On the existence of three dimensional Room frames and Howell cubes

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Abstract

A Howell design of side s and order $2n+2$, or more briefly an $H(s, 2n+2)$ is an $s \times s$ array in which each cell is either empty or contains an unordered pair of elements from some $2n+2$ set V such that (1) every element of V occurs in precisely one cell of each row and each column, and (2) every unordered pair of elements from V is in at most one cell of the array. It follows immediately from the definition of an $H(s, 2n+2)$ that $n+1 \leq s \leq 2n+1$. A d-dimensional Howell design $H_d(s, 2n+2)$ is a d-dimensional array of side s such that (1) every cell is either empty or contains an unordered pair of elements from some $2n+2$ set V, and (2) each two-dimensional projection is an $H(s, 2n+2)$. The two boundary cases are well known designs: an $H_d(2n+1, 2n+2)$ is a Room d-cube of side $2n+1$ and the existence of d mutually orthogonal latin squares of order $n+1$ implies the existence of an $H_d(n+1, 2n+2)$. In this paper, we investigate the existence of Howell cubes, $H_3(s, 2n+2)$. We completely determine the spectrum for $H_3(2n, 2n+\alpha)$ where $\alpha \in \{2, 4, 6, 8\}$. In addition, we establish the existence of 3-dimensional Room frames of type 2^v for all $v \geq 5$ with only a few small possible exceptions for v.

1 Introduction

A *Howell design* of side s and order $2n+2$, or more briefly an $H(s, 2n+2)$ is an $s \times s$ array in which each cell is either empty or contains an unordered pair of elements from some $2n+2$ set V such that

(1) every element of V occurs in precisely one cell of each row and each column, and

(2) every unordered pair of elements from V is in at most one cell of the array.

It follows immediately from the definition of an $H(s, 2n+2)$ that $n+1 \leq s \leq 2n+1$. An $H(2n+1, 2n+2)$ is also known as a Room square of side $2n+1$, an $RS(2n+1)$. At the lower extreme, if there exists a pair of mutually orthogonal latin squares of side $n+1$, then there is an $H(n+1, 2n+2)$. Howell designs have been studied extensively, and the existence of these designs was established in the early 1980's, [17, 4].

Theorem 1.1. [17, 4] *Let n be a non-negative integer and let s be a positive integer such that $n+1 \leq s \leq 2n+1$. Then there exists an $H(s, 2n+2)$ if and only if $(s, 2n+2) \neq (2,4), (3,4), (5,6)$, or $(5,8)$*

An $H^*(s, 2n+2)$ is an $H(s, 2n+2)$ in which there is a subset of V, say X, of cardinality $2n+2-s$, such that no pair of elements from X is in any cell of the design. We note that there exist $H^*(s, 2n+2)$ for s even with two exceptions: there is no $H^*(2,4)$ and there is no $H^*(6,12)$, [4].

It is often useful to think about a Howell design in terms of its underlying graph. Let G be an s-regular graph on $2n+2$ vertices. A *one-factor* in G is a set of pairwise disjoint edges which between them contain each vertex exactly once. A *one-factorization* of G is a partition of all of the edges of G into pairwise disjoint one-factors. Two one-factorizations F_1 and F_2 are *orthogonal* if any one-factor in F_1 and any one-factor in F_2 have at most one edge in common. It is easy to see that the rows and columns of an $H(s, 2n+2)$ form a pair of orthogonal one-factorizations of the underlying s-regular graph on $2n+2$ vertices. Similarly, we can use a pair of orthogonal one-factorizations F_1 and F_2 to construct an $H(s, 2n+2)$. We index the rows and columns of an $s \times s$ array by the numbers 1 through s and let $F_i = \{f_i^1, f_i^2, \ldots, f_i^s\}$ for $i = 1, 2$. In the cell labeled (i, j), we place the pair of symbols (edge) from $f_1^i \cap f_2^j$ for each $i, j = 1, 2, \ldots, s$. If $f_1^i \cap f_2^j = \emptyset$, the cell is left empty. We note that in graph theory terms an $H^*(s, 2n+2)$ is an $H(s, 2n+2)$ whose underlying graph contains a maximal independent set of size $2n+2-s$.

