The Hamilton-Waterloo problem: The case of triangle-factors and one Hamilton cycle

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Abstract

The Hamilton-Waterloo problem is to determine the existence of a 2-factorization of K_{2n+1} in which r of the 2-factors are isomorphic to a given 2-factor R and s of the 2-factors are isomorphic to a given 2-factor S, with $r+s=n$. In this paper we consider the case when R is a triangle-factor, S is a Hamilton cycle and $s=1$. We solve the problem completely except for 14 possible exceptions. This solves a major open case from the 2004 paper of Horak, Nedela, and Rosa.

1 Introduction

Let K_n denote the complete graph on n vertices. A *factor* in K_n is a spanning subgraph. A $k-$factor is a $k-$regular spanning subgraph of K_n and a $k-$factorization is a set of $k-$factors whose edge sets partition the graph. A fundamental question pervasive in combinatorial design theory is the *factorization problem*, i.e whether or not there exists a factorization of K_n where each of the factors are of a prescribed type. In this paper we will strictly be concerned with the existence of certain 2-factorizations. The earliest question concerning 2-factorizations (dating back to the Rev. T.P. Kirkman in 1850) is of the existence a 2-factorization of K_n where each 2-factor is the union of $\frac{n}{3}$ disjoint 3-cycles – a so-called *Kirkman triple system* or KTS(n) (Kirkman constructed a KTS(15)). It was shown in 1971 by Ray-Chadhuri and Wilson [11] and independently by Lu (see [10]) that a KTS(n) exists if and only $n \equiv 3 \pmod 6$. Generalizing to higher k, a *resolvable* $k-$cycle system of order n is a 2-factorization of K_n in which each 2-factor consists exclusively of $k-$cycles. In 1989, Alspach, Schellenberg, Stinson and Wagner [2] proved that the necessary conditions are sufficient for the existence of a resolvable $k-$cycle system of order n, namely that n is odd and that $k \equiv n \bmod (2n)$.

The well-known *Oberwolfach Problem* was first formulated by Ringel at a meeting in Oberwolfach in 1967. We denote the 2-regular graph consisting of exactly α_i m_i-cycles for $i=1,2,\ldots,t$ by $[m_1^{\alpha_1}, m_2^{\alpha_2}, \ldots, m_t^{\alpha_t}]$. The graph $[m^\alpha]$ is called a C_m-factor of K_n where $n = m \cdot \alpha$, also when $m = 3$ it will be called a *triangle-factor*. The problem of determining whether K_n (n odd) has a 2-factorization in which each 2-factor is isomorphic to $[m_1^{\alpha_1}, m_2^{\alpha_2}, \ldots, m_t^{\alpha_t}]$ is the *Oberwolfach Problem* and is denoted by $\mathrm{OP}(n; m_1^{\alpha_1}, m_2^{\alpha_2}, \ldots, m_t^{\alpha_t})$. In words, the $\mathrm{OP}(n; m_1^{\alpha_1}, m_2^{\alpha_2}, \ldots, m_t^{\alpha_t})$ asks whether it is possible to seat n participants at a conference over a series of $\frac{n-1}{2}$ days in such a way that each person sits next to each other person exactly once where there are precisely α_i tables seating m_i people for each $1 \le i \le t$. Considerable effort has been expended on this problem. The following was taken from an excellent survey of the known results on the Oberwolfach Problem given in [4]. Original references are generally given in that survey.

Theorem 1.1 *(See [4]) Other than* $\mathrm{OP}(4,5)$ *and* $\mathrm{OP}(3,3,5)$ *neither of which has a solution, the following Oberwolfach problems all have solutions.*

1. $\mathrm{OP}(mt; m^t)$ *for all* $t \ge 1$ *and* $m \ge 3$ *with* m *and* t *odd* [2];
2. $\mathrm{OP}(n; m_1^{\alpha_1}, m_2^{\alpha_2}, \ldots, m_t^{\alpha_t})$ *for* $n = \alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_t m_t \le 17$;
3. $\mathrm{OP}(3k+4; 3^k, 4)$ *for all odd* $k \ge 1$;
4. $\mathrm{OP}(3k+5; 3^k, 5)$ *for all even* $k \ge 4$;
5. $\mathrm{OP}(n; r^k, n-kr)$ *for* $n \ge 6kr - 1$, $k \ge 1$, $r \ge 3$;
6. $\mathrm{OP}(n; r, n-r)$ *for* $r = 3,4,5,6,7,8,9$ *and* $n \ge r+3$;
7. $\mathrm{OP}(n; r, r, n-2r)$ *for* $r = 3,4$ *and* $n \ge 2r+3$;
8. $\mathrm{OP}(2r+1; r, r+1)$ *for* $r \ge 3$;
9. $\mathrm{OP}(8s+3; 3, 4s, 4s)$ *for* $s \ge 1$;
10. $\mathrm{OP}(4\alpha+2s+1; 4^\alpha, 2s+1)$ *for* $s \ge 1$, $\alpha \ge 0$;

The Oberwolfach Problem is extended further in the so-called *Hamilton-Waterloo problem*. In this problem it is now assumed that the conference takes place in two venues (Hamilton and Waterloo) with different fixed configurations of tables at each site. More specifically, the *Hamilton-Waterloo problem*, denoted $\mathrm{HW}(r, s; m, k)$, is the problem of determining whether K_n (n odd) has a 2-factorization in which exactly r of the 2-factors are C_m-factors and s of the 2-factors are C_k-factors. Clearly a necessary condition for the existence of an $\mathrm{HW}(r, s; m, k)$ is that if $n = 2(r+s)+1$ is the number of points, then m divides n when $r > 0$, and k divides n when $s > 0$.

There is much less literature on the Hamilton-Waterloo problem than on the Oberwolfach problem. The first paper on this topic, [1], settled the problem for all odd $n \le 17$ and in addition proved that the necessary conditions for the existence of an $\mathrm{HW}(r, s; m, k)$ are sufficient when $(m, k) \in \{(3,5), (3,15), (5,15)\}$ except that an $\mathrm{HW}(6,1;3,5)$ does not exist and the case $\mathrm{HW}(\frac{v-3}{2}, 1; 3, 5)$ is unresolved for $n \equiv 0 \pmod{15}$ with $n > 15$. In a recent paper [6] it is shown that the necessary conditions for the existence of an $\mathrm{HW}(r, s; 3, 4)$ are sufficient with 7 possible exceptions. It seems somewhat fitting that the seating arrangement in Hamilton should be one big cycle (a Hamilton cycle). There are two papers that treat this case. The paper by D. Bryant [3] considers a slight (more general) variant of the Hamilton-Waterloo problem. In that paper it is proven that for all odd $n \geq 11$, K_n has a 2-factorization in which three of the 2-factors are isomorphic to any three given 2-regular graphs of order n, and the remaining 2-factors are all Hamilton cycles. These are termed *Hamilton cycle rich* 2-factorizations.

In the paper by Horak, Nedela and Rosa [9] the authors consider the case of finding a 2-factorization of K_n consisting of Hamilton cycles (at Hamilton) and of triangle-factors (at Waterloo), so they are considering the case of HW$(r, s; n, 3)$. In this case the necessary condition is just that $n \equiv 3 \pmod 6$, when $s \geq 1$. Note that $0 \leq r \leq \frac{n-1}{2}$. So since $s + r = \frac{n-1}{2}$ it is then convenient to just give the number of Hamilton cycles r in the HW$(r, s; n, 3)$ when determining the spectrum for this problem. The following two theorems are proven in [9].

