Holey Factorizations

J. H. Dinitz
Department of Mathematics
University of Vermont
Burlington VT 05405

D. K. Garnick
Department of Computer Science
Bowdoin College
Brunswick ME 04011

Abstract: Holey factorizations of $K_{v_1,v_2,\ldots,v_n}$ are a basic building block in the construction of Room frames. In this paper we give some necessary conditions for the existence of holey factorizations and give a complete enumeration for nonisomorphic sets of orthogonal holey factorizations of several special types.

1 Introduction

The use of holes in designs is one of the most powerful tools in combinatorial design theory. The purpose of this paper is to study designs called holey factorizations. These objects are essentially one-factorizations of K_n with holes. They have been used implicitly in the construction of Room frames, but have never been studied systematically. We will give some general necessary conditions for the existence of holey factorizations, will discuss existence of holey factorizations of certain types and will give a complete enumeration of all nonisomorphic sets of orthogonal holey factorizations of several small types. We begin with the definitions.

Let V be a set of v vertices, and let $\{V_1, \ldots, V_n\}$ be a partition of V, where $|V_i| = v_i$. Let $K_{v_1,\ldots,v_n}$ denote the complete multipartite graph with vertices V and with parts $\{V_1, \ldots, V_n\}$. These parts are called the *holes* since $K_{v_1,\ldots,v_n} = K_v \setminus \bigcup_{i=1}^{n}\{xy | x, y \in V_i\}$. A *holey factor* of $K_{v_1,\ldots,v_n}$, missing hole V_i, is a one-factor of the graph $K_{v_1,\ldots,v_n} \setminus V_i$ (i.e. a set of edges such that each vertex of $K_{v_1,\ldots,v_n} \setminus V_i$ is on exactly one of these edges and there is no edge between any two vertices in the same hole). A *holey factorization* of $K_{v_1,\ldots,v_n}$ is a partition of the edges of the graph into holey factors such that for each $1 \le i \le n$ there are exactly v_i holey factors missing hole V_i.

The *type* of a holey factorization with holes $\{V_1, V_2, \ldots, V_n\}$ is the multiset $\{|V_1|, |V_2|, \ldots, |V_n|\}$. We will say that a holey factorization has *type* $T = t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ if there are u_i V_j's of cardinality t_i, $1 \le i \le k$. If a holey factorization has type t^n for some t and n, then it is called *uniform*.

Holey factorizations represent a special case of a general object called a frame (not to be confused with a Room frame). A *frame* is a group-divisible design (X, G, B) whose block set admits a partition into *holey* parallel classes, each holey parallel class being a partition of $X \setminus G_i$ for some group $G_i \in G$. A holey factorization is a frame where *every* block has size 2. For further information pertaining to frames in general see [12].

A holey factorization of type $1^1 3^4$ is given in Figure 1. Note that the holes $V_1 = \{1\}$, $V_2 = \{2, 3, 4\}$, $V_3 = \{5, 6, 7\}$, $V_4 = \{8, 9, a\}$, and $V_5 = \{b, c, d\}$ are written on the left of each holey factorization in square brackets.

[1]	{9 c}	{5 b}	{2 d}	{3 6}	{7 a}	{4 8}
[234]	{8 b}	{a d}	{5 c}	{7 9}	{1 6}	
[234]	{1 c}	{9 b}	{5 d}	{6 a}	{7 8}	
[234]	{5 9}	{a c}	{1 b}	{7 d}	{6 8}	
[567]	{2 a}	{1 8}	{9 d}	{4 b}	{3 c}	
[567]	{a b}	{8 d}	{4 c}	{1 3}	{2 9}	
[567]	{8 c}	{1 d}	{2 b}	{4 9}	{3 a}	
[89*a*]	{6 d}	{7 c}	{3 b}	{1 2}	{4 5}	
[89*a*]	{6 b}	{1 7}	{2 c}	{4 d}	{3 5}	
[89*a*]	{3 d}	{7 b}	{6 c}	{1 4}	{2 5}	
[*bcd*]	{5 8}	{1 a}	{3 9}	{2 7}	{4 6}	
[*bcd*]	{6 9}	{4 a}	{2 8}	{3 7}	{1 5}	
[*bcd*]	{4 7}	{5 a}	{3 8}	{1 9}	{2 6}	

Figure 1: A holey factorization of type $1^1 3^4$

Two holey factorizations F and G, both of type T, are said to be *orthogonal* if

1. for any two edges of the underlying graph (say e_1 and e_2), if e_1 and e_2 are in the same holey factor in F, then they are different holey factors of G; and

2. any holey factor in F and any holey factor in G which are missing the same hole have no edges in common.

Our interest in holey factorizations stems from their connection to Room frames. The following theorem gives this connection. (For results on Room frames see [4] or [5]).

Theorem 1.1 *The existence of a pair of orthogonal holey factorizations of type T is equivalent to the existence of a Room frame of type T.*

The proof is simple. Given a Room frame of type T, the rows and the columns are both holey factorizations of type T. Clearly the row holey factorization and the column holey factorization are orthogonal. This construction can be reversed also.

In Figure 2 we give a Room frame of type $1^1 3^4$ to help clarify the above connection. Note that the rows of this frame contain the holey factors in the holey factorization given in Figure 1.

Another connection which we wish to point out is given in the following theorem. Note that in order for a holey factorization of type 1^n to exist, then necessarily n must be odd; each factor consists of $(n-1)/2$ edges.

Theorem 1.2 *The existence of a holey factorization of type 1^n is equivalent to the existence of a one-factorization of K_{n+1}.*

Proof. Given the holey factorization F, say each holey factor f_i is missing the vertex x_i. Add the new vertex ∞ to the graph and let $h_i = f_i \cup \{x_i, \infty\}$. Then $\{h_1, h_2, \ldots, h_n\}$ is a one-factorization of the graph K_{n+1}. This construction is easily reversed. ∎

	$9c$		$5b$	$2d$				36		$7a$	48	
				$8b$	ad			$5c$			79	16
				$1c$	$9b$		$5d$				$6a$	78
59						ac	$1b$		$7d$	68		
$2a$	18	$9d$						$4b$	$3c$			
		ab	$8d$				$4c$				13	29
$8c$								$1d$	$2b$	49		$3a$
	$6d$		$7c$			$3b$				12		45
$6b$		17			$2c$	$4d$				35		
$3d$	$7b$	$6c$			14						25	
		58	$1a$	39				27	46			
			69	$4a$		28	37		15			
47	$5a$				38	19	26					

Figure 2: A Room frame of type $1^1 3^4$

Results on one-factorizations of K_n can be found in [10] or [20].

This paper is organized a follows: Section 2 describes some necessary conditions for the existence of a holey factorization of type T, in Section 3 we prove the existence of many types of holey factorizations, and in Section 4 we discuss some results on the enumeration of nonisomorphic holey factorizations with small types, and of sets of orthogonal holey factorizations of types 2^4 and 2^5.

2 Necessary conditions for existence

A holey factorization must have at least three holes; if a holey factorization had two holes, then a factor missing hole V_i would need to contain edges connecting pairs of vertices in the other hole. The following theorem applies to holey factorizations with exactly three holes.

Theorem 2.1 *If a holey factorization has exactly three holes then necessarily all the holes must be of the same size.*

The proof follows immediately from the observation that the edges in any of the factors missing hole V_i must form a matching between the vertices in the remaining two holes. The following is a necessary parity condition.

