{r'} + 2$ for all $r' < r$ then $\ell_r = 2$, otherwise $\ell_r = 4$.
For $k \equiv 1 \bmod 3$: if $r \ne 3r' - \ell_{r'}$ for all $r' < r$ then $\ell_r = 0$, otherwise $\ell_r = 4$.
For $k \equiv 2 \bmod 3$: if $r \ne 3r' - \ell_{r'}$ for all $r' < r$ then $\ell_r = 0$, otherwise $\ell_r = 2$.

As an aid to verification we give the translations.

From	To	Add	Comments
(2a)	(2j)	$7k-2$	
(2b)	(2k)	$2k+1$	
(2c)	(2ℓ)	$6k-1$	
(2d)	(2m)	$6k+1$	if $k \equiv 2 \bmod 4$
(2d)	(2n)	$6k+2$	if $k \equiv 1 \bmod 4$
(2d)	(2o)	$6k+3$	if $k \equiv 0 \bmod 4$
(2d)	(2p)	$6k+4$	if $k \equiv 3 \bmod 4$
(2e)	(2q)	$5k+s_k$	
(2f)	(2r)	$2r$	for $1 \le r \le k-1$
(2g)	(2s)	$2r+1$	for $1 \le r \le k-1$
(2h)	(2t)	$17k+4r+2$ to the triples $\{0, -(2r+1), -(10k+r+1)\}$	
			for $1 \le r \le k-1$
(2i)	(2u)	$21k+r+6+s_k$	for $1 \le r \le k-1$ if $r \not\equiv 0 \bmod 3$
(2i)	(2v)	$13k+r-l_r+4$	for $1 \le r \le k-1$ if $r \equiv 0 \bmod 3$

We complete the proof by giving the small cases.
$v = 7$ ($k = 0$): (0,1,3)

$v = 31$ ($k = 1$): (3,6,12), (5,9,16), (8,13,21), (10,11,25), (15,17,27)

$v = 55$ ($k = 2$): (0,1,27), (2,7,17), (3,9,26), (5,12,23), (8,49,51), (10,14,34), (11,19,32), (13,22,38), (18,40,37)

$v = 79$ ($k = 3$): (2,9,24), (3,11,37), (4,13,27), (5,15,40), (7,18,34), (12,71,73), (14,72,76), (16,22,52), (17,29,48), (19,20,58), (23,36,60), (25,54,57), (28,56,61)

$v = 103$ ($k = 4$): (2,11,31), (3,13,48), (4,15,34), (5,17,51), (6,19,37), (7,21,54), (9,24,45), (14,91,93), (16,92,96), (20,28,68), (23,39,64), (26,27,77), (30,47,79), (32,71,74), (35,73,78), (38,75,82), (59,65,87)

Theorem 3 *There exists a disjoint (24k+13,3,1) difference family for* $k \ge 0$.

Once again begin with a (24k+13,3,1) difference family that does not have disjoint blocks and then translate the blocks. The initial difference family is given below. It is obtained from the O'Keefe triple system of order $4k+2$ found in [1], Theorem 8.3.5. This construction requires that $k \geq 7$.

(3a)	0	1	$11k+7$	
(3b)	0	$2k+3$	$12k+7$	
(3c)	0	$4k+1$	$10k+5$	
(3d)	0	$2k+2r+3$	$11k+r+7$	$1 \leq r \leq k-2$
(3e)	0	$2r+1$	$10k+r+5$	$1 \leq r \leq k$
(3f)	0	$2r$	$6k+r+4$	$1 \leq r \leq 2k+1$

From the difference family above we obtain the following disjoint difference family by linear translation of the triples.

$(3g)$	$7k-3$	$20k+3$	$20k+4$	
$(3h)$	$7k+2$	$9k+5$	$19k+9$	
$(3i)$	$5k+8$	$9k+9$	$15k+13$	
$(3j)$	$7k$	$11k$	$15k+4$	
$(3k)$	$15k+2$	$19k+4$	$23k+7$	
(3ℓ)	$13k+10$	$15k+10$	$20k+14$	
$(3m)$	$13k+8$	$15k+6$	$20k+11$	
for all $1 \leq r \leq k-2$:				
$(3n)$	$9k+2r$	$11k+4r+3$	$20k+3r+7$	
for all $1 \leq r' \leq k-1$:				
$(3o)$	$r'+1$	$2k+3r'+1$	$7k+2r'+5$	if $k \equiv 1 \bmod 3$
$(3p)$	$r'-1$	$2k+3r'-1$	$7k+2r'+3$	if $k \equiv 0,2 \bmod 3$
for $1 \leq r \leq k$:				
$(3q)$	$11k+2r$	$11k+4r+1$	$21k+3r+5$	if $k \equiv 0,1 \bmod 3$
$(3r)$	$11k+2r+4$	$11k+4r+5$	$21k+3r+9$	if $k \equiv 2 \bmod 3$
for all $1 \leq r \leq k-2$:				
$(3s)$	$k+r$	$k+3r$	$7k+2r+4$	if $r \equiv 1,2 \bmod 3$
$(3t)$	$15k+r-\ell_r+5$	$15k+3r-\ell_r+5$	$9k+2r-\ell_r+1$	if $r \equiv 0 \bmod 3$

where $0 < r < k$ and $\ell_r = \begin{cases} 0 & \text{if } r \neq 3r' - \ell_{r'} \text{ for all } r' < r \\ 2 & \text{if } r = 3r' - \ell_{r'} \text{ for some } r' < r \end{cases}$

The translations from the first difference family to the second one now follow.