Theorem 1.2 *[9] (a) Let $n = 6k + 3$ and assume that $k \equiv 1 \pmod 3$, then there is a solution to the Hamilton-Waterloo problem HW$(r, s; n, 3)$ with triangle-factors and exactly r Hamilton cycles for every $0 \leq r \leq \frac{n-1}{2}$, except possibly when $r = 1$.*

(b) Let $n = 6k+3$ and assume that $k \equiv 0, 2 \pmod 3$, then there is a solution to the Hamilton-Waterloo problem HW$(r, s; n, 3)$ with triangle-factors and exactly r Hamilton cycles for every $\frac{n+3}{6} \leq r \leq \frac{n-1}{2}$, except possibly when $r = \frac{n+3}{6} + 1$.

Theorem 1.3 *[9] Let $n \equiv 3 \pmod 6$. There is a solution to the Hamilton-Waterloo problem HW$(1, s; n, 3)$ with triangle-factors and exactly one Hamilton cycle when $n = a \cdot 3^m$ where $a \in \{5, 7, 13, 19\}$ and $m \geq 1$. There is no $HW(1, 3; 9, 3)$.*

It is Theorem 1.3 which we will improve in this paper. In fact this paper is completely dedicated to the case of the Hamilton-Waterloo problem with triangle-factors and exactly one Hamilton cycle. As noted in [9], this is the most difficult case of this problem. We will base our work on several recursive constructions as well as direct constructions for some small orders. In Section 2 we give a few recursive constructions. Section 3 gives direct constructions for some small orders as well as a recursive construction for an important solution on 75 points. In Section 4 we give the main recursive construction and solve the case when $n \equiv 3 \pmod{12}$ and finally in Section 5 we solve the case when $n \equiv 9 \pmod{12}$.

The following is our main result.

Theorem 1.4 *There is a solution to the Hamilton-Waterloo problem on n points with triangle-factors and exactly one Hamilton cycle for all $n \equiv 3 \pmod 6$ except when $n = 9$ and with the possible exceptions of $n \in \{93, 111, 123, 129, 141, 153, 159, 177, 183, 201, 207, 213, 237, 249\}$.*

2 Some Recursive Constructions

In this section we present several recursive constructions which give solutions to the Hamilton-Waterloo problem. As we are only dealing with the case of triangle-factors and exactly one Hamilton cycle we will define some notation designed for this purpose. For the remainder of this paper we use the following definition.

Definition 2.1 *An HW(n) is a solution to the Hamilton-Waterloo problem (a 2-factorization of K_n) with $\frac{n-3}{2}$ triangle-factors and exactly one Hamilton cycle.*

An HW($n : a, b$) is a solution to the Hamilton-Waterloo problem (a 2-factorization of K_n) with exactly $\frac{n-3}{2}$ triangle-factors and one 2-factor consisting of one $a-$cycle and one $b-$cycle.

The backbone and primary ingredient in most of the recursive constructions are resolvable group divisible designs, transversal designs and frames. The necessary definitions and background on these objects can be found in Part IV of [5] but we will give the basic definitions here also.

Let K and G be sets of positive integers. *A group divisible design of order v* (K-GDD) is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$, where $\mathcal{V}$ is a finite set of cardinality v, $\mathcal{G}$ is a partition of $\mathcal{V}$ into parts (*groups*) whose sizes lie in G, and $\mathcal{B}$ is a family of subsets (*blocks*) of $\mathcal{V}$ that satisfy (1) if $B \in \mathcal{B}$ then $|B| \in K$, (2) every pair of distinct elements of $\mathcal{V}$ occurs in exactly one block or one group, but not both, and (3) $|\mathcal{G}| > 1$. If $v = a_1g_1 + a_2g_2 + \ldots + a_sg_s$, and if there are a_i groups of size g_i, $i = 1, 2, \ldots, s$, then the K-GDD is of *type* $g_1^{a_1}g_2^{a_2}\ldots g_s^{a_s}$ and denoted as a K-GDD$(g_1^{a_1}g_2^{a_2}\ldots g_s^{a_s})$. When $K = \{k\}$, a K-GDD is denoted as a $k-$GDD. A *transversal design* TD(k, n) is a k-GDD of type n^k. So a GDD is a transversal design if and only if each block meets every group in exactly one point.

Let $\mathcal{B}$ be a set of blocks in a GDD. A *parallel class* or *resolution class* is a collection of blocks that partition the point-set $\mathcal{V}$ of the design. A GDD or TD is *resolvable* if the blocks of the design can be partitioned into parallel classes. A resolvable GDD is denoted by RGDD.

Let k be a positive integer. A $k-$*frame* is a triple $(V, \mathcal{G}, \mathcal{B})$ where V is a set of cardinality v, $\mathcal{G}$ is a partition of V into parts (*groups*), and $\mathcal{B}$ is a collection of blocks of size k that can be partitioned into a collection $\mathcal{P}$ of partial resolution classes of V that satisfies the conditions: (1) The complement of each partial resolution class P of $\mathcal{P}$ is a group $G \in \mathcal{G}$; (2) Each unordered pair $\{x, y\} \subseteq V$ that does not lie in some group G of $\mathcal{G}$ lies in precisely one block of $\mathcal{B}$; and (3) No unordered pair $\{x, y\} \subseteq V$ that lies in some group G of $\mathcal{G}$ also lies in a block of $\mathcal{B}$. As with GDD's we say that a $k-$frame is of *type* $g_1^{a_1}g_2^{a_2}\ldots g_s^{a_s}$ if $v = a_1g_1 + a_2g_2 + \ldots + a_sg_s$, and if there are a_i groups of size g_i for $i = 1, 2, \ldots, s$.

In this paper we will require the existence of 3-frames of the following types: $2^4, 6^112^4$, $6^112^5, 6^412^2, 6^512^1, 6^6, 12^4, 12^5$, and 12^6. Existence of all of these frames (except for the one of type 6^412^2) is given in Section IV.5.2 of [5]. A 3-frame of type 6^412^2 can be constructed from a 4-GDD of type 3^46^2 (see [5], Table IV.4.10) by inflating each block with a 3-frame of type 2^4.

We need to define one final ingredient for our first construction, namely a transversal design missing a sub-transversal design. As we need only one specific instance of this ingredient we will just give that specific definition here (again the more general definition is available at [5]). A TD(k, n)-TD(k,2) is a TD(k, n) (on the pointset $\mathcal{V}$) with two designated points in each group (the *hole*) which satisfies the following properties: (1) Any pair of distinct elements of $\mathcal{V}$ that occurs in a group does not occur in any block; and (2) If a pair of distinct elements from $\mathcal{V}$ comes from distinct groups and each element occurs in the hole of its respective group, then that pair occurs in no block of $\mathcal{B}$; otherwise, it occurs in exactly one block.

Our first recursive construction enables us to multiply an HW(n) by v if there exists a KTS(v), a Kirkman triple system of order v. It is an extension of a result in [9] which covers the case when $v = 3$. We will give that result as Theorem 2.4.

Construction 2.2 *Assume there exists a KTS(v), a TD($4, n$)-TD($4, 2$), a 3-RGDD(2^v) and a HW(n), then there exists an HW(vn).*

Proof. Let $D = (\mathbf{X}, \mathcal{A})$ be a KTS(v) on the points $\mathbf{X} = \mathbb{Z}_v$. Inflate each of these points by a factor of n and denote the new points as ordered pairs in $\mathbb{Z}_v \times \mathbb{Z}_n$. From the TD($4, n$)-TD(4,2), delete the points in last group to create a TD($3, n$)$-$TD(3,2) (call this T) which is resolvable into $n-2$ parallel classes of triples and 2 partial parallel classes of triples (these will be missing the points in the hole). Now, for each block $\{a, b, c\} \in \mathcal{A}$ place the blocks of T on the points $\{a, b, c\} \times \mathbb{Z}_n$ where the sub TD$(3, 2)$ is on the points $\{a, b, c\} \times \{0, 1\}$. So we see that each resolution class of D yields $n-2$ parallel classes of triples in $\mathbb{Z}_v \times \mathbb{Z}_n$. Also, since each resolution class contributes 2 partial resolutions and there are $\frac{v-1}{2}$ resolution classes, we get $v-1$ partial resolution classes of triples (each missing the symbols in $\mathbb{Z}_v \times \{0, 1\}$).