Theorem 2.2 *If there exists a holey factorization of type $t_1^{u_1} t_2^{u_2} \dots t_n^{u_n}$, then $t_i \equiv v \pmod 2$ for each i where $v = \sum_{i=1}^{n}(t_i \times u_i)$ is the number of vertices in the underlying graph.*

Proof. Since for each i there must be a holey factor on $v - t_i$ vertices, then $v - t_i$ must be even. Thus $t_i \equiv v \pmod 2$. ∎

The next theorem gives a condition on the sizes of the holes.

Theorem 2.3 *If there exists a holey factorization on v vertices of type $T = t_1t_2t_3 \dots t_n$, then $v \geq 2t_i + t_j$, for any i and j,*

Proof. Let V_i be a hole of size t_i and let V_j be a hole of size t_j. A holey factor missing V_j must pair up the vertices in V_i with a subset of vertices from $G - V_i - V_j$. Thus $t_i \leq v - t_i - t_j$. ∎

Theorems 2.2 and 2.3 are probably close to being sufficient conditions for the existence of holey factorizations. However we exhibit below an infinite class of holey factorizations that is allowed by the previous three theorems yet can not exist. We thank R. M. Wilson for helpful discussions concerning the next theorem.

Theorem 2.4 *There exists a holey factorization of type $a^2t_1t_2t_3 \dots t_k$ where $a = t_1 + t_2 + t_3 + \dots + t_k$ if and only if there exists a holey factorization of type $t_1t_2t_3 \dots t_k$.*

Proof. Let U_1 and U_2 be the two holes of size a and let $V_1, V_2, V_3, \dots V_k$ be the holes of sizes $t_1, t_2, t_3, \dots, t_k$, respectively. Now note that the $2a$ holey factors missing either U_1 or U_2 must contain every edge from $V_1, V_2, V_3, \dots V_k$ to U_1 and U_2. Thus, ignoring the edges between U_1 and U_2, the holey factors missing the holes $V_1, V_2, V_3, \dots V_k$ constitute a holey factorization of type $t_1t_2t_3 \dots t_k$. Thus, if there exists a holey factorization of type $a^2t_1t_2t_3 \dots t_k$, then there exists a holey factorization of type $t_1t_2t_3 \dots t_k$.

Conversely, begin with a holey factorization of type a^3 which exists by Theorem 3.1 below. Use Theorem 3.6 to fill in one hole of size a with a holey factorization of type $t_1t_2t_3 \dots t_k$. The resulting frame has type $a^2t_1t_2t_3 \dots t_k$. ∎

Corollary 2.5 *There does not exist a holey factorization of type $a^2b^1c^1$ if $b + c \leq a$.*

Proof. If $b + c < a$, then this holey factorization does not exist by Theorem 2.3. If $b+c = a$, the frame does not exist by Theorem 2.4 since there is no holey factorization of type b^1c^1. ∎

Theorem 2.4 provides an infinite class of nonexistent holey factorizations which satisfy the necessary conditions of Theorems 2.2 and 2.3. Essentially, given a type T ($|T| = t$) for which no holey factorization exists, Theorem 2.4 states that there also does not exist a holey factorization of type t^2T. This process can then be repeated to show that there does not exist a holey factorization of type $(3t)^2t^2T$, etc. For example, since there is no holey factorization of type 2^2, there is no holey factorization of type 4^22^2. This in turn implies that there is no holey factorization of type $12^24^22^2$. There is also never a holey factorization of type t^2abc where $a + b + c = t$ and $a \neq b$, since holey factorizations of type abc exist only if $a = b = c$ (Theorem 2.1).

The next theorem extends Theorem 2.3 to the case of d orthogonal holey factorizations. The case of $d = 2$ was originally proven in [17].

Theorem 2.6 *If there exists a set of d orthogonal holey factorization on v vertices of type $T = t_1t_2t_3 \dots t_n$, then $v \geq (d + 1)t_i + t_j$, for any i and j.*

Proof. Let V_i be a hole of size t_i and let V_j be a hole of size t_j (containing the vertex x). Also let $\{F_1, F_2, \ldots F_d\}$ be a set of d pairwise orthogonal holey factorizations. Since the holey factors missing V_i must contain an edge on the vertex x, and since all of the holey factors in $F_1, F_2, \ldots$ and F_d which are missing hole V_i contain no edges in common, we deduce that there must be at least $d \times t_i$ vertices other than the vertices in V_i or V_j. So $d \times t_i \leq v - t_i - t_j$ and hence $v \geq (d+1)t_i + t_j$. ∎

3 Existence of holey factorizations

In this section we will briefly discuss the existence of holey factorizations of certain specified types. Two particular types of Room frames have been extensively studied by researchers; these are frames of type h^u and of type $1^{u-v}v^1$. There have been numerous papers concerning the existence of these frames over the past 15 years. Recently, both spectra (for Room frames) have been essentially determined [19]. In contrast to the apparent difficulty in finding the spectra of the frames of these types, the first two theorems of this section give complete answers to the spectra of holey factorizations of these types. The remainder of this section will discuss some holey factorizations of other types.

The first result concerns uniform holey factorizations (type h^u). The proof can be found in [12] (Theorem 1.4). Note that the conditions are necessary by Theorem 2.2.

Theorem 3.1 (Rees-Stinson) *There exists a holey factorization of type h^u if and only if $u \geq 3$ and $h(u-1) \equiv 0 \bmod 2$.*

Holey factorizations of type $1^{u-v}v^1$ correspond to one-factorizations of K_{u+1} which contain sub one-factorizations of K_{v+1}. This problem has also been solved completely.

Theorem 3.2 (Cruse [1]) *There exists a holey factorization of type $1^{u-v}v^1$ if and only if u and v are odd and $u \geq 2v+1$.*

Since the existence of a Room frame of type T implies the existence of a holey factorization of type T (via Theorem 1.1), results on the existence of Room frames apply to holey factorizations also. We summarize the list of known frames in the following theorem.

Theorem 3.3 *There exist frames (and hence holey factorizations) of the following types*

1. $1^a 3^b$ *for all* $a+b = 5, 7$ *or* 9, *except for* $(a,b) = (2,3), (3,2), (4,1), (5,0), (5,2)$ *or* $(6,1)$, [5]
2. $2^a 4^b$ *for all* $a+b \in \{6, 7, \ldots, 14, 31, 42, 43, 44\}$ *or if* $a+b \geq 48$, [5]
3. $2^n u^1$ *if and only if u is even and* $n \geq u+1$, *except possibly for* $2^{19}18^1$. [6] [19]

The following three constructions are recursive constructions for holey factorizations.

If T is the type $t_1^{u_1}t_2^{u_2}\dots t_n^{u_n}$ and m is an integer, then mT is defined to be the type $(mt_1)^{u_1}(mt_2)^{u_2}\dots(mt_n)^{u_n}$. The following recursive construction is referred as the Inflation Construction. It essentially "blows up" every hole by use of a Latin square of order m (thought of as a one-factorization of $K_{n,n}$). We leave the details to the reader.

Construction 3.4 *Suppose there exists a holey factorization of type T and suppose that m is a positive integer, then there exists a holey factorization of type mT.*

An interesting fact to note in the above construction is that m can be *any* positive integer. In the analogous theorem for frames there is a restriction that $m \neq 2$ or 6.

The Fundamental Holey Factorization Construction follows immediately from the Fundamental Frame Construction [17].

Construction 3.5 (Fundamental Holey Factorization Construction) *Let $(X, \mathcal{G}, \mathcal{A})$ be a group divisible design having type T, and let $w : X \to \mathbf{Z}^+ \cup \{0\}$ (we say that w is a* weighting*). For every $A \in \mathcal{A}$, suppose there is a holey factorization having type $\{w(x) : x \in A\}$. Then there is a holey factorization of type $\{\sum_{x\in G} w(x) : G \in \mathcal{G}\}$.*

The final recursive construction that we present here is a Filling in the Hole Construction. The proof is immediate.