We illustrate these definitions with a small example. An $H(6, 8)$ is displayed below.

67	12	58		34	
24	57	13		68	
38		27	45		16
15		46	37		28
	48		26	17	35
	36		18	25	47

An $H(6,8)$.

The definition of Howell design can be extended to higher dimensions. A *d-dimensional Howell design* $H_d(s, 2n+2)$ is a d-dimensional array of side s such that

(1) every cell is either empty or contains an unordered pair of elements from some $2n+2$ set V, and

(2) each two-dimensional projection is an $H(s, 2n+2)$.

An $H_2(s, 2n+2)$ is just an $H(s, 2n+2)$. An $H_3(s, 2n+2)$ is called a *Howell cube.* A set $\mathcal{F} = \{F_1, F_2, \ldots, F_d\}$ of d one-factorizations of a graph is called a set of d mutually orthogonal one-factorizations if the one-factorizations of $\mathcal{F}$ are all pairwise orthogonal. It is straightforward to see that $H_d(s, 2n+2)$ is equivalent to a set of d mutually orthogonal one-factorizations of the underlying s-regular graph on $2n+2$ vertices. Note that each of the two dimensional projections of an $H_d(s, 2n+2)$ have the rows and colums indexed by two of the orthogonal one-factorizations.

Example 1.2. *The following three one-factorizations are orthogonal one-factorizations of a 12 regular graph on 14 vertices (the cocktail party graph). Hence they yield a* $H_3(12, 14)$.

F_1

f_1^1 : {{4,7},{1,3},{9,2},{8,12},{10,13},{5,6},{11,14}}
f_1^2 : {{12,14},{5,8},{2,4},{10,3},{9,1},{11,13},{6,7}}
f_1^3 : {{1,14},{6,9},{3,5},{11,4},{10,2},{12,13},{7,8}}
f_1^4 : {{8,9},{2,14},{7,10},{4,6},{12,5},{11,3},{1,13}}
f_1^5 : {{9,10},{3,14},{8,11},{5,7},{1,6},{12,4},{2,13}}
f_1^6 : {{3,13},{10,11},{4,14},{9,12},{6,8},{2,7},{1,5}}
f_1^7 : {{2,6},{4,13},{11,12},{5,14},{10,1},{7,9},{3,8}}
f_1^8 : {{3,7},{5,13},{12,1},{6,14},{11,2},{8,10},{4,9}}
f_1^9 : {{5,10},{4,8},{6,13},{1,2},{7,14},{12,3},{9,11}}
f_1^{10} : {{6,11},{5,9},{7,13},{2,3},{8,14},{1,4},{10,12}}
f_1^{11} : {{11,1},{7,12},{6,10},{8,13},{3,4},{9,14},{2,5}}
f_1^{12} : {{12,2},{8,1},{7,11},{9,13},{4,5},{10,14},{3,6}}

F_2

f_2^1 : {{4,7},{12,14},{8,9},{3,13},{2,6},{5,10},{11,1}}
f_2^2 : {{5,8},{1,14},{9,10},{4,13},{3,7},{6,11},{12,2}}
f_2^3 : {{1,3},{6,9},{2,14},{10,11},{5,13},{4,8},{7,12}}
f_2^4 : {{2,4},{7,10},{3,14},{11,12},{6,13},{5,9},{8,1}}
f_2^5 : {{9,2},{3,5},{8,11},{4,14},{12,1},{7,13},{6,10}}
f_2^6 : {{10,3},{4,6},{9,12},{5,14},{1,2},{8,13},{7,11}}
f_2^7 : {{8,12},{11,4},{5,7},{10,1},{6,14},{2,3},{9,13}}
f_2^8 : {{10,13},{9,1},{12,5},{6,8},{11,2},{7,14},{3,4}}
f_2^9 : {{11,13},{10,2},{1,6},{7,9},{12,3},{8,14},{4,5}}
f_2^{10} : {{5,6},{12,13},{11,3},{2,7},{8,10},{1,4},{9,14}}
f_2^{11} : {{6,7},{1,13},{12,4},{3,8},{9,11},{2,5},{10,14}}
f_2^{12} : {{11,14},{7,8},{2,13},{1,5},{4,9},{10,12},{3,6}}