For each $g \in \mathbb{Z}_v$ place the blocks of an HW(n) on the points $\{g\} \times \mathbb{Z}_n$ in such a way that the points $(g, 0)$ and $(g, 1)$ are adjacent in the Hamilton cycle (denoted H_g).

On the set of points $S = \mathbb{Z}_v \times \{0, 1\}$, we first define new groups $\mathcal{G} = \{\{(a, 0), (a + 1, 1)\} \mid a \in \mathbb{Z}_v\}$ where addition is in $\mathbb{Z}_v$. Place the blocks of a 3-RGDD(2^v) on S, respecting the groups of $\mathcal{G}$. Here we note that each of the $v-1$ parallel classes of triples in the 3-RGDD(2^v) can be added to a partial parallel class of triples from above to form a parallel classes of triples in the new design.

Finally, we construct the Hamilton cycle in $\mathbb{Z}_v \times \mathbb{Z}_n$. For each $g \in \mathbb{Z}_v$ delete the edge $(g, 0)(g, 1)$ in the Hamilton cycle H_g to construct a Hamilton path H_g^* in $\{g\} \times \mathbb{Z}_n$ beginning at $(g, 1)$ and ending at $(g, 0)$. Also let e_i denote the edge between $(i, 0)$ and $(i + 1, 1)$. Now sew all these Hamilton paths together to get a Hamilton cycle C as follows: $C = H_0^* e_0 H_1^* e_1 H_2^* e_2 \ldots H_{v-1}^* e_{v-1}$.

We first note that this construction yields only 3-cycles and one Hamilton cycle. It is also easy to check that each pair of points in $\mathbb{Z}_v \times \mathbb{Z}_n$ occurs exactly once in either a triangle or adjacent in the Hamilton cycle. Finally note that the triples are resolvable into parallel classes. Each resolution class of the KTS gives $n-2$ parallel classes of triples for a total of $\frac{v-1}{2} \cdot (n-2)$ parallel classes of this type. The HW(n)'s together contribute$\frac{n-3}{2}$ parallel classes of triples and there are $v-1$ parallel classes formed by the classes in the 3-RGDD(2^v) combined with the partial parallel classes from the TD($3, n$)-TD(3,2). The total number of parallel classes of triples is $\frac{vn-3}{2}$ as required. □

In the above construction, if exactly one of the HW(n) is replaced by an HW($n : a, b$) we obtain the following construction.

Construction 2.3 *Assume there exists a KTS(v), a TD($4, n$)-TD($4, 2$), a 3-RGDD(2^v), an HW(n) and an HW($n : a, b$), then there exists an HW($vn : vn - b, b$).*

The following is Lemma 11 in [9].

Theorem 2.4 *[9] If there exists an HW(n), then there is an HW(3n).*

We now have a multiplication theorem which generalizes Theorem 2.4.

Theorem 2.5 *If there exists an HW(n), then there is an HW(vn) for all $v \equiv 3$ (mod 6).*

Proof. When $v = 3$ the result is from Theorem 2.4. If $v > 3$, the result is from Construction 2.2 since all of the necessary ingredients (a KTS(v), a TD($4, n$)-TD(4,2), a 3-RGDD(2^v)) all are known to exist (See [5]). ∎

It is apparent that one ingredient in the proof of Theorem 2.4 above is an HW(n). In fact, exactly three copies of the design are used in the construction. If exactly one of those copies is replaced by an HW($n : n-3, 3$), the next theorem results.

Theorem 2.6 *If there exists an HW(n) and a HW($n : n-3, 3$), there exists an HW($3n : 3n-3, 3$).*

For the following construction as well as for several others which will follow we use the idea of *sewing* together long cycles. Let G be a group divisible design and say that each point in G has been inflated by some amount. Let b be an inflated block with weight w_b and g be a (inflated) group of weight w_g that intersects b in three points x, y, z. Place the blocks of an HW(w_b) on the points of b in such a way that the Hamilton cycle is $H_b = (yP_bxz)$ where P_b is a path from y to x in b which contains each of the points of b (except z) exactly once. Place the blocks of an HW(w_g) on the points of g in such a way that the Hamilton cycle is $H_g = (xzP_gy)$ where P_g is a path from z to y in g which contains each of the points of g (except x) exactly once. Now we *sew* together H_b and H_g to form the big cycle $H = (xzP_gyP_b)$. Note that H contains all the points in $H_b \cup H_g$ and all the same edges, except xy and yz. However, these edges appear in triangles in the HW(w_g) and the HW(w_b), respectively.

To summarize the above paragraph, when two cycles H_b and H_g are *sewn* together a new single cycle is formed and all the same of edges are still covered either in the new cycle or in triangles of the original HW's.

Our master design in the next construction is a resolvable holey group divisible design. Let X be a set of $3mn$ points which is partitioned into 3-subsets X_{ij}, $1 \le i \le m$, $1 \le j \le n$. Let $\mathcal{A}$ be a collections of 3-subsets of X (the *blocks*) which satisfy the following conditions: (1) every pair of points $x \in X_{i_1j_1}$ and $y \in X_{i_2j_2}$ is contained in exactly one block if $i_1 \ne i_2$ and $j_1 \ne j_2$ and in no block if either $i_1 = i_2$ or $j_1 = j_2$, and (2) the blocks are resolvable into parallel classes. Then $(X, \mathcal{A})$ is a *resolvable holey group divisible design* and denoted as a 3-RHGDD of type $(m, 3^n)$. The subsets $\bigcup_{j=1}^n X_{ij}$, where $1 \le i \le m$, are called the *groups* while the subsets $\bigcup_{i=1}^m X_{ij}$, where $1 \le j \le n$, are the *holes*.

Construction 2.7 *Assume there exists a 3-RHGDD(k,3^n), a 3-RGDD(3^k), an HW($3n$) and an HW($3k$), then there exists an HW($3nk$).*

Proof. Begin with the master design, the 3-RHGDD$(k, 3^n)$. Put the blocks of the HW$(3n)$ in each of the groups making sure that in each case the Hamilton cycle contains the three vertices in the first hole (X_{i1}, for each $1 \le i \le k$) in the manner described above. Note that $\frac{3n-3}{2}$ parallel classes of triples result from taking the union of the parallel classes of triples from the HW$(3n)$ in each group. Now put the blocks of the HW$(3k)$ in the first hole, again being careful to place the Hamilton cycle appropriately. Now sew together the horizontal and the vertical Hamilton cycles to get one big cycle containing all the points.

In all the other holes place the blocks of a $3-$RGDD(3^k). The union of one parallel class of blocks from each of these designs with a parallel classes of triples from the HW$(3k)$ in the first hole forms a parallel class of blocks in the entire design. The total number of parallel classes of triples thus formed is $\frac{3k-3}{2}$.

It is straightforward to check that all pairs are covered in either the Hamilton cycle or in a triple. There are $\frac{1}{2}(3n-3)(k-1)$ parallel classes in the 3-RHGDD$(k, 3^n)$, which when added to the earlier mentioned parallel classes gives a total of $3kn - 3$ parallel classes of triples, the desired number. ∎

Again, the following is an easy generalization of the above construction where one HW$(3n)$ is replaced by an HW$(3n : a, b)$.

Construction 2.8 *Assume there exists a 3-RMGDD(n^k), a 3-RGDD(3^k), an HW($3n$), an HW($3n : a, b$) and a HW($3k$), then there exists an HW($3nk : 3nk - a, a$).*

From Constructions 2.7 and 2.8 we get the following two theorems since all the necessary ingredients exist. The existence of 3-RHGDD(k,3^n) is given in [12], while existence results on 3-RGDD(3^k) can be found in [5].