Construction 3.6 *Suppose that there exists a holey factorization of type $T = t_1^{u_1}t_2^{u_2}\dots t_n^{u_n}$ and a holey factorization of type $S = s_1^{v_1}s_2^{v_2}\dots s_m^{v_m}$ where $t_1 = \sum_{i=1}^{m}(s_i \times v_i)$, then there exists a holey factorization of type $t_1^{u_1-1}(s_1^{v_1}s_2^{v_2}\dots s_m^{v_m})t_2^{u_2}\dots t_n^{u_n}$.*

There are many more recursive constructions that can be used to construct holey factorizations. Essentially, any recursive construction for Room frames will work for holey factorizations. This includes the Filling in the Holes and Fundamental Construction above, as well as the use of holey transversals, and the use of frame starters and intransitive frame starters. It is beyond our scope to discuss these additional constructions here. The interested reader is referred to [5], [17], [18], and [19].

The next two theorems follow from Theorem 3.3, where the missing cases were constructed using a modification of the hill-climbing algorithm for one-factorizations (described in [4]) to holey factorizations. This algorithm is extremely fast and effective in constructing holey factorizations. Files containing these factorizations are available from the authors.

Theorem 3.7 *There exists a holey factorization of type 1^a3^b for all $a + b \in \{5, 7, 9\}$.*

Theorem 3.8 *There exists a holey factorization of type 2^a4^b if and only if $a + b \geq 4$ with $(a, b) \neq (2, 2)$ or if $(a, b) = (3, 0)$ or $(0, 3)$.*

Note that the conditions in Theorem 3.8 are necessary by Theorems 2.1 and 2.2. The holey factorizations of types 2^3 and 4^3 exist by Theorem 3.1. The remaining holey factorizations which were missing from Theorem 3.3(2) were again constructed by the authors using the hill-climbing algorithm for holey factorizations.

We have also generated holey factorizations for other small orders by use of the hill-climbing algorithm for holey factorizations. We record these in the next theorem.

Theorem 3.9 *There exist holey factorizations of the following types:* $2^4 6^1$, $2^3 6^2$, $2^2 6^3$, $2^1 6^4$, *and* $2^3 4^1 6^1$.

In the next theorem we will be concerned with holey factorizations of type $2^n u^1$. From Theorems 2.2 and 2.3 we have that necessarily u must be even and $n \geq \frac{1}{2}u + 1$. From Theorem 3.3(3) there is a holey factorization of type $2^n u^1$ if $n \geq u + 1$. In the theorem below we will show essentially that if $n \geq \frac{2}{3}u$, then there exists a holey factorization of type $2^n u^1$.

Theorem 3.10 *There exists a holey factorization of type* $2^n u^1$ *if*

1. $n \geq \frac{1}{2}u + 1$, $u \equiv 2 \bmod 4$ *and* n *is even,*
2. $n \geq \frac{2}{3}u$ *and* $u \equiv 0 \bmod 6$,
3. $n \geq \frac{2}{3}(u + 2)$ *and* $u \equiv 4 \bmod 6$, *and*
4. $n \geq \frac{2}{3}(u + 4)$ *and* $u \equiv 2 \bmod 6$.

Proof. Statement 1 follows from Theorem 3.2 and Theorem 3.4 with $m = 2$.

To prove statement 2, give weight 2 to every point in the first three groups of a transversal design TD(5,t), give weights 2, 4 or 6 to every point in the fourth group and give weight 6 to every point in the last group. (Note that $t \neq 2, 3, 6, 10$). Apply Construction 3.5, using the fact that holey factorizations of type $2^4 6^1$, $2^3 4^1 6^1$ and $2^3 6^2$ all exist by Theorem 3.9. Now fill in the first three groups with holey factorizations of type 2^t. Also fill in the fourth group with a holey factorization of type 2^a for $t \leq a \leq 3t$. Thus for every $4t \leq s \leq 6t$ we have constructed a holey factorization of type $2^s(6t)^1$. Let $u = 6t$ to get that there is a holey factorization of type $2^s u^1$ for all $\frac{2}{3}u \leq s \leq u$, except for the cases $2^i 12^1$ for $8 \leq i \leq 12$, $2^i 18^1$ for $12 \leq i \leq 18$, $2^i 36^1$ for $24 \leq i \leq 36$, and $2^i 60^1$ for $40 \leq i \leq 60$.

To handle the case $u = 36$, first begin with transversal designs TD(5,12). Give weight 1 to every point in the first three groups, give weight either 1 or 3 to the points in the fourth group and give weight 3 to every point in the last group. Apply Construction 3.5, using the fact that holey factorizations of type $1^a 3^{5-a}$ exist for $a = 3, 4$ by Theorem 3.7. The construction now proceeds as above by filling in the first four groups with holey factorizations of type 2^a for $6 \leq a \leq 18$. The case $u = 60$ is done similarly beginning with a $TD(5, 20)$. All of the missing cases for $u = 12$ and $u = 18$ (including $2^{19} 18^1$) were constructed on the computer by use of the hill-climbing algorithm. The final result now follows from Theorem 3.3(3).

Statement 3 is proven in a similar manner. Now give all the points in the last group of the TD(5,t) weight 6 except for one point which receives weight 4. The proof is then the same as above. Again the exceptional cases occur when $t = 2, 3, 6, 10$. The cases for $t = 2$ and $t = 3$ ($u = 10$ and $u = 16$, respectively) were handled by the computer. For $t = 6$ and $t = 10$ ($u = 34$ and $u = 58$, respectively) use the same construction as above for $u = 36$ and $u = 60$ except in the last group give weight 1 to exactly one point.

Statement 4 follows this same pattern. Here we give all the points in that last group weight 6 except for one point which receives weight 2. The proof is again the same as above where once more the exceptional cases occur when $t = 2, 3, 6, 10$. The cases for $t = 2$ and $t = 3$ ($u = 8$ and $u = 14$, respectively) were handled by the computer. For $t = 6$ and $t = 10$ ($u = 32$ and $u = 56$, respectively) use the same construction as above for $u = 36$ and $u = 60$ except in the last group give weight 1 to exactly two points. ∎

4 Enumerating nonisomorphic holey factorizations

In this section we will enumerate the nonisomorphic holey factorizations of types 2^3, 3^3, 2^4 and 2^5. In addition we will enumerate nonisomorphic sets of orthogonal holey factorizations of type 2^5. We begin by defining isomorphic holey factorizations.

Two holey factorizations F and H of a graph G, say $F = \{f_1, f_2, \ldots, f_k\}$, $H = \{h_1, h_2, \ldots, h_k\}$, are called *isomorphic* if there exists a map ϕ from the vertex-set of G onto itself such that $\{f_1\phi, f_2\phi, \ldots, f_k\phi\} = \{h_1, h_2, \ldots, h_k\}$. Here $f_i\phi$ is the set of all the edges $\{x\phi, y\phi\}$ where $\{x, y\}$ is an edge in f_i.

The exact number of nonisomorphic one-factorizations (OFs) of K_n is known only for even $n \leq 12$. It is easy to see that there is a unique one-factorization of K_2, K_4, and K_6. There are exactly six for K_8; these were found by Dickson and Safford [2] and a full exposition is given in [21]. In 1973, Gelling [7] proved that there are exactly 396 isomorphism classes of OFs of K_{10}. Recently, Dinitz, Garnick, and McKay [3] determined that there are 526,915,620 nonisomorphic OFs of K_{12}.

It is easy to check that there is a unique holey factorization of each of the types 1^1, 1^3 and 1^5. The automorphism group of each of the six one-factorizations of K_8 is transitive on points (see [21]). Thus by deleting a point to form a holey factorization of type 1^7, we see that there are also exactly six nonisomorphic holey factorizations of this type.