From	To	Add	Comments
(3a)	(3g)	$20k+3$	
(3b)	(3h)	$7k+2$	
(3c)	(3i)	$5k+8$	
(3d)	(3n)	$9k+2r$	
(3e)	(3q)	$11k+2r$	if $k \equiv 0,1 \bmod 3$
(3e)	(3r)	$11k+2r+4$	if $k \equiv 2 \bmod 3$
(3f)	(3s)	$k+r$	if $r \equiv 1,2 \bmod 3$ and $1 \leq r \leq k-2$
(3f)	(3t)	$15k+3r-\ell_r+5$	to the triples $\{0, -(2r), -(6k+r+4)\}$ for $1 \leq r \leq k-2$, $r \equiv 0 \bmod 3$

From	To	Add	Comments
(3f)	(3m)	$13k+8$	if $r=k-1$
(3f)	(3ℓ)	$13k+10$	if $r=k$
(3f)	(3o)	$r'+1$	for $k+1 \leq r \leq 2k-1$ (let $r=k+r'$) if $k \equiv 1 \bmod 3$
(3f)	(3p)	$r'-1$	for $k+1 \leq r \leq 2k-1$ (let $r=k+r'$) if $k \equiv 0,2 \bmod 3$
(3f)	(3j)	$7k$	if $r=2k$
(3f)	(3k)	$15k+2$	if $r=2k+1$

The proof is completed with the small cases below.
$v=13$ ($k=0$) (0,1,4), (2,4,10)

$v=37$ ($k=1$) (0,8,21), (1,23,34), (2,9,19), (4,10,29), (11,16,25), (27,28,30)

$v=61$ ($k=2$): (0,6,19), (11,43,44), (12,22,33), (15,24,40), (16,23,47), (18,26,38), (28,31,54), (30,35,57), (34,36,51), (37,41,55)

$v=85$ ($k=3$): (0,8,26), (1,11,28), (4,6,27), (16,30,45), (18,63,64), (21,33,49), (22,31,65), (23,36,58), (29,40,70), (34, 66, 72), (35,38,71), (37,42,74), (39,46,77), (44,48,68)

$v=109$ ($k=4$): (2,12,35), (3,15,37), (4,18,39), (5,7,34), (6,10,36), (19,33,54), (21,39,58), (25,83,84), (26,43,71), (28,44,64), (29,51,90), (30,41,85), (31,55,93), (43,81,87), (48,53,95), (50,57,98), (52,61,101), (62,70,94)

$v=133$ ($k=5$): (0,12,40), (1,15,42), (2,18,44), (3,21,46), (6,8,41), (7,11,43), (26,48,71), (31,102,103), (33,54,88), (35,55,79), (37,50,104), (47,62,110), (49,66,113), (51,70,116), (52,96,102), (61,64,117), (63,68,120), (65,72,123), (67,76,126), (69,80,129), (73,81,111), (75,85,114)

$v=157$ ($k=6$): (39,123,124), (38,63,103), (42,66,94), (92,118,145), (88,100,134), (86,96,131), (56,73,130), (58,77,133), (60,81,136), (62,85,139), (0,14,47), (1,17,49), (2,20,51), (3,23,53), (4,26,55), (70,75,137) (72,79,140) (74, 83,143) (76,87,146) (78,91,149), (7,9,48), (8,12,50), (98,104,61), (10,18,54), (6,21,85), (31,34,97)

Theorem 4 *There exists a disjoint (24k+19,3,1) difference family for* $k \geq 0$.

Again begin with a ($24k$+19,3,1) difference family that does not have disjoint blocks and then translate the blocks. The initial difference family is given below. It is obtained from the O'Keefe triple system of order $4k$+3 found in [1], Theorem 8.3.5. This construction requires that $k \geq 3$.

(4a)	0	1	$9k+8$	
(4b)	0	$2k+1$	$12k+10$	
(4c)	0	$4k+3$	$10k+8$	
(4d)	0	$2r+1$	$10k+r+9$	$1 \leq r \leq k-1$
(4e)	0	$2k+2r+1$	$11k+r+8$	$1 \leq r \leq k$
(4f)	0	$2r$	$6k+r+5$	$1 \leq r \leq 2k+1$

From the difference family above we obtain the following difference family by linear translation of the triples.