Theorem 2.9 *If there exists an HW($3n$) and a HW($3k$), then there exists an HW($3nk$).*

Theorem 2.10 *If there exists an HW($3n$), an HW($3n : a, b$) and a HW($3k$), then there exists an HW($3nk : 3nk - a, a$).*

3 Small orders

In this section we construct solutions to the Hamilton-Waterloo problem for some small orders. The constructions are all direct constructions obtained by the aid of a computer except for the last one which is recursive.

Proposition 3.1 *There exists an HW$(21 : 18, 3)$.*

On the point set $V = \mathbb{Z}_9 \times \{0, 1\} \cup \{x, y, z\}$, let C_0 denote the following set of seven triples (where $(a, b) \in \mathbb{Z}_9 \times \{0, 1\}$ is denoted a_b): $C_0 = \{\{0_0, 1_0, 4_0\}, \{3_0, 5_0, 0_1\}$, $\{7_0, 1_1, 3_1\}$, $\{2_1, 5_1, 6_1\}, \{x, 2_0, 4_1\}, \{y, 6_0, 7_1\}, \{z, 8_0, 8_1\}\}$.

Then C_0 forms a parallel class of triples in V. Eight other parallel classes of triples are formed by developing C_0 mod$(9, -)$. It is easy to check that the two mixed differences, 7 and 8 are unused. The final parallel class consists of the 3-cycle (x, y, z) along with the 18-cycle constructed from the mixed differences 7 and 8. The 18-cycle is $(0_0, 8_1, 1_0, 0_1, 2_0, 1_1, 3_0, \ldots 6_1, 8_0, 7_1)$. ∎

Proposition 3.2 *There exists an HW*(27).

On the point set $V = \mathbb{Z}_9 \times \{0,1,2\}$, let $C_0 = \{\{0_0, 1_0, 3_0\}, \{2_0, 6_0, 0_1\}, \{1_1, 2_1, 4_1\}, \{3_1, 7_1, 0_2\}, \{1_2, 2_2, 4_2\}, \{5_0, 3_2, 8_2\}, \{4_0, 8_1, 6_2\}, \{7_0, 6_1, 7_2\}, \{8_0, 5_1, 5_2\}\}$.

Then C_0 forms a parallel class of triples in V. Eight other parallel classes of triples are formed by developing C_0 mod$(9, -)$. The three blocks, $\{0_0 0_1 4_2\}, \{1_0 3_1 2_2\}$, and $\{2_0 7_1 1_2\}$ will each generate one parallel class of triples when developed mod$(9, -)$. Finally, one can check that pairs of elements of the form $(x_0, (x+1)_1)$, $(y_0, (y+5)_2)$ and $(z_1, (z+5)_2)$ (for $x, y, z \in \mathbb{Z}_9$) have yet to appear together in a block. By Lemma 1 of [9] these sets of edges can be ordered to form a Hamilton cycle in V. □

We now extend our notation slightly for Propositions 3.3 and 3.8. Define a HW$(n : a, b, c)$ to be a solution to the Hamilton-Waterloo problem (a 2-factorization of K_n) with exactly $\frac{n-3}{2}$ triangle-factors and one 2-factor consisting of exactly one $a-$cycle, one $b-$cycle and one $c-$cycle.

Proposition 3.3 *There exists an HW*$(27 : 12, 12, 3)$.

On the point set $V = \mathbb{Z}_{24} \cup \{x, y, z\}$, let $C = \{\{1, 2, 7\}, \{3, 5, 20\}, \{6, 9, 22\}\}$. Let

$$C_0 = C \cup\ (C + 12)\ \cup \{\{x, 0, 4\}, \{y, 11, 23\}, \{z, 12, 16\}\} \quad \text{and}$$
$$C_1 = C \cup\ (C + 12)\ \cup \{\{z, 0, 4\}, \{y, 11, 23\}, \{x, 12, 16\}\}.$$

Then C_0 is a parallel class of triples which when developed by adding all the elements of $\{0, 1, 2, 3, 8, 9, 10, 11\}$ (modulo 24) yields 8 parallel classes of triples. Also, when C_1 is developed by adding all the elements of $\{4, 5, 6, 7\}$ (modulo 24) there are 4 additional parallel classes formed (for a total of 12). The triples in these 12 parallel classes contain every pair of elements in V except those with difference 10 and those with both elements from the set $\{x, y, z\}$. These missing pairs form two disjoint 12-cycles which together with the 3-cycle (x, y, z) comprise the last parallel class. □

Proposition 3.4 *There exists an HW*(33).

On the point set $V = \mathbb{Z}_{11} \times \{0,1,2\}$, let

$C_0 = \{\{0_0, 1_0, 3_0\}, \{2_0, 6_0, 0_1\}, \{4_0, 9_0, 1_1\}, \{2_1, 3_1, 5_1\}, \{4_1, 9_1, 0_2\}, \{6_1, 10_1, 3_2\}, \{1_2, 2_2, 5_2\}, \{5_0, 6_2, 8_2\}, \{10_0, 4_2, 9_2\}, \{7_0, 8_1, 7_2\}, \{8_0, 7_1, 10_2\}\}$.

Then C_0 forms a parallel class of triples in V. Ten additional parallel classes of triples are formed by developing C_0 mod$(11, -)$.

Each of the four blocks, $\{0_0, 0_1, 6_2\}, \{1_0, 3_1, 8_2\}, \{2_0, 6_1, 6_2\}$ and $\{3_0, 10_1, 0_2\}$ will generate one parallel class of triples when developed mod$(11, -)$. Finally, one can check that pairs of elements of the form $(x_0, (x+6)_1)$, $(y_0, (y+9)_2)$ and $(z_1, (z+9)_2)$ (for $x, y, z \in \mathbb{Z}_{11}$) have yet to appear together in a block. By Lemma 1 of [9] these sets of edges can be ordered to form a Hamilton cycle in V. □

Proposition 3.5 *There exists an HW*$(33 : 30, 3)$.

On the point set $V = \mathbb{Z}_{15} \times \{0,1\} \cup \{x,y,z\}$, let C_0 denote the following set of triples:

$C_0 = \{\{0_0,1_0,3_0\},\{2_0,6_0,11_0\},\{0_1,1_1,3_1\},\{4_0,12_0,2_1\},\{5_0,4_1,8_1\},\{8_0,9_1,14_1\},$
$\{10_0,5_1,12_1\},\{13_0,7_1,13_1\},\{7_0,11_1,x\},\{9_0,6_1,y\},\{14_0,10_1,z\}\}$

Then C_0 forms a parallel class of triples in V. Fourteen other parallel classes of triples are formed by developing C_0 mod$(15,-)$. It is easy to check that the two mixed differences, 7 and 8 are unused. The final parallel class consists of the 3-cycle (x,y,z) along with the 30-cycle constructed from the mixed differences 7 and 8. The 18-cycle is $(0_0,8_1,1_0,9_1,2_0,\ldots 6_1,14_0,7_1)$. □

Proposition 3.6 *There exists an* $HW(39:36,3)$.

On the point set $V = \mathbb{Z}_{36} \cup \{x,y,z\}$, let $C = \{\{1,2,4\},\{3,7,15\},\{5,14,28\},$ $\{8,13,29\}, \{9,16,35\}\}$. Now let

$C_0 = C \cup (C+18) \cup \{\{x,0,6\},\{y,12,30\},\{z,18,24\}$ and
$C_1 = C \cup (C+18) \cup \{\{z,0,6\},\{y,12,30\},\{x,18,24\}$.