In the next theorem we will count the number of distinct holey factorizations of type t^3. The underlying graph here is $K_{t,t,t}$. Let V_1, V_2, and V_3 be the three holes. The number of distinct ways to pick the holey factors missing hole V_i is obviously the number of distinct one-factorizations of $K_{t,t}$. This in turn is the number of distinct latin squares of side t with the first column fixed. We get the following theorem.

Theorem 4.1 *The number of distinct holey factorizations of type t^3, with fixed holes, is $[L(t)]^3$, where $L(t)$ is the number of distinct Latin squares of order t with the first column fixed.*

Since $L(2) = 1$ and $L(3) = 2$, then it follows that there is a unique holey factorization of 2^3 and 8 distinct holey factorizations of type 3^3.

Theorem 4.2 *There are 2 nonisomorphic holey factorizations of type* 3^3*.*

Proof. Without loss of generality, one can fix the first six factors, as well as one edge in the seventh factor, in a holey factorization of type 3^3. There are two choices for the second edge in the seventh factor, and that choice determines the rest of the factorization. Thus, there are at most two isomorphism classes of holey factorizations of type 3^3; representatives, A and B, of the classes are shown in Figure 3, where the holes are indicated as triples in square brackets. If we choose any three holey

	A:		B:
A:	[123] 47 58 69	B:	[123] 47 58 69
	[123] 48 59 67		[123] 48 59 67
	[123] 49 57 68		[123] 49 57 68
	[456] 17 28 39		[456] 17 28 39
	[456] 18 29 37		[456] 18 29 37
	[456] 19 27 38		[456] 19 27 38
	[789] 14 25 36		[789] 14 26 35
	[789] 15 26 34		[789] 15 24 36
	[789] 16 24 35		[789] 16 25 34

Figure 3: The two nonisomorphic holey factorizations of type 3^3

factors f, g, and h from A such that each of the three is missing a different hole, then $f \bigcup g \bigcup h$ forms either three disjoint triangles, or a single 9-cycle. However, if we choose any three holey factors f, g, and h from B, again such that each is missing a different hole, then $f \bigcup g \bigcup h$ forms a triangle and a disjoint 6-cycle. Thus, A and B are nonisomorphic, and there are two equivalence classes of 3^3 holey factorizations. ∎

In order to count holey factorizations on a larger number of holes we use the computer. We construct nonisomorphic holey factorizations by use of an *orderly algorithm*; it generates the nonisomorphic holey factorizations of type T in a lexicographic order defined in the following way. First, if $F = \{f_1, f_2, \ldots, f_n\}$ is a holey factorization, then we must have $f_i < f_j$ for all $i < j$. Then we say that $F > H = \{h_1, h_2, \ldots, h_n\}$ if there is some k, $1 \leq k \leq n$, such that $f_i = h_i$ for all $i < k$ and $f_k > h_k$. We define a similar ordering on partial holey factorizations consisting of fewer than n factors.

The algorithm builds up each factorization by adding one factor at a time and rejects a partial factorization if it is not the lowest representative (lexicographically) of all the partial factorizations in its isomorphism class. In this way, the algorithm generates only the lowest representative of any isomorphism class of factorizations and as such never generates any holey factorizations which are isomorphic to each other. This approach saves both time and space over algorithms which first generate distinct (but possibly isomorphic) factorizations and then use methods to winnow isomorphs.

This type of algorithm has been used in other combinatorial searches including enumerating Latin squares [11], strong starters [8], one-factorizations of small graphs [13, 15], perfect one-factorizations of K_{14} [14], and Howell designs of small order

[13]. Our algorithm below is essentially the one that was described in [3] to count the nonisomorphic one-factorizations of K_{12}. However, our algorithm here reduces the search space by assuming a first fixed factor consisting of the hyper-edges that correspond to the holes.

Theorem 4.3 *There are* 2 *nonisomorphic holey factorizations of type* 2^4 *and, for a fixed set of holes, there are* 40 *distinct holey factorizations of type* 2^4.

Using the orderly algorithm we generated the lexicographically lowest representatives of the two isomorphism classes of holey factorizations of type 2^4. The computation required about .1 seconds at a rate of 20 mips. The two factorizations are shown in Figure 4.

1:	[1 2] 35 47 68	2:	[1 2] 35 47 68
$\lvert Aut\rvert = 12$	[1 2] 36 48 57	$\lvert Aut\rvert = 48$	[1 2] 38 46 57
	[3 4] 15 28 67		[3 4] 15 28 67
	[3 4] 17 26 58		[3 4] 17 26 58
	[5 6] 14 27 38		[5 6] 13 27 48
	[5 6] 18 24 37		[5 6] 18 24 37
	[7 8] 13 25 46		[7 8] 14 25 36
	[7 8] 16 23 45		[7 8] 16 23 45

Figure 4: The two nonisomorphic holey factorizations of type 2^4

We determined the orders of the automorphism groups by counting the number of permutations that mapped a factorization onto itself while checking that the factorization was the lexicographically lowest representative of its isomorphism class. We checked the results by using *nauty* [9] to determine the order of the automorphism group of the line graph corresponding to the factorization. If we label the nonisomorphic holey factorizations of type 2^4 as F_i, $1 \leq i \leq 2$, then the number of distinct holey factorizations of type 2^4, with fixed holes, can be computed as

$$\sum_{i=1}^{2} \frac{4!\ 2^4}{|Aut(F_i)|} = 40$$

We checked this result by fixing the holes, and using backtracking to generate all distinct factorizations.

Theorem 4.4 *There are* 747 *nonisomorphic holey factorizations of type* 2^5 *and, for a fixed set of holes, there are* $2,253,312$ *distinct holey factorizations of type* 2^5.

Again, we used the orderly algorithm to generate the lexicographically lowest representatives of the isomorphism classes of holey factorizations of type 2^5. The computation required about 20 minutes at a rate of 20 mips. The file containing these 747 holey factorizations is available from the authors.

We computed the orders of the automorphism groups in the same ways as for the 2^4 factorizations. We computed the number of distinct holey factorizations of type 2^5, with fixed holes, as

$$\sum_{i=1}^{747} \frac{5!\ 2^5}{|Aut(F_i)|} = 2,253,312$$

where $\{F_i | 1 \leq i \leq 747\}$ is the set of nonisomorphic holey factorizations of type 2^5. Checking this result with backtracking required 5.5 hours of cpu time at 20 mips.

We now turn our attention to nonisomorphic *sets* of orthogonal holey factorizations. There does not exist a pair of orthogonal holey factorizations of type 2^4 [16]. We will show below that there are pairs (in fact triples) of orthogonal holey one-factorizations of type 2^5.

Two sets of orthogonal holey factorizations $\bar{F}$ and $\bar{H}$ of type T, say $\bar{F} = \{F_1, F_2, \ldots, F_s\}$, $\bar{H} = \{H_1, H_2, \ldots, H_s\}$, are called *isomorphic* if there exists a map ϕ from the vertex-set of G onto itself such that $\{F_1\phi, F_2\phi, \ldots, F_s\phi\} = \{H_1, H_2, \ldots, H_s\}$. When $s = 2$, this corresponds to the notion of isomorphic Room frames of type T.

We extend the ordering used for holey factorizations to an ordering of sets of holey factorizations. First, if $\bar{F} = \{F_1, F_2, \ldots, F_s\}$ is a set of orthogonal holey factorizations, then we must have $F_i < F_j$ for all $i < j$. Then we say that $\bar{F} > \bar{H}$ if there is some t, $1 \leq t \leq s$, such that $F_i = H_i$ for all $i < t$ and $F_t > H_t$. Using this ordering we obtained the following results.