F_3

f_3^1 : {{1,3},{6,7},{10,2},{12,5},{8,11},{4,13},{9,14}}
f_3^2 : {{2,4},{7,8},{11,3},{1,6},{9,12},{5,13},{10,14}}
f_3^3 : {{11,14},{3,5},{8,9},{12,4},{2,7},{10,1},{6,13}}
f_3^4 : {{12,14},{4,6},{9,10},{1,5},{3,8},{11,2},{7,13}}
f_3^5 : {{1,14},{5,7},{10,11},{2,6},{4,9},{12,3},{8,13}}
f_3^6 : {{2,14},{6,8},{11,12},{3,7},{5,10},{1,4},{9,13}}
f_3^7 : {{10,13},{3,14},{7,9},{12,1},{4,8},{6,11},{2,5}}
f_3^8 : {{11,13},{4,14},{8,10},{1,2},{5,9},{7,12},{3,6}}
f_3^9 : {{4,7},{12,13},{5,14},{9,11},{2,3},{6,10},{8,1}}
f_3^{10} : {{9,2},{5,8},{1,13},{6,14},{10,12},{3,4},{7,11}}
f_3^{11} : {{8,12},{10,3},{6,9},{2,13},{7,14},{11,1},{4,5}}
f_3^{12} : {{5,6},{9,1},{11,4},{7,10},{3,13},{8,14},{12,2}}

The three two-dimensional projections for this $H_3(12,14)$ are:

4,7		1,3		9,2		8,12	10,13		5,6		11,14
12,14	5,8		2,4		10,3		9,1	11,13		6,7	
	1,14	6,9		3,5		11,4		10,2	12,13		7,8
8,9		2,14	7,10		4,6		12,5		11,3	1,13	
	9,10		3,14	8,11		5,7		1,6		12,4	2,13
3,13		10,11		4,14	9,12		6,8		2,7		1,5
2,6	4,13		11,12		5,14	10,1		7,9		3,8	
	3,7	5,13		12,1		6,14	11,2		8,10		4,9
5,10		4,8	6,13		1,2		7,14	12,3		9,11	
	6,11		5,9	7,13		2,3		8,14	1,4		10,12
11,1		7,12		6,10	8,13		3,4		9,14	2,5	
	12,2		8,1		7,11	9,13		4,5		10,14	3,6

Projection 1 of the $H_3(12,14)$. Rows indexed by F_1, columns by F_2

1,3		11,14				10,13		4,7	9,2	8,12	5,6
6,7	2,4		12,14				11,13		5,8	10,3	9,1
10,2	7,8	3,5		1,14				12,13		6,9	11,4
12,5	11,3	8,9	4,6		2,14				1,13		7,10
8,11	1,6	12,4	9,10	5,7		3,14				2,13	
	9,12	2,7	1,5	10,11	6,8		4,14				3,13
4,13		10,1	3,8	2,6	11,12	7,9		5,14			
	5,13		11,2	4,9	3,7	12,1	8,10		6,14		
		6,13		12,3	5,10	4,8	1,2	9,11		7,14	
			7,13		1,4	6,11	5,9	2,3	10,12		8,14
9,14				8,13		2,5	7,12	6,10	3,4	11,1	
	10,14				9,13		3,6	8,1	7,11	4,5	12,2

Projection 2 of the $H_3(12,14)$. Rows indexed by F_1, columns by F_3

		8,9	12,14	2,6	5,10			4,7		11,1	3,13
4,13			9,10	1,14	3,7	6,11			5,8		12,2
1,3	5,13			10,11	2,14	4,8	7,12			6,9	
	2,4	6,13			11,12	3,14	5,9	8,1			7,10
8,11		3,5	7,13			12,1	4,14	6,10	9,2		
	9,12		4,6	8,13			1,2	5,14	7,11	10,3	
		10,1		5,7	9,13			2,3	6,14	8,12	11,4
12,5			11,2		6,8	10,13			3,4	7,14	9,1
10,2	1,6			12,3		7,9	11,13			4,5	8,14
9,14	11,3	2,7			1,4		8,10	12,13			5,6
6,7	10,14	12,4	3,8			2,5		9,11	1,13		
	7,8	11,14	1,5	4,9			3,6		10,12	2,13	