Then C_0 is a parallel class of triples which when developed by adding all the elements of $\{0,1,\ldots 5,12,13,\ldots 17\}$ (modulo 36) yields 12 parallel classes of triples. Also, when C_1 is developed by adding all the elements of $\{6,7,\ldots 11\}$ (modulo 48) there are 6 additional parallel classes formed (for a total of 18). These triples in these 18 parallel classes contain every pair of elements in V except those with difference 11 and those with both elements from the set $\{x,y,z\}$. These missing pairs form a 36-cycle which together with the 3-cycle (x,y,z) comprises the last parallel class. □

Proposition 3.7 *There exists an* $HW(51)$.

On the point set $V = \mathbb{Z}_{17} \times \{0,1,2\}$, let

$C_0 = \{\{0_0,1_0,3_0\},\{2_0,6_0,11_0\},\{4_0,10_0,0_1\},\{5_0,12_0,3_1\},\{1_1,2_1,4_1\},\{5_1,9_1,14_1\},$
$\{6_1,12_1,0_2\},\{8_1,15_1,1_2\},\{2_2,3_2,5_2\},\{4_2,8_2,13_2\},\{7_0,6_2,12_2\},\{8_0,9_2,16_2\},$
$\{9_0,10_1,7_2\},\{13_0,13_1,15_2\},\{14_0,16_1,14_2\},\{15_0,7_1,11_2\},\{16_0,11_1,10_2\}\}$.

Then C_0 forms a parallel class of triples in V. Sixteen additional parallel classes of triples are formed by developing C_0 mod$(17,-)$.

Each of the seven blocks, $\{0_0,3_1,3_2\}$, $\{0_0,4_1,10_2\}$, $\{0_0,5_1,14_2\}$, $\{0_0,6_1,7_2\}$, $\{0_0,11_1,6_2\},\{0_0,14_1,4_2\},\{0_0,16_1,12_2\}$ will generate one parallel class of triples when developed mod$(17,-)$. Finally, one can check that pairs of elements of the form $(x_0,(x+10)_1)$, $(y_0,(y+8)_2)$ and $(z_1,(z+6)_2)$ (for $x,y,z \in \mathbb{Z}_{17}$) have yet to appear together in a block. By Lemma 1 of [9] these sets of edges can be ordered to form a Hamilton cycle in V. □

Proposition 3.8 *There exists an* $HW(51:24,24,3)$.

On the point set $V = \mathbb{Z}_{48} \cup \{x,y,z\}$, let $C = \{\{1,2,4\},\{3,7,12\},\{5,11,37\},\{6,17,44\},$ $\{9,21,38\},\{10,23,43\},\{15,22,40\}\}$. Now let

$C_0 = C \cup (C+24) \cup \{\{x,0,8\},\{y,18,42\},\{z,24,32\}\}$ and
$C_1 = C \cup (C+24) \cup \{\{z,0,8\},\{y,18,42\},\{x,24,32\}\}$.

Then C_0 is a parallel class of triples which when developed by adding all the elements of $\{0,1,2,\ldots 7,16,17,\ldots,23\}$ (modulo 48) yields 16 parallel classes of triples. Also, when C_1 is developed by adding all the elements of $\{8, 9, \ldots, 15\}$ (modulo 48) there are 8 additional parallel classes formed (for a total of 24). The triples in these 24 parallel classes contain every pair of elements in V except those with difference 14 and those with both elements from the set $\{x, y, z\}$. These missing pairs form two disjoint 24-cycles which together with the 3-cycle (x, y, z) comprises the last parallel class. ▯

Proposition 3.9 *There exists an* $HW(69)$.

On the point set $V = \mathbb{Z}_{23} \times \{0, 1, 2\}$, let

$$C_0 = \{\{0_0, 1_0, 3_0\}, \{2_0, 6_0, 11_0\}, \{4_0, 10_0, 17_0\}, \{5_0, 13_0, 0_1\}, \{7_0, 18_0, 1_1\}, \{2_1, 3_1, 5_1\}, \{4_1, 8_1, 13_1\}, \{6_1, 12_1, 19_1\}, \{7_1, 15_1, 0_2\}, \{9_1, 20_1, 1_2\}, \{2_2, 3_2, 5_2\}, \{4_2, 8_2, 13_2\}, \{6_2, 12_2, 19_2\}, \{8_0, 7_2, 15_2\}, \{9_0, 9_2, 20_2\}, \{12_0, 10_1, 10_2\}, \{14_0, 11_1, 17_2\}, \{15_0, 17_1, 16_2\}, \{16_0, 21_1, 22_2\}, \{19_0, 18_1, 21_2\}, \{20_0, 16_1, 14_2\}, \{21_0, 14_1, 11_2\}, \{22_0, 22_1, 18_2\}\}.$$

Then C_0 forms a parallel class of triples in V. Twenty two additional parallel classes of triples are formed by developing C_0 mod$(23, -)$.

Each of the ten blocks, $\{0_0, 1_1, 8_2\}$, $\{0_0, 3_1, 12_2\}$, $\{0_0, 4_1, 14_2\}$, $\{0_0, 7_1, 18_2\}$, $\{0_0, 8_1, 20_2\}, \{0_0, 9_1, 4_2\}, \{0_0, 11_1, 16_2\}, \{0_0, 13_1, 15_2\}, \{0_0, 14_1, 5_2\}, \{0_0, 15_1, 9_2\}$ will generate one parallel class of triples when developed mod$(23, -)$. Finally, one can check that pairs of elements of the form $(x_0, (x + 12)_1)$, $(y_0, (y + 10)_2)$ and $(z_1, (z + 13)_2)$ (for $x, y, z \in \mathbb{Z}_{23}$) have yet to appear together in a block. By Lemma 1 of [9] these sets of edges can be ordered to form a Hamilton cycle in V. ▯

Proposition 3.10 *There exists an* $HW(87)$.

On the point set $V = \mathbb{Z}_{29} \times \mathbb{Z}_3$, let $C_0 = \{\{0_0, 1_0, 3_0\}, \{2_0, 6_0, 11_0\}, \{4_0, 10_0, 17_0\}, \{5_0, 13_0, 23_0\}, \{7_0, 19_0, 8_1\}, \{9_0, 24_0, 12_1\}, \{14_0, 16_1, 20_2\}, \{15_0, 21_1, 28_2\}\}$. Let C_1 and C_2 be cyclic shifts of C_0 mod $(-, 3)$ and let $D = \{\{18_0, 26_1, 22_2\}, \{22_0, 27_1, 25_2\}, \{25_0, 18_1, 26_2\}, \{26_0, 25_1, 18_2\}, \{27_0, 22_1, 27_2\}\}$.

Then $C = C_0 \cup C_1 \cup C_2 \cup D$ forms a parallel class of blocks in V. Twenty eight additional parallel classes of triples are formed by developing C modulo $(29, -)$.

Each of the blocks in the following set generates a parallel class when developed mod$(29, -)$ $\{\{0_0, 0_1, 14_2\}, \{0_0, 9_1, 18_2\}, \{0_0, 10_1, 20_2\}, \{0_0, 11_1, 24_2\}, \{0_0, 12_1, 2_2\}, \{0_0, 13_1, 5_2\}, \{0_0, 14_1, 9_2\}, \{0_0, 15_1, 15_2\}, \{0_0, 19_1, 16_2\}, \{0_0, 20_1, 19_2\}, \{0_0, 21_1, 7_2\}, \{0_0, 25_1, 8_2\}, \{0_0, 26_1, 17_2\}\}$.

Finally, one can check that pairs of elements of the form $(x_0, (x+27)_1)$, $(y_0, (y+10)_2)$ and $(z_1, (z+11)_2)$ (for $x, y, z \in \mathbb{Z}_{29}$) have yet to appear together in a block. By Lemma 1 of [9] these sets of edges can be ordered to form a Hamilton cycle in V. ▯

Proposition 3.11 *There exists an* $HW(87 : 84, 3)$.