Theorem 4.5 *1. There are* 64 *nonisomorphic sets of two orthogonal holey factorizations of type* 2^5.

2. There are 28 *nonisomorphic sets of three orthogonal holey factorizations of type* 2^5.

3. There are no nonisomorphic sets of four orthogonal holey factorizations of type 2^5.

We generated the 64 mutually orthogonal pairs of factorizations in the following way. Let F_i, $1 \leq i \leq 747$ be the lexicographically lowest nonisomorphic factorizations as described in Theorem 4.4. We attempted to pair each F_i, $1 \leq i \leq 747$, with each possible permutation of F_j, $i \leq j \leq 747$. We checked each successful pairing to see if it was the lexicographically lowest such pair in its isomorphism class.

The Appendix displays the 64 pairs as Room frames of type 2^5. We counted the sets of three and four orthogonal holey factorizations in a similar way. The Appendix also lists the sets of three orthogonal factorizations of type 2^5.

References

[1] A.B. Cruse, On embedding incomplete symmetric Latin squares, *J. Combin. Theory (A)* **16** (1974), 18–22.

[2] L.E. Dickson and F.H. Safford, Solution to problem 8 (group theory). *Amer. Math. Monthly* **13** (1906), 150–151.

[3] J.H. Dinitz, D.K. Garnick, and B.D. McKay, There are 526,915,620 nonisomorphic one-factorizations of K_{12}, *J. Comb. Des.*, **2** (1994), 273–285.

[4] J.H. Dinitz and D.R. Stinson, "Room Squares and Related Designs", *Contemporary Design Theory*, J.H. Dinitz and D.R. Stinson, eds., Wiley, New York, 1992, pp. 137–204.

[5] J.H. Dinitz and D.R. Stinson, A few more Room frames, *Graphs Matrices and Designs*, R. Rees (Editor), Dekker, New York, 1993, pp. 133–146.

[6] G. Ge and L.Zhu, Existence of Room frames of type $2^n u^1$, *J. Comb. Math. and Comb. Comput.*, (to appear).

[7] E.N. Gelling and R.E. Odeh, On 1-factorizations of the complete graph and the relationship to round-robin schedules. *Congressus Num.* **9**(1974), 213–221.

[8] W.L. Kocay, D.R. Stinson and S.A. Vanstone, On strong starters in cyclic groups, *Discrete Math.* **56** (1985), 45–60.

[9] B.D. McKay, *nauty* User's Guide, Computer Science Dept., Australian National University, Canberra, Australia.

[10] E. Mendelsohn and A. Rosa, One-factorizations of the complete graph – A survey, *J. Graph Theory* **9** (1985), 43–65.

[11] R.C. Read, Every one a winner, *Annals of Discrete Math.* **2** (1978), 107–120.

[12] R.S. Rees and D.R. Stinson, Frames with block size four, *Can. J. Math.* **44** (1992), 1030–1049.

[13] E. Seah and D.R. Stinson, An enumeration of non-isomorphic one-factorizations and Howell designs for the graph K_{10} minus a one-factor, *Ars Combin.* **21** (1986), 145–161.

[14] E. Seah and D.R. Stinson, Some perfect one-factorizations for K_{14}. *Ann. Discrete Math.* **34** (1987), 419–436.

[15] E. Seah and D.R. Stinson, On the enumeration of one-factorizations of the complete graph containing prescribed automorphism groups. *Math Comp.* **50** (1988), 607–618.

[16] D.R. Stinson, The non-existence of a (2,4)-frame, *Ars Combin.* **11** (1981), 99–106.

[17] D.R. Stinson, Some constructions for frames, Room squares, and subsquares, *Ars Combin.* **12** (1981), 229–267.

[18] D.R. Stinson and L. Zhu, Towards the spectrum of Room squares with subsquares, *J. Comb. Theor Ser. A* **63** (1993), 129–142.

[19] D.R. Stinson, L. Zhu and J.H. Dinitz, On the spectra of certain classes of Room frames, (preprint).

[20] W.D. Wallis, One-factorizations of complete graphs, in *Contemporary Design Theory*, pp. 593–631, J.H. Dinitz and D.R. Stinson, eds., John Wiley and Sons, New York, 1992.

[21] W.D. Wallis, A.P. Street and J.S. Wallis, *Combinatorics: Room squares, sun-free sets, Hadamard matrices* Lect. Notes Math. **292**, Springer-Verlag, Berlin, 1972.

Appendix: Nonisomorphic sets of orthogonal holey factorizations of type 2^5

The 64 sets of two orthogonal holey factorizations of type 2^5 are displayed below as Room frames. Beneath the index of each Room frame is the pair of indices of the constituent holey factorizations. For example, Room frame 1 is derived from holey factorizations 1 and 690. The first holey factorization of each pair, which yields the rows of the Room frame, is always in canonical (lexicographically lowest) form; the second holey factorization, yielding the columns, is not necessarily in canonical form.

1:
1
690

				79	68		35	24	
		69	78			34			25
59	48							17	06
				16	07	58	49		
26						19	08		37
	27	18	09					36	
	39	04	15	28					
38					29			05	14
		57	46		13	02			
47	56			03			12		

2:
1
690

				68	79		35	24	
		69	78			34			25
59	48			17	06				
						58	49	16	07
26						19	08	37	
	27	18	09						36
	39	04	15		28				
38				29				05	14
		57	46		13	02			
47	56			03			12		

3:
1
690

		68	79			35		24	
				78	69		34		25
59	48							17	06
				16	07	49	58		
26		19	08						37
	27					18	09	36	
	39	04	15		28				
38				29				05	14
		57	46		13	02			
47	56			03			12		

4:
1
690

		79	68				35		24
				78	69	34		25	
59	48							17	06
				16	07	58	49		
26						19	08		37
	27	18	09					36	
	39				28			04	15
38		05	14	29					
		46	57		13	02			
47	56			03			12		

5:
1
690

				79	68	35			24
		78	69				34	25	
59	48							17	06
				16	07	49	58		
26		19	08						37
	27					18	09	36	
	39			28				04	15
38		05	14		29				
		46	57		13	02			
47	56			03			12		

6:
1
690

				68	79	35			24
		78	69				34	25	
59	48			17	06				
						49	58	16	07
26		19	08					37	
	27					18	09		36
	39				28			04	15
38		05	14	29					
		46	57		13	02			
47	56			03			12		

7:
1
690

			79	68		24	35		
		69			78			34	25
59				17	06		48		
	49					58		16	07
26	37		08			19			
		18					09	27	36
	28	04	15	39					
38					29			05	14
		57	46	02	13				
47	56					03	12		

8:
1
746

				68	79	24	35		
		69	78					34	25
59	48			17	06				
						58	49	16	07
26	37					19	08		
		18	09					27	36
		04	15	39	28				
38	29							05	14
		57	46	02	13				
47	56					03	12		

9:
1
746

			79	68				24	35
		69	78			34	25		
59	48							17	06
				16	07	58	49		
26	37					19	08		
		18	09					36	27
		04	15	28	39				
38	29							05	14
		57	46			02	13		
47	56			03	12				

10: 1 690

			79	68				35	24
		69			78	25	34		
	59			17	06	48			
49							58	16	07
26	37		08			19			
		18					09	27	36
	28			39				04	15
38		05	14		29				
57	46			02	13				
		47	56			03	12		

11: 1 746

				68	79	24	35		
		78	69					34	25
				17	06	59	48		
49	58							16	07
26	37	19	08						
						18	09	27	36
		04	15	39	28				
38	29							05	14
57	46			02	13				
		56	47			03	12		