$(4g)$	$20k+11$	$20k+12$	$5k$	
$(4h)$	$18k+8$	$20k+9$	$6k-1$	
$(4i)$	$15k+8$	$19k+11$	$k-3$	
$(4j)$	$k-2$	$5k-1$	$13k+6$	
$(4k)$	$13k+4$	$15k+6$	$20k+10$	if $k \not\equiv 2 \bmod 3$
for $1 \leq r \leq k-1$:				
(4ℓ)	$11k+2r$	$11k+4r+1$	$21k+3r+9$	if $k \equiv 0,2 \bmod 3$
$(4m)$	$11k+2r+4$	$11k+4r+5$	$21k+3r+13$	if $k \equiv 1 \bmod 3$
$(4n)$	$9k+2r+2$	$11k+4r+3$	$20k+3r+10$	
for $1 \leq r' \leq k$:				
$(4o)$	$r'-4 \pmod v$	$2k+3r'-2$	$7k+2r'+2$	
for $1 \leq r \leq k$:				
$(4p)$	$k+r$	$k+3r$	$7k+2r+5$	if $r \not\equiv 0 \bmod 3$ and $k \not\equiv 2 \bmod 3$
$(4q)$	$k+r-2$	$k+3r-2$	$7k+2r+3$	if $r \not\equiv 0 \bmod 3$ and $k \equiv 2 \bmod 3$
for $1 \leq r \leq k+1$:				
$(4r)$	$15k+3r+6$	$15k+r+6$	$9k+2r+1$	if $r \equiv 3,6 \bmod 9$
$(4s)$	$15k+3r-\ell_r+4$	$15k+r-\ell_r+4$	$9k+2r-\ell_r-1$	if $r \equiv 0 \bmod 9$

where $1 \leq r \leq k+1$ and $\ell_r = \begin{cases} 0 & \text{if } r \neq 3r'-\ell_r \text{ for some } r'<r, \\ 2 & \text{if } r = 3r'-\ell_r \text{ for some } r'<r. \end{cases}$

We give the translations from the first difference family to the second one.

From	To	Add	Comments
(4a)	(4g)	$20k+11$	
(4b)	(4h)	$18k+8$	
(4c)	(4i)	$15k+8$	
(4d)	(4ℓ)	$11k+2r$	if $k \equiv 0,2 \bmod 3$
(4d)	(4m)	$11k+2r+4$	if $k \equiv 1 \bmod 3$
(4e)	(4n)	$9k+2r+2$	if $1 \leq r \leq k-1$
(4e)	(4j)	$k-2$	if $r=k$
(4f)	(4p)	$k+r$	$1 \leq r \leq k$, $r \not\equiv 0 \bmod 3$ and $k \not\equiv 2 \bmod 3$
(4f)	(4q)	$k+r-2$	$1 \leq r \leq k$, $r \not\equiv 0 \bmod 3$ and $k \equiv 2 \bmod 3$
(4f)	(4k)	$13k+4$	if $r=k+1$ and $k \not\equiv 2 \bmod 3$
(4f)	(4r)	$15k+3r+6$	to the triples $\{0,-(2r),-(6k+r+4)\}$ for $1 \leq r \leq k+1$, $r \equiv 3,6 \bmod 9$

From	To	Add	Comments
(4f)	(4s)	$15k + 3r - \ell_r + 4$	to the triples $\{0, -(2r), -(6k + r + 4)\}$ for $1 \leq r \leq k + 1$, $r \equiv 0 \bmod 9$
(4f)	(4o)	$r' - 4$	for $k + 2 \leq r \leq 2k + 1$ (let $r = k + 1 + r'$)

To complete the proof we now provide the small cases.

$v = 19$ ($k = 0$): (0,1,5), (6,8,14), (7,10,17)

$v = 43$ ($k = 1$): (31,32,5), (6,9,28), (23,30,41), (33,38,10), (8,12,21), (40,3,11), (2,4,14)

$v = 67$ ($k = 2$): (51,52, 10), (44,49,11), (0,9,32), (24,27,54), (22,29,53), (64,5,18), (65,8,20), (1,3,19), (2,6,21), (45,39,25), (7,46,57)

3 Conclusion

We have shown that for all $v \equiv 1 \pmod 6$ there exists a $(v, 3, 1)$ disjoint difference family. We believe that very similar techniques could be employed for the case of $v \equiv 3 \pmod 6$, but have left that for later work.

In checking the correctness of these difference families, we have used a program which generates the prescribed triples and then tests them for disjointness of both their elements and their differences. This program can now be used to generate the triples in the $(v, 3, 1)$ disjoint difference families given in this paper for $v \equiv 1 \pmod 6$. It is written in Pascal and is available at `http://www.emba.uvm.edu/~dinitz/disjoint.pas`.

4 Addendum

Peter Rodney passed away in October 1995 while this paper was in preparation. He was 30 years old. J.D. dedicates this paper to his memory.

References

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