On the point set $V = \mathbb{Z}_{84} \cup \{x, y, z\}$, let $C = \{\{1, 2, 4\}, \{3, 7, 12\}, \{5, 11, 18\}, \{6, 16, 24\}, \{8, 19, 34\}, \{9, 25, 62\}, \{10, 31, 70\}, \{13, 40, 75\}, \{15, 38, 74\}, \{17, 36, 69\}, \{21, 41, 71\}, \{22, 39, 68\}, \{23, 35, 79\}\}$.

Now let $C_0 = C \cup\ (C + 42)\ \cup \{\{x, 0, 14\}, \{y, 30, 72\}, \{z, 42, 56\}\}$ and let

$C_1 = C \cup\ (C+42)\ \cup \{\{z,0,14\},\{y,30,72\},\{x,42,56\}\}$. Then C_0 is a parallel class of triples which when developed by adding all the elements of $\{0,1,\ldots 13,28,29,\ldots,41\}$ (modulo 84) yields 28 parallel classes of triples. Also, when C_1 is developed by adding all the elements of $\{14,15,\ldots,27\}$ (modulo 84) there are 14 additional parallel classes formed (for a total of 42). The triples in these 42 parallel classes contain every pair of elements in V except those with difference 41 and those with both elements from the set $\{x,y,z\}$. These missing pairs form an 84-cycle and the 3-cycle (x,y,z) comprising the last parallel class. ∎

The following is a recursive construction for an HW(75) and a HW(75:72,3). We will take special note of the structure of the resulting designs as they will be major ingredients in our main recursive construction in the next section.

Proposition 3.12 *There exists an* $HW(75)$ *and a* $HW(75:72,3)$*.*

Proof. The master design $\mathcal{D} = (\mathcal{X},\mathcal{A})$ is a 5-GDD(4^6) on the symbols $\mathbb{Z}_6 \times \mathbb{Z}_4$. Note that this GDD is actually a frame and is easily obtainable by deleting a point from the (25,5,1)-BIBD. Inflate each point in this design by 3 and add 3 points at infinity to get a total of 75 points. Denote the point set by $V = \mathbb{Z}_6 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \cup \{x,y,z\}$ where now group i contains the points $\{i\} \times \mathbb{Z}_4 \times \mathbb{Z}_3$.

Let b_0 be a block in the master design which misses the first group. On the points $b_0 \times \mathbb{Z}_3$ put an HW(15) being sure that for each $x \in b_0$ that $(x,0),(x,2)$ and $(x,1),(x,2)$ are both edges in the Hamilton circuit H_0. Let b_1 be a block in $\mathcal{D}$ with the property that it misses the 6^{th} group and intersects b_0 in the 5^{th} group. On the points $b_1 \times \mathbb{Z}_3$ put a 2-factorization of K_{15} consisting of one 2-factor with 3 C_3's and a C_6 and all other 2-factors consisting of just 3-cycles (such a factorization is given in [8]). This factorization should be placed so that the 6-cycle, C, is in the first two groups and each of the remaining three 3-cycles is in a different group. C should contain the edges $(p,0),(p,2)$ and $(p,1),(p,2)$ and $(q,0),(q,2)$ and $(q,1),(q,2)$ where p and q are the points in b_1 in the first two groups. For each of the remaining blocks $b \in \mathcal{A}$ place a 3-RGDD(3^5) on the points $b \times \mathbb{Z}_3$ (respecting the groups).

Now, on the first group plus the points $\{x,y,z\}$ put an HW(15) being certain that both the edges $(p,0),(p,1)$ and $(p,0),(p,2)$ are in the Hamilton cycle. On the second group plus the points $\{x,y,z\}$ put an HW(15:12,3), where in the last 2-factor the 3-cycle is (x,y,z) and the 12-cycle contains the edges $(q,0),(q,1)$ and $(q,0),(q,2)$ as well as $(a,0),(a,1)$ and $(a,0),(a,2)$ where a is the point in b_0 and the second group. In each remaining group g_i plus the points $\{x,y,z\}$ put an HW(15:12,3), where in the last 2-factor the 3-cycle is again (x,y,z) and the 12-cycle contains the edges $(a_i,0),(a_1,1)$ and $(a_i,0),(a_i,2)$ where $a_i \in b_0 \cap g_i$. As in Construction 2.7 we "sew" the long cycles together to obtain a Hamilton cycle on all of V.

Each of the other 2-factors are formed by taking the union of a triangle-factor from each group g_i and adding to that the triangle-factors that arise from a parallel class of blocks missing g_i. Since there are 6 triangle-factors in each group we see that this construction yields 36 triangle-factors, the required number. It is not hard to check that indeed we have constructed the desired HW(75).

To construct an $HW(75:72,3)$, instead of placing an HW(15) on the first group as above, place an HW(15:12,3) on the points of that group in the obvious manner.

The only thing that is changed is that now the last 2-factor consists of a 72-cycle and a 3-cycle instead of a Hamilton cycle. ∎

In order to solve the existence problem for HW($6n+3$) we will be working in congruence classes modulo 12. To facilitate the recursion we define the following sets:

$S = \{t \mid \text{there exists an } HW(12t+3)\}$
$S_3 = \{t \mid \text{there exists an } HW(12t+3 : 12t, 3)\}$
$M = \{t \mid \text{there exists an } HW(12t+9)\}$

The following three propositions summarize our results for small values in the sets S, S_3, and M.

Proposition 3.13 $\{1,3,5,6,7,8,11,12,16,19\} \subset S_3$.

An HW($12t+3$: $12t$,3) is given in [8] for $t=1$. When $t=3$ and 7 direct constructions for an HW($12t+3 : 12t, 3$) are given above in Propositions 3.6 and 3.11, respectively. Justification for the remaining orders is given in the following table.

t	12t+3	Justification
5	63	Theorem 2.6. HW(21) from [9], HW(21:18,3) from Proposition 3.1
6	75	Proposition 3.12
8	99	Theorem 2.6. HW(33) and HW(33:30,3) from Propositions 3.4 and 3.5
11	135	Theorem 2.6. HW(45) and HW(45:42,3) from Theorem 2.6 with $n=15$
12	147	Theorem 2.10 with $n=k=7$. HW(21) from [9], HW(21:18,3) from Proposition 3.1
16	195	Theorem 2.10 with $3k=15$ and $3n=39$. HW(15) and HW(39) from [9], HW(39:36,3) from Proposition 3.6
19	231	Theorem 2.10 with $3k=33$ and $3n=21$

Proposition 3.14 $\{1,2,3,4,5,6,7,8,11,12,14,16,19,20,21\} \subset S$.

An HW($12t+3$) is given in [9] for $t=1,3,5,11$, and 14. When $t=2,4,7$ direct constructions for an HW($12t+3$) are given above in Propositions 3.2, 3.7, and 3.10, respectively. Justification for the other orders is given in the following table.

t	12t+3	Justification
6	75	Proposition 3.12
8	99	Theorem 2.4. HW(33) from Prop. 3.4
12	147	Theorem 2.9 with $n=k=7$.
16	195	Theorem 2.9 with $3n=15$ and $3k=39$. HW(39) from [9]
19	231	Theorem 2.9 with $3n=21$ and $3k=33$
20	243	Theorem 2.4 (twice), with $n=27$.
21	255	Theorem 2.9 with $3n=15$ and $3k=51$. HW(51) from Proposition 3.7

Proposition 3.15 $\{1,2,3,4,5,6,8,9,13,15,18,21\} \subset M$.