12: 1 746

		68	79			24	35		
				78	69			25	34
						59	48	17	06
49	58			16	07				
26	37	19	08						
						18	09	36	27
				39	28			04	15
38	29	05	14						
57	46			02	13				
		47	56			03	12		

13: 2 690

				79	68		35	24	
		69	78			34			25
59	48							17	06
				16	07	58	49		
26						19	08		37
	27	18	09					36	
	39	04	15	28					
38					29			05	14
47	56				13	02			
		57	46	03			12		

14: 2 690

		68	79			35		24	
				78	69		34		25
59	48							17	06
				16	07	49	58		
26		19	08						37
	27					18	09	36	
	39	04	15		28				
38				29				05	14
47	56				13	02			
		57	46	03			12		

15: 2 690

		68	79			35		24	
				69	78		34		25
59	48			17	06				
						49	58	16	07
26		19	08					37	
	27					18	09		36
	39	04	15	28					
38					29			05	14
47	56				13	02			
		57	46	03			12		

16: 2 690

		79	68				35		24
				78	69	34		25	
59	48							17	06
				16	07	58	49		
26						19	08		37
	27	18	09					36	
	39				28			04	15
38		05	14	29					
47	56				13	02			
		46	57	03			12		

17: 2 690

		79	68				35		24
				69	78	34		25	
59	48			17	06				
						58	49	16	07
26						19	08	37	
	27	18	09						36
	39			28				04	15
38		05	14		29				
47	56				13	02			
		46	57	03			12		

18: 2 690

				68	79	35			24
		78	69				34	25	
59	48			17	06				
						49	58	16	07
26		19	08					37	
	27					18	09		36
	39				28			04	15
38		05	14	29					
47	56				13	02			
		46	57	03			12		

19: 2 690

			79	68		24	35		
		69			78			34	25
59				17	06		48		
	49					58		16	07
26	37		08			19			
		18					09	27	36
	28	04	15	39					
38					29			05	14
47	56			02	13				
		57	46			03	12		

20: 2 690

		79			68	24	35		
			69	78				25	34
59							48	17	06
	49			16	07	58			
26	37		08			19			
		18					09	36	27
	28			39				04	15
38		05	14		29				
47	56			02	13				
		46	57			03	12		

21: 2 746

				68	79	24	35		
		69	78					34	25
59	48			17	06				
						58	49	16	07
26	37					19	08		
		18	09					27	36
		04	15	39	28				
38	29							05	14
47	56			02	13				
		57	46			03	12		

22: 2 746

				79	68			24	35
		69	78			34	25		
59	48							17	06
				16	07	58	49		
26	37					19	08		
		18	09					36	27
		04	15	28	39				
38	29							05	14
47	56					02	13		
		57	46	03	12				

23: 2 746

		68	79					24	35
				78	69	25	34		
59	48							17	06
				16	07	49	58		
26	37	19	08						
						18	09	36	27
		04	15	39	28				
38	29							05	14
47	56			02	13				
		57	46			03	12		

24: 2 746

				68	79			35	24
		78	69			25	34		
59	48			17	06				
						49	58	16	07
26	37	19	08						
						18	09	27	36
				39	28			04	15
38	29	05	14						
47	56			02	13				
		46	57			03	12		

25:
4
312

				79	68			35	24
		69	78			34	25		
				06		59	48	17	
49	56				07	18			
		08					19	26	37
27	38		09						16
58	29			13				04	
		14		28	39				05
36	47		15			02			
		57	46		12		03		

26:
6
312

				79	68			35	24
		69	78			34	25		
				06		59	48		17
49	56				07	18			
		08					19	27	36
37	28		09					16	
58	39			12				04	
		14		38	29				05
		57	46		13	02			
26	47		15				03		

27:
7
670

		79			68		35	24	
			69	78		34			25
	48				17	59		06	
57				19		08			46
	39	16					28		07
26			18				09	37	
49		58		02					13
38		04			29		15		
	56		47		03	12			
	27		05	36			14		

28:
7
673

		79			68		35	24	
			69	78		34			25
48					17	59		06	
57				19		08			46
39		16					28		07
26			18				09	37	
	49	58		02					13
	38	04			29			15	
	56		47		03	12			
	27		05	36			14		

29:
9
746

				68	79			35	24
		78	69			25	34		
59	48							17	06
				19	08			46	57
		16	07			39	28		
26	37					18	09		
		49	58					02	13
38	29	05	14						
47	56			03	12				
				27	36	04	15		

30:
19
746

				68	79			35	24
		69	78			25	34		
				17	06	48	59		
57	46					19	08		
38	29							16	07
		18	09					27	36
49	58			02	13				
				39	28			04	15
		47	56			03	12		
26	37	05	14						

31:
20
670

		79		68			35	24	
			78		69	34			25
59					17		48	06	
	46			19		08		57	
38			16			29			07
	27	18					09		36
	58		49	02				13	
	39		05		28				14
47		56			03		12		
26		04		37		15			

32:
20
673

		79		68			35	24	
			78		69	34			25
	59				17		48	06	
	46			19		08		57	
	38		16			29			07
	27	18					09		36
58			49	02				13	
39			05		28				14
47		56			03		12		
26		04		37		15			

33:
52
55

				79	68			35	24
		69	78			25	34		
59				06		18		47	
48					07		19		56
	27	08				39		16	
	36		09				28		17
	49		15	38				02	
	58	14			29				03
26		57			13	04			
37			46	12			05		

34:
52
58

				79	68	35	24		
		69	78					34	25
59				06			18		47
48					07	19		56	
	27	08					39		16
	36		09			28		17	
	49		15	38				02	
	58	14			29				03
26		57			13	04			
37			46	12			05		

35:
52
52

				79	68			35	24
		69	78			25	34		
	59			06		18		47	
	48				07		19		56
	27	08				39		16	
	36		09				28		17
49			15	38				02	
58		14			29				03
26		57			13	04			
37			46	12			05		

36:
52
58

				79	68	35	24		
		69	78					34	25
	59			06			18		47
	48				07	19		56	
	27	08					39		16
	36		09			28		17	
49			15	38				02	
58		14			29				03
26		57			13	04			
37			46	12			05		

37:
55
58

				79	68			35	24
		69	78			25	34		
59				06			18	47	
48					07	19			56
37		08					29	16	
26			09			38			17
	58	14			39			02	
	49		15	28					03
	36	57			12	04			
	27		46	13			05		

38:
55
55

				79	68	35	24		
		69	78					34	25
	59			06		18			47
	48				07		19	56	
37		08				29			16
26			09				38	17	
58		14			39			02	
49			15	28					03
	36	57			12	04			
	27		46	13			05		

39:
55
58

				79	68			35	24
		69	78			25	34		
	59			06			18	47	
	48				07	19			56
37		08					29	16	
26			09			38			17
58		14			39			02	
49			15	28					03
	36	57			12	04			
	27		46	13			05		

40: 58 58

				79	68	35	24		
		69	78					34	25
59				06			18		47
48					07	19		56	
26		08					39	17	
37			09			28			16
	49		15	38				02	
	58	14			29				03
	36	57		12		04			
	27		46		13		05		

41: 58 58

		79	68					35	24
				78	69	25	34		
59				06		18		47	
48					07		19		56
26		08				39			17
37			09				28	16	
	49	15			38			02	
	58		14	29					03
	36		57		12	04			
	27	46		13			05		

42: 58 58

				79	68	35	24		
		69	78					34	25
	59			06			18		47
	48				07	19		56	
26		08					39	17	
37			09			28			16
49			15	38				02	
58		14			29				03
	36	57		12		04			
	27		46		13		05		