Projection 3 of the $H_3(12,14)$. Rows indexed by F_2, columns by F_2

Let $\nu(s,2n+2)$ denote the maximum value of d such that an $H_d(s,2n+2)$ exists. Information on $\nu(s,2n+2)$ can be found in [6]. We summarize some of the known results in the next proposition. Note that in general (with 4 exceptions), from Theorem 1.1 above we have that $\nu(s,2n+2) \geq 2$.

Proposition 1.3. [2, 11, 16] *Values for $\nu(s,2n)$, $2n \leq 12$*

$\nu(2,4) = \nu(3,4) = \nu(5,6) = \nu(5,8) = 1.$

$\nu(3,6) = \nu(4,6) = \nu(7,10) = 2.$

$\nu(4,8) = \nu(7,8) = \nu(6,10) = 3.$

$\nu(5,10) = \nu(8,10) = \nu(9,10) = 4.$

$\nu(6,12) \geq 3,\ \nu(7,12) \geq 3,\ \nu(11,12) \geq 5.$

Concerning the existence of d-dimensional Howell designs $H_d(s, 2n+2)$, with $d > 2$, there are three known infinite classes. Recall that $n+1 \leq s \leq 2n+1$. At the lower extreme, if there exists a set of d mutually orthogonal latin squares of side $n+1$, then we can construct an $H_d(n+1, 2n+2)$. So, for example, we can construct Howell cubes, $H_3(n+1, 2n+2)$ for n a positive integer, $n \geq 3$ and $n \neq 5, 9$, [3]. Since an $H_3(6, 12)$ was constructed by E. Brickell in [2], we have the lower boundary for Howell cubes with only one possible exception.

Theorem 1.4. *Let $n \geq 3$ be a positive integer. Then there exists an $H_3(n+1, 2n+2)$ except possibly for $n = 9$.*

At the other extreme, an $H_d(2n+1, 2n+2)$ is a Room d-cube of side $2n+1$. The asymptotic existence of Room d-cubes was established in 1971 by Gross, Mullin, and Wallis [12] and the existence of Room 5-cubes was established in 1987 by Dinitz [5]. The following theorem summarizes what is known about the existence of Room d-cubes where $d = 3, 4, 5$. For further information on the existence of Room d-cubes, see [7].

Theorem 1.5. [5] *For all odd $n \geq 11$, except possibly for $n = 15$, there exists a Room 5-cube of side n, an $H_5(n, n+1)$. There exist $H_4(n, n+1)$ for $n = 9, 15$ and there exists an $H_3(7, 8)$.*

In addition to the boundary cases, one infinite class of d-dimensional Howell designs has been constructed for $d \geq 3$. These designs are constructed using hyperovals in finite projective planes.

Theorem 1.6. [13] *Let $m \geq 2$ be a positive integer, then there exists an $H_{2^{m-1}}(2^m, 2^m+2)$.*

In this paper, we investigate the existence of Howell cubes of even side. Our main recursive construction for Howell cubes uses 3-dimensional Room frames. In the next section, we introduce the idea of sets of orthogonal intransitive frame starters and we describe constructions and existence results for 3-dimensional Room frames. In particular, we show that there exist 3-dimensional Room frames of type 2^n for n a positive integer, $n \geq 5$ with at most 5 possible exceptions for n. We use direct and recursive constructions in Section 3 to establish the existence of Howell cubes $H_3(n, n+\alpha)$ for $\alpha = 2, 4, 6, 8$. This completes the existence of Howell cubes for the upper end of the spectrum.