An HW($12t+9$) is given in [9] for $t=1,3,4,9$ and 15. When $t=2$ and $t=5$ direct constructions for an HW(33) and a HW(69) are given above in Propositions 3.4 and 3.9, respectively. Justification for the other orders is given in the following table.

t	12t+9	Justification
6	81	Theorem 2.4. HW(27) from Proposition 3.2
8	105	Theorem 2.9 with $n = 5$ and $k = 7$
13	165	Theorem 2.9 with $n = 5$ and $k = 11$
18	225	Theorem 2.5 with $n = 15$
21	261	Theorem 2.4. HW(87) from Proposition 3.10

4 The spectrum: the 3 modulo 12 case

The following construction will be our main recursive construction. In each case the master design is a TD(6,n) and we carefully sew the long cycles together as in Constructions 2.7 and 3.12.

Construction 4.1 *Assume there exists a TD$(6, n)$ and $1 \leq k_1, k_2 \leq n$.*
a) If $\{n, k_1\} \subset S_3$ and $k_2 \in S_3 \cup \{2, 4\}$, then $4n + k_1 + k_2 \in S_3$.
b) If $\{n, k_2\} \subset S_3 \cup \{2, 4\}$ and $k_1 \in S$, then $4n + k_1 + k_2 \in S$.

Proof of a) Let $\mathcal{D}$ be the master design, a TD(6,n). Give weight 12 to all the points in the last four groups. Let b be a designated block in $\mathcal{D}$ and give weight 12 to the points in b that are in the first two groups. We first assume that $k_2 \neq 2, 4$. Give $k_1 - 1$ additional points weight 12 group 1, give $k_2 - 1$ additional points weight 12 in group 2 and give all other points weight 0. Additionally add three infinite points $\{x, y, z\}$. Now, in each inflated block of $\mathcal{D}$ (except b) put the blocks of a 3-frame of type either $12^4, 12^5$ or 12^6 (respecting the groups). On the first and second groups put the blocks of an HW$(12k_1 + 3 : 12k_1, 3)$ and a HW$(12k_2 + 3 : 12k_2, 3)$, respectively where the 3-cycle in the non-triangle 2-factor is (x, y, z). On the final four groups put the blocks of a HW$(12n + 3 : 12n, 3)$, where again the 3-cycle in the non-triangle 2-factor is (x, y, z). Let L_i denote the long cycle in each of these groups

Now consider the inflated block b, it consists of 6 groups each of size 12. As was done to construct the blocks of the HW(75:72,3) in Proposition 3.12, begin with the master design $(X, \mathcal{A})$, a 5-GDD(4^6) and inflate each point in this design by 3. Let b_0 be a block in $\mathcal{A}$ which misses the first group. On the points of the inflated b_0 put an HW(15) with Hamilton circuit H_0 as before. Again let b_1 be a block in $\mathcal{A}$ with the property that it misses the 6^{th} group and intersects b_0 in the 5^{th} group. On the points of the inflated b_1 put a 2-factorization of K_{15} consisting of one 2-factor with a C_6 and 3 C_3's and where all other 2-factors are triangle-factors [8]. This factorization should be placed so that the 6-cycle C is in the first two groups and each of the remaining three 3-cycles is in a different group. Finally, for each of the remaining blocks $B \in \mathcal{A}$ place a 3-RGDD(3^5) on the points of the inflated B (respecting the groups).

To make a cycle of length $12(4n + k_1 + k_2)$, sew L_1 to C, C to L_2 and each of $L_2, L_3, \ldots L_6$ to H_0 in precisely the same manner as was done in Proposition 3.12. Add the cycle (x, y, z) to this long cycle to get a 2-factor consisting of a triangle and a $12(4n + k_1 + k_2)$-cycle.

Let g_i be a group in the master design. To form a triangle factor, take one of the triangle factors from the HW that was placed on the inflated points of $g_i \cup \{x, y, z\}$ and add to that a triangle-factor that arises from a parallel class of blocks missing g_i.

It is straightforward to check that this is indeed a triangle factor of the entire design. It can also be checked that the union of these 2-factors is indeed a 2-factorization.

Now we deal with the case where $k_2 = 2$ or 4. Proceed exactly as above except now on the points of the inflated b_1 put a 2-factorization of K_{15} with one 2-factor consisting of a C_9 and 2 C_3's and where all other 2-factors are triangle-factors [8]. This factorization should be placed so that the 9-cycle C is in the first three groups and each of the remaining two 3-cycles is in a different group. Nothing else changes. Obviously every 2-factor except the one with the long cycle will be exactly as above.

Now we construct the long cycle of length $12(4n+k_1+k_2)$. Let H_0 and $L_1, L_3, \ldots L_6$ be as above. Let the two long cycles of the HW$(12k_2+3 : 6k_2, 6k_2, 3)$ which was placed in the second group be called M_0 and M_1. Assume that M_0 intersects the inflated b_0 in precisely three points and M_1 intersects the inflated b_1 in precisely three points. The long cycle is then constructed by sewing L_1 to C then sewing C to M_1, then C to L_3, then L_3 to H_0, then H_0 to M_0, then H_0 in turn to each of L_6, L_5, L_4. The result is a cycle containing $12(4n + k_1 + k_2)$ points. Add the cycle (x, y, z) to this long cycle to get a 2-factor consisting of a triangle and a $12(4n + k_1 + k_2)$-cycle.

Proof of b) This is identical to the above except the cycle L_1 in the first group now contains the points x, y, z. ∎

The following construction is very similar to Construction 4.2 and most of the details will be omitted.

Construction 4.2 *Assume there exist a $TD(6, n)$, a $HW(6n + 9 : 6n + 6, 3)$ and assume $1 \leq i \leq n$.*

(a) If there exists an HW($6n + 6i + 9$), then there exists an HW($36n + 6i + 39$).

(b) If there exists an HW($6n+6i+9 : 6n+6i+6, 3$), then there exists an HW($36n+6i + 39 : 36n + 6i + 36, 3$).

Proof. Begin with the master design $\mathcal{D}$, a TD(6,n). Give weight 12 to all the points in one designated block b and give weight 6 to all the points in the first five groups (except those points in b). Finally in the last group give weight 12 to i additional points and weight 6 to $n - i - 1$ of the points. Again we add three infinite points $\{x, y, z\}$.

Now, in each inflated block of $\mathcal{D}$ (except b) put the blocks of a 3-frame of type either $6^5 12^1$, 6^6, or $6^4 12^2$ (respecting the groups). The existence of these frames was discussed in Section 2.

On the points of the last five groups put the blocks of an HW($6n$+9 : $6n$+6, 3) where the 3-cycle in the non-triangle 2-factor is (x, y, z). In the first group plus the points $\{x, y, z\}$ put the blocks of a HW($6n$+$6i$+9) for part a or an HW($6n$+$6i$+9 : $6n$+$6i$+6, 3) for part b, where again the 3-cycle in the non-triangle 2-factor is (x, y, z).

Now do exactly as was done in Construction 4.1, putting an HW(75:72,3) on the points of the inflated block b plus $\{x, y, z\}$ and carefully sewing together all the long cycles. ∎

Corollary 4.3 $\{18, 20\} \in S_3$, $18 \in S$ *and* $19 \in M$.

Proof. In Construction 4.2 let $n = 5$. Use $i = 0, 4$ to prove $\{18, 20\} \in S_3$, $i = 4$ to get $18 \in S$ and $i = 3$ to get that $19 \in M$. ∎

We are now in position to determine the set $S_3 = \{t \mid \text{there exists an } HW(12t+3 : 12t, 3)\}$. Recall that from Proposition 3.13 we already have that $\{1, 3, 5, 6, 7, 8, 11, 12, 16, 19\} \subset S_3$.