43: 58 58

		79	68					24	35
				78	69	34	25		
47					18	59		06	
56				19			48		07
39			17			08			26
28		16					09	37	
	38		49	02				15	
	29	58			03				14
	57	04		36		12			
	46		05		27		13		

44: 188 208

		68		79		35	24		
		49	78	36					25
57				18	69	04			
						19	58	46	07
39		17	06			28			
	38						09	27	16
48			59		12			03	
	29				08			15	34
	56		14	02	37				
26	47	05					13		

45: 217 304

		68	79				24	35	
		49		78	36				25
	57			69	18			04	
						58	09	16	47
						19	38	27	06
26	39	17	08						
59	28		14	03					
48		05			29				13
37	46					02	15		
			56	12	07	34			

46: 268 746

		79	68					35	24
		48	59	26	37				
				78	69	04	15		
				19	08			46	57
						39	28	17	06
27	36					18	09		
49	58							02	13
38	29	05	14						
56	47			03	12				
		16	07			25	34		

47: 269 746

		79	68					35	24
		48	59	26	37				
				78	69	04	15		
				19	08			46	57
						39	28	17	06
27	36					18	09		
49	58							02	13
38	29	05	14						
	47	16		03		25			
56			07		12		34		

48: 282 746

		79	68					24	35
				26	37	48	59		
78	69					15	04		
				19	08			56	47
						39	28	17	06
36	27	18	09						
49	58							03	12
		05	14	38	29				
		46	57			02	13		
25	34			07	16				

49: 282 705

			68		79	35			24
			59		26	48		37	
69				78			15	04	
47					08	19		56	
	39	06					28		17
	27	18					09		36
	58	49		03				12	
38			14	29					05
	46	57			13	02			
25			07	16			34		

50: 283 746

		79	68					24	35
				26	37	48	59		
78	69					15	04		
				19	08			56	47
						39	28	17	06
36	27	18	09						
49	58							03	12
		05	14	38	29				
	34		57		16	02			
25		46		07			13		

51: 283 705

			68		79	35			24
			59		26	48		37	
69				78			15	04	
47					08	19		56	
	39	06					28		17
	27	18					09		36
	58	49		03				12	
38			14	29					05
		57		16		02	34		
25	46		07		13				

52: 284 746

		79	68					24	35
				26	37	48	59		
78	69					15	04		
				19	08			56	47
						39	28	17	06
36	27	18	09						
	58		14		29			03	
49		05		38					12
	34		57		16	02			
25		46		07			13		

53: 285 746

		79	68					24	35
				26	37	48	59		
78	69					15	04		
				19	08			56	47
	27	18				39			06
36			09				28	17	
49	58							03	12
		05	14	38	29				
	34		57		16	02			
25		46		07			13		

54: 286 746

		79	68					24	35
				26	37	48	59		
78	69					15	04		
				19	08			56	47
	27	18				39			06
36			09				28	17	
	58		14		29			03	
49		05		38					12
	34		57		16	02			
25		46		07			13		

55: 287 746

		79	68					35	24
				37	26	48	59		
78	69					15	04		
				19	08			47	56
36		18				29			07
	27		09				38	16	
	58		14		39			02	
49		05		28					13
25		46			17	03			
	34		57	06			12		

56: 312 312

		79	68					35	24
				37	26	59	48		
69				18		04			57
78					19		05	46	
	27	08					39		16
	36		09			28		17	
	49		15		38			02	
	58	14		29					03
25			47	06		13			
34		56			07		12		

57: 312 690

		79	68					35	24
				37	26	59	48		
	69			18		04			57
78					19		05	46	
	27	08				39			16
36			09			28		17	
49			15		38			02	
	58	14		29					03
25			47	06		13			
	34	56			07		12		

58: 319 319

		68	79					35	24
				26	37	48	59		
69				18			04		57
78					09	15		46	
	27	19					38		06
	36		08			29		17	
	58		14	39				02	
	49	05			28				13
25		47			16	03			
34			56	07			12		

59: 319 746

		68	79					35	24
				26	37	48	59		
	69			18			04		57
78					09	15		46	
	27	19					38		06
36			08			29		17	
	58		14	39				02	
49		05			28				13
25		47			16	03			
	34		56	07			12		

60: 342 342

		79	68					35	24
				37	26	58	49		
69				18		04			57
78					19		05	46	
	27	08				39			16
	36		09				28	17	
	59	14			38			02	
	48		15	29					03
25			47	06			13		
34		56			07	12			

61: 342 342

				79	68	35	24		
		49	58					37	26
69						04	18		57
78						19	05	46	
	27	08	16				39		
	36	17	09			28			
	59			38				02	14
	48				29			15	03
25			47	06	13				
34		56		12	07				

62: 343 347

		79	68					35	24
				37	26	58	49		
69					18	04			57
78				19			05	46	
	36	08				29		17	
	27		09				38		16
	48		15		39			02	
	59	14		28					03
25			47	06		13			
34		56			07		12		

63: 690 690

			79	36			58		24
		68			37	49		25	
	69			18			04		57
78					09	15		46	
	27	19				38			06
26			08				39	17	
59			14		28			03	
	48	05		29					13
	35	47			16	02			
34			56	07			12		

64: 746 746

			79	36			58		24
		68			27	49		35	
69				18			04		57
	78				09	15		46	
37		19				28			06
	26		08				39	17	
	59		14		38			02	
48		05		29					13
25		47			16	03			
	34		56	07			12		

The 28 sets of three orthogonal holey factorizations of type 2^5 are listed below in the following way. First, the index of the triple is given as $\#i$, for $1 \leq i \leq 28$. Beneath the index of the triple is the parenthesized index of the Room frame that yields the first two factorizations of the triple. The third holey factorization is fully listed with the holes bracketed. Beneath the index of the Room frame are the indices of the three factorizations; as with the Room frames, only the first factorization is necessarily in canonical form. Thus, for example, triple number 6 consists of Room frame 8 (which in turn consists of canonical factorization 1 together with a factorization isomorphic to canonical factorization 746) and the fully listed factorization which is isomorphic to canonical factorization 746.

#1:
(1)
1
690
690

[01] 28 36 49 57
[01] 29 37 46 58
[23] 05 19 47 68
[23] 08 14 56 79
[45] 02 17 38 69
[45] 06 12 39 78
[67] 03 18 24 59
[67] 09 13 25 48
[89] 04 16 27 35
[89] 07 15 26 34

#2:
(2)
1
690
690

[01] 28 36 49 57
[01] 29 37 46 58
[23] 05 19 47 68
[23] 08 14 56 79
[45] 02 16 39 78
[45] 07 12 38 69
[67] 03 18 24 59
[67] 09 13 25 48
[89] 04 17 26 35
[89] 06 15 27 34

#3:
(3)
1
690
690

[01] 28 36 49 57
[01] 29 37 46 58
[23] 05 18 47 69
[23] 09 14 56 78
[45] 02 17 39 68
[45] 06 12 38 79
[67] 03 19 25 48
[67] 08 13 24 59
[89] 04 16 27 35
[89] 07 15 26 34

#4:
(4)
1
690
690

[01] 28 36 49 57
[01] 29 37 46 58
[23] 04 19 56 78
[23] 08 15 47 69
[45] 02 17 39 68
[45] 06 12 38 79
[67] 03 18 24 59
[67] 09 13 25 48
[89] 05 16 27 34
[89] 07 14 26 35

#5:
(5)
1
690
690

[01] 28 36 49 57
[01] 29 37 46 58
[23] 04 18 56 79
[23] 09 15 47 68
[45] 02 17 38 69
[45] 06 12 39 78
[67] 03 19 25 48
[67] 08 13 24 59
[89] 05 16 27 34
[89] 07 14 26 35