2 Room Frames

Let S be a set, and let $\{S_1, S_2, \ldots, S_n\}$ be a partition of S. An $\{S_1, S_2, \ldots, S_n\}$-Room frame is an $|S| \times |S|$ array, F, indexed by S, which satisfies the following properties:

(1) every cell of F is either empty or contains an unordered pair of symbols of S,

(2) the subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are referred to as *holes*),

(3) each symbol $x \notin S_i$ occurs once in row (or column) s, for any $s \in S_i$,

(4) the pairs occurring in F are $\{s, t\}$, where $(s, t) \in (S \times S) \setminus \cup_{i=1}^{n} (S_i \times S_i)$.

An $\{S_1, S_2, \ldots, S_n\}$-Room frame F is said to be *skew* if at most one of the cells (i, j) and (j, i) $(i \neq j)$ is nonempty. The *type* of a Room frame is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. We usually use an "exponential" notation to describe types: a type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ denotes that there are u_i holes of size t_i, $1 \leq i \leq k$. The *order* of a Room frame is $|S|$.

To illustrate these definitions, we display a Room frame of type 2^5 in Example 2.1. The design is defined on $\mathbb{Z}_5 \times \mathbb{Z}_2$. We use an abbreviated notation in the figure: an element (x, y) in $\mathbb{Z}_5 \times \mathbb{Z}_2$ is written xy. (So for example, the pair $21, 41$ is $(2,1), (4,1)$.) Note that the holes $S_i = \{(i, 0), (i, 1)\}$ for $i = 0, 1, \ldots, 4$.

Example 2.1. *A (skew) Room frame of type 2^5 defined on $\mathbb{Z}_5 \times \mathbb{Z}_2$,* [20] .

		21 41			40 31	10 20			30 11
			20 40	41 30			11 21	31 10	
	40 21			31 01			00 21	20 30	
41 20					30 00	01 40			21 31
30 40			00 31			41 11			10 01
	31 41	01 30					40 10	11 00	
	20 11	40 00			10 41			01 21	
21 10			41 01	11 40					00 20
11 31			30 21	00 10			20 01		
	10 30	31 20			01 11	21 00			

A *d-dimensional $\{S_1, S_2, \ldots, S_n\}$-Room frame F* of order $|S|$ is a d-dimensional array of side $|S|$ which satisfies the following properties:

(1) every cell of F is either empty or contains an unordered pair of symbols of S,

(2) each two-dimensional projection is a $\{S_1, S_2, \ldots, S_n\}$-Room frame.

For convenience, we will call a d-dimensional Room frame a *Room d-frame.*

It will be helpful to think of a Room frame in terms of an underlying graph and a set of factorizations of that graph. We give this equivalence for a Room frame of type 2^n – it is easy to generalize to any type. A *holey one-factor* in the graph $K_{2n} - nK_2$ is a matching consisting of $n-1$ edges (on $2n-2$ vertices) that does not contain a vertex in one of the missing K_2's (these are called the "holes"). A holey one-factorization of the graph $K_{2n} - nK_2$ is a partition of the edges into $2n$ holey one-factors. Two holey one-factorizations F and G are *orthogonal* if two edges which appear in the same holey one-factor of F appear in different holey one-factors of G. It is straightforward to see that the existence of d pairwise orthogonal holey one-factorizations of $K_{2n} - nK_2$ is equivalent to a d-dimensional Room frame of type 2^n.

The Room frame displayed in Example 2.1 has a third orthogonal holey one-factorization (sometimes called an *orthogonal resolution*). Recall that the holes are $S_i = \{i0, i1\}$ for $i = 0, 1, 2, 3, 4$. We list the orthogonal resolution by listing the holey one-factors associated with S_i for $i = 0, 1, 2, \ldots, 4$ below. So this example yields a Room 3-frame of type 2^5, or equivalently three orthogonal holey one-factorizations of the graph $K_{10} - 5K_2$.

hole	resolutions			
00,01	20,30	21,31	41,11	40,10
	30,21	31,20	11,40	10,41
10,11	30,40	31,41	01,21	00,20
	40,31	41,30	21,00	20,01
20,21	40,00	41,01	11,31	10,30
	00,41	01,40	31,10	30,11
30,31	00,10	01,11	21,41	20,40
	10,01	11,00	41,20	40,21
40,41	10,20	11,21	31,01	30,00
	20,11	21,10	01,30	00,31