Proposition 4.4 *If $22 \leq t \leq 133$, then $t \in S_3$.*

Proof. We will use Construction 4.1(a). The following table gives the values for the ingredients needed in the construction. We require that there exists a TD(6,n) and that $\{n, k_1\} \subset S_3$ and $k_2 \in S_3 \cup \{2, 4\}$ where $1 \leq k_1, k_2 \leq n$. The result is that $4n + k_1 + k_2 \in S_3$.

n	$k_1 \in$	$k_2 \in$	$4n + k_1 + k_2$
5	$\{1, 5\}$	$\{1, 2, 3, 4, 5\}$	$22 - 30$
7	$\{1, 7\}$	$\{1, \ldots, 7\}$	$30 - 42$
8	$\{1, 8\}$	$\{1, \ldots, 8\}$	$34 - 48$
11	$\{1, 7, 11\}$	$\{1, \ldots, 8\}$	$46 - 63$
12	$\{1, 7, 12\}$	$\{1, \ldots, 8\}$	$50 - 68$
16	$\{1, 7, 12, 16\}$	$\{1, \ldots, 8\}$	$66 - 88$
19	$\{1, 7, 12, 19\}$	$\{1, \ldots, 8\}$	$78 - 103$
25	$\{1, 7, 12, 19, 25\}$	$\{1, \ldots, 8\}$	$102 - 133$

□

We are now able to finish the spectrum of S_3.

Theorem 4.5 *There exists a $HW(12m + 3 : 12m, 3)$ (i.e. $m \in S_3$) for all $m \geq 1$ except possibly for $m \in \{2, 4, 9, 10, 13, 14, 15, 17, 21\}$.*

Proof. If $m \leq 133$ the theorem follows from Propositions 3.13, 4.4 and Corollary 4.3. So assume now that $m > 133$. Assume by way of induction that $k \in S_3$ for all $22 \leq k < m$. Now write $m = 4n + b$ where $b = 2, 3, 4, 5$, it follows that $n > 32$. Since there exists a TD$(6, n)$ for all $n > 22$ we can use Construction 4.1(a) to show that $m \in S_3$, by letting $k_1 = 1$ and $k_2 = b - 1$. □

We are now ready to determine the set $S = \{t \mid \text{there exists an } HW(12t+3)\}$. Remember, from Proposition 3.14 we already have that $\{1, 2, 3, 4, 5, 6, 8, 11, 12, 14, 16, 19, 20, 21\} \subset S$. We again use Construction 4.1 for the backbone of the work.

Proposition 4.6 *If $22 \leq t \leq 133$, then $t \in S$.*

Proof. This proof is very similar to Proposition 4.6 except here we use Construction 4.1(b). The following table gives the values for the ingredients needed in the construction. We require that there exists a TD(6,n) and that $\{n, k_2\} \subset S_3 \cup \{2, 4\}$ and $k_1 \in S$ for $1 \leq k_1, k_2 \leq n$. The result is that $4n + k_1 + k_2 \in S_3$.

n	$k_1 \in$	$k_2 \in$	$4n + k_1 + k_2$
5	$\{1, 5\}$	$\{1, 2, 3, 4, 5\}$	$22 - 30$
7	$\{1, 7\}$	$\{1, \ldots, 7\}$	$30 - 42$
8	$\{1, 8\}$	$\{1, \ldots, 8\}$	$34 - 48$
11	$\{1, 7, 11\}$	$\{1, \ldots, 8\}$	$46 - 63$
12	$\{1, 7, 12\}$	$\{1, \ldots, 8\}$	$50 - 68$
16	$\{1, 7, 12, 16\}$	$\{1, \ldots, 8\}$	$66 - 88$
19	$\{1, 7, 12, 19\}$	$\{1, \ldots, 8\}$	$78 - 103$
25	$\{1, 7, 12, 19, 25\}$	$\{1, \ldots, 8\}$	$102 - 133$

□

Theorem 4.7 *There exists a HW$(12m+3)$ (i.e. $m \in S$) for all $m \geq 1$, except possibly for $m \in \{9, 10, 13, 15, 17\}$.*

Proof. If $m \leq 133$ the theorem follows from Propositions 3.14, 4.6 and Corollary 4.3. So assume now that $m > 133$. Assume by way of induction that $k \in S$ for all $19 \leq k < m$. Now write $m = 4n + b$ where $b = 2, 3, 4, 5$, it follows that $n \geq 32$. Since there exists a TD$(6, n)$ for all $n > 22$ we can use Construction 4.1 to show that $m \in S$, by letting $k_1 = 1$ and $k_2 = b - 1$.

5 The spectrum: the 9 modulo 12 case

To construct HW$(12n + 9)$, we can do essentially Construction 4.1, but by giving weight 6 to exactly to one point in the first group we get a total number of points that is congruent to 9 modulo 12.

Lemma 5.1 *Assume there exists a $TD(6, n)$, $1 \leq k_1 \leq n - 1$ and $1 \leq k_2 \leq n$. If $\{n, k_2\} \subset S_3 \cup \{2, 4\}$ and $k_1 \in M$, then $4n + k_1 + k_2 \in M$.*

Proof. This is Construction 4.1(b), except exactly one point in the first group must receive weight 6. Here we use the fact that there exist 3-frames of type $12^4 6^1$ and $12^5 6^1$ (see [5]). □

We now can proceed as in the previous section to find the spectrum of HW(12t+9), i.e. the set M. Remember that from Proposition 3.15 we already have that $\{1, 2, 3, 4, 5, 6, 8, 9, 13, 15, 18, 21\} \subset M$.

Proposition 5.2 *If $22 \leq t \leq 132$, then $t \in M$.*

Proof. This proof is again very similar to Proposition 4.6 except here we use Lemma 5.1. The following table gives the values for the ingredients needed in the construction. We require that there exists a TD(6,n), and for $1 \leq k_1, k_2 \leq n$, that $\{n, k_2\} \subset S_3 \cup \{2, 4\}$ and $k_1 \in M$. The result is that $4n + k_1 + k_2 \in M$.

n	$k_1 \in$	$k_2 \in$	$4n + k_1 + k_2$
5	$\{1, 4\}$	$\{1, 2, 3, 4, 5\}$	$22 - 29$
7	$\{1, 6\}$	$\{1, \ldots, 7\}$	$30 - 41$
8	$\{1, 6\}$	$\{1, \ldots, 8\}$	$34 - 46$
11	$\{1, 9\}$	$\{1, \ldots, 8\}$	$46 - 61$
12	$\{1, 9\}$	$\{1, \ldots, 8\}$	$50 - 65$
16	$\{1, 9, 15\}$	$\{1, \ldots, 8\}$	$66 - 87$
19	$\{1, 9, 15, 18\}$	$\{1, \ldots, 8\}$	$78 - 102$
25	$\{1, 9, 15, 23, 24\}$	$\{1, \ldots, 8\}$	$102 - 132$

Theorem 5.3 *There exists an HW*$(12m + 9)$ *(i.e.* $m \in M$*) for all* $m \geq 1$ *except possibly for* $m \in \{7, 10, 11, 12, 14, 16, 17, 19, 20\}$.

Proof. If $m \leq 132$ the theorem follows from Propositions 3.15, 5.2 and Corollary 4.3. So assume now that $m > 132$. Assume by way of induction that $k \in M$ for all $22 \leq k < m$. Now write $m = 4n + b$ where $b = 2, 3, 4, 5$, it follows that $n \geq 32$. Since there exists a TD$(6, n)$ for all $n > 22$ we can use Lemma 5.1 with $k_1 = 1$ and $k_2 = b - 1$ to show that $m \in M$.

6 Conclusion

In this paper we considered the Hamilton-Waterloo problem in the case where there is exactly Hamilton cycle and all other 2-factors are triangle-factors. The necessary condition for such a decomposition is that $n \equiv 3 \pmod 6$. We have shown that this necessary condition is sufficient except when $n = 9$ and possibly for 14 additional cases, namely when $n \in \{93, 111, 123, 129, 141, 153, 159, 177, 183, 201, 207, 213, 237, 249\}$.

In a companion paper [7] we will extend the results of Horak, Nedela, and Rosa [9] given in Theorem 1.2(b) in the case where $n \equiv 3 \pmod{18}$.

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