#6:
(8)
1
746
746

[01] 27 39 46 58
[01] 28 36 49 57
[23] 05 19 47 68
[23] 08 14 56 79
[45] 03 16 29 78
[45] 07 12 38 69
[67] 02 18 34 59
[67] 09 13 25 48
[89] 04 17 26 35
[89] 06 15 24 37

#7:
(9)
1
746
746

[01] 27 39 46 58
[01] 28 36 49 57
[23] 05 19 47 68
[23] 08 14 56 79
[45] 02 17 38 69
[45] 06 13 29 78
[67] 03 18 24 59
[67] 09 12 35 48
[89] 04 16 25 37
[89] 07 15 26 34

#8:
(10)
1
690
690

[01] 27 39 48 56
[01] 29 36 47 58
[23] 04 19 57 68
[23] 09 15 46 78
[45] 03 16 28 79
[45] 07 12 38 69
[67] 02 18 35 49
[67] 08 13 24 59
[89] 05 17 26 34
[89] 06 14 25 37

#9:
(11)
1
746
746

[01] 27 39 48 56
[01] 28 36 47 59
[23] 05 18 46 79
[23] 09 14 57 68
[45] 03 16 29 78
[45] 07 12 38 69
[67] 02 19 34 58
[67] 08 13 25 49
[89] 04 17 26 35
[89] 06 15 24 37

#10:
(12)
1
746
746

[01] 27 39 48 56
[01] 28 36 47 59
[23] 04 18 57 69
[23] 09 15 46 78
[45] 03 17 29 68
[45] 06 12 38 79
[67] 02 19 34 58
[67] 08 13 25 49
[89] 05 16 24 37
[89] 07 14 26 35

#11:
(13)
2
690
690

[01] 28 36 49 57
[01] 29 37 46 58
[23] 05 19 47 68
[23] 08 14 56 79
[45] 02 17 38 69
[45] 06 12 39 78
[67] 03 18 24 59
[67] 09 13 25 48
[89] 04 16 27 35
[89] 07 15 26 34

#12:
(14)
2
690
690

[01] 28 36 49 57
[01] 29 37 46 58
[23] 05 18 47 69
[23] 09 14 56 78
[45] 02 17 39 68
[45] 06 12 38 79
[67] 03 19 25 48
[67] 08 13 24 59
[89] 04 16 27 35
[89] 07 15 26 34

#13:
(15)
2
690
690

[01] 28 36 49 57
[01] 29 37 46 58
[23] 05 18 47 69
[23] 09 14 56 78
[45] 02 16 38 79
[45] 07 12 39 68
[67] 03 19 25 48
[67] 08 13 24 59
[89] 04 17 26 35
[89] 06 15 27 34

#14:
(16)
2
690
690

[01] 28 36 49 57
[01] 29 37 46 58
[23] 04 19 56 78
[23] 08 15 47 69
[45] 02 17 39 68
[45] 06 12 38 79
[67] 03 18 24 59
[67] 09 13 25 48
[89] 05 16 27 34
[89] 07 14 26 35

#15:
(18)
2
690
690

[01] 28 36 49 57
[01] 29 37 46 58
[23] 04 18 56 79
[23] 09 15 47 68
[45] 02 16 39 78
[45] 07 12 38 69
[67] 03 19 25 48
[67] 08 13 24 59
[89] 05 17 26 34
[89] 06 14 27 35

#16:
(20)
2
690
690

[01] 27 39 46 58
[01] 29 36 48 57
[23] 04 19 56 78
[23] 09 15 47 68
[45] 03 17 28 69
[45] 06 12 38 79
[67] 02 18 34 59
[67] 08 13 25 49
[89] 05 16 24 37
[89] 07 14 26 35

#17:
(21)
2
746
746

[01] 27 39 46 58
[01] 28 36 49 57
[23] 05 19 47 68
[23] 08 14 56 79
[45] 03 16 29 78
[45] 07 12 38 69
[67] 02 18 34 59
[67] 09 13 25 48
[89] 04 17 26 35
[89] 06 15 24 37

#18:
(22)
2
746
746

[01] 27 39 46 58
[01] 28 36 49 57
[23] 05 19 47 68
[23] 08 14 56 79
[45] 02 17 38 69
[45] 06 13 29 78
[67] 03 18 24 59
[67] 09 12 35 48
[89] 04 16 25 37
[89] 07 15 26 34

#19:
(23)
2
746
746

[01] 27 39 46 58
[01] 28 36 49 57
[23] 05 18 47 69
[23] 09 14 56 78
[45] 03 17 29 68
[45] 06 12 38 79
[67] 02 19 35 48
[67] 08 13 24 59
[89] 04 16 25 37
[89] 07 15 26 34

#20:
(24)
2
746
746

[01] 27 39 46 58
[01] 28 36 49 57
[23] 04 18 56 79
[23] 09 15 47 68
[45] 03 16 29 78
[45] 07 12 38 69
[67] 02 19 35 48
[67] 08 13 24 59
[89] 05 17 26 34
[89] 06 14 25 37

#21:
(33)
52
55
58

[01] 28 39 46 57
[01] 29 38 47 56
[23] 04 16 58 79
[23] 05 17 49 68
[45] 02 18 37 69
[45] 03 19 26 78
[67] 08 13 24 59
[67] 09 12 35 48
[89] 06 15 27 34
[89] 07 14 25 36

#22:
(33)
52
55
58

[01] 28 39 47 56
[01] 29 38 46 57
[23] 04 16 58 79
[23] 05 17 49 68
[45] 02 18 37 69
[45] 03 19 26 78
[67] 08 13 24 59
[67] 09 12 35 48
[89] 06 15 27 34
[89] 07 14 25 36

#23:
(34)
52
58
52

[01] 28 39 47 56
[01] 29 38 46 57
[23] 04 16 58 79
[23] 05 17 49 68
[45] 02 18 37 69
[45] 03 19 26 78
[67] 08 12 34 59
[67] 09 13 25 48
[89] 06 14 27 35
[89] 07 15 24 36

#24:
(37)
55
58
55

[01] 28 39 47 56
[01] 29 38 46 57
[23] 04 16 58 79
[23] 05 17 49 68
[45] 02 19 36 78
[45] 03 18 27 69
[67] 08 13 24 59
[67] 09 12 35 48
[89] 06 14 25 37
[89] 07 15 26 34

#25: (40) 58 58 58

[01] 28 39 46 57
[01] 29 38 47 56
[23] 04 16 58 79
[23] 05 17 49 68
[45] 02 19 36 78
[45] 03 18 27 69
[67] 08 12 34 59
[67] 09 13 25 48
[89] 06 15 24 37
[89] 07 14 26 35

#26: (40) 58 58 58

[01] 28 39 47 56
[01] 29 38 46 57
[23] 04 16 58 79
[23] 05 17 49 68
[45] 02 19 36 78
[45] 03 18 27 69
[67] 08 12 34 59
[67] 09 13 25 48
[89] 06 15 24 37
[89] 07 14 26 35

#27: (41) 58 58 58

[01] 28 39 46 57
[01] 29 38 47 56
[23] 04 17 58 69
[23] 05 16 49 78
[45] 02 18 36 79
[45] 03 19 27 68
[67] 08 13 24 59
[67] 09 12 35 48
[89] 06 14 25 37
[89] 07 15 26 34

#28: (41) 58 58 58

[01] 28 39 47 56
[01] 29 38 46 57
[23] 04 17 58 69
[23] 05 16 49 78
[45] 02 18 36 79
[45] 03 19 27 68
[67] 08 13 24 59
[67] 09 12 35 48
[89] 06 14 25 37
[89] 07 15 26 34