The primary direct method for constructing Room d-frames with one hole size h is to use starters and adders. Let G be an additive abelian group of order g, let H be a subgroup of G of order h where $g - h$ is even and let $n = g/h$. An *h^n–frame starter* in $G - H$ is a set of pairs $S = \{\{s_i, t_i\} \mid i = 1, 2, \ldots, \frac{g-h}{2}\}$ satisfying

(1) $\{s_i, t_i \mid i = 1, 2, \ldots, \frac{g-h}{2}\} = G - H$ and

(2) $\{\pm(s_i - t_i) \mid i = 1, 2, \ldots, \frac{g-h}{2}\} = G - H$

An *adder* A for S is a set of $\frac{g-h}{2}$ distinct elements of $G-H$, $A = \{a_i \mid i = 1, 2, \ldots, \frac{g-h}{2}\}$ such that $\{s_i + a_i, t_i + a_i \mid i = 1, 2, \ldots, \frac{g-h}{2}\} = G - H$. An adder A is *skew* provided $a_i \neq -a_j$ for any i, j.

Lemma 2.2. [18] *If there exists an h^n–frame starter in $G - H$ and an adder A for S, then there is an h^n Room frame. Furthermore, if A is skew, then there is a skew h^n Room frame.*

A frame starter $S = \{\{s_i, t_i\} \mid i = 1, 2, \ldots, \frac{g-h}{2}\}$ in $G - H$ is said to be *strong* provided $s_i + t_i \notin H$ for $i = 1, 2, \ldots, \frac{g-h}{2}$ and $s_i + t_i = s_j + t_j$ implies $i = j$. A strong starter S is called a *skew-strong* starter if $s_i + t_i \neq -(s_j + t_j)$ for any i, j.

An equivalent method to constructing Room frames from starters and adders is to construct them via orthogonal starters, which we now define. Let $S = \{\{s_i, t_i\} \mid i = 1, 2, \ldots, \frac{g-h}{2}\}$ and $T = \{\{u_i, v_i\} \mid i = 1, 2, \ldots, \frac{g-h}{2}\}$ be two frame starters in $G - H$. S and T are called a pair of *orthogonal frame starters* if there is an adder A so that $S + A = T$. A set of frame starters $Q = \{S_1, S_2, \ldots, S_t\}$ is called a set of t orthogonal frame starters if each pair of starters is orthogonal.

The *patterned frame starter* is the starter $P = \{\{s_i, t_i\} \mid i = 1, 2, \ldots, \frac{g-h}{2}\}$ where $s_i = -t_i$ for all i. The patterned frame starter can exist in $G - H$ only if $|G|$ is odd. If S is a strong-frame starter in $G - H$ and $|G|$ is odd, then $\{S, -S, P\}$ is a set of 3 orthogonal frame starters and can be used to construct a 3-dimensional Room frame of type h^n, where $n = g/h$ [9].

Lemma 2.3. [9] *If there exists a strong frame starter S in $G - H$ with $|G|$ odd, then there is a Room 3-frame of type h^n where $n = g/h$.*

To illustrate the connection between frame starters and Room frames we note that the the four pairs $\{(2,1),(4,1)\}, \{(4,0),(3,1)\}, \{(1,0),(2,0)\}$ and $\{(3,0),(1,1)\}$, form a frame starter S in $(\mathbb{Z}_5 \times \mathbb{Z}_2) \setminus (\{0\} \times \mathbb{Z}_2)$. An adder for this frame starter is $A = \{(4,0),(3,1),(2,0),(1,1)\}$. $S + A$ is the frame starter $\{(1,1),(3,1)\}, \{(2,1),(1,0)\}, \{(3,0),(4,0)\}, \{(4,1),(2,0)\}$ which is orthogonal to S. Indexing the rows by the translates of S and the columns by the translates of $S + A$ (in the group $\mathbb{Z}_5 \times \mathbb{Z}_2$) yields the Room frame of type 2^5 given in Example 2.1. Since we have exhibited three orthogonal frame starters of type 2^5 we have shown that there exists a Room 3-frame of type 2^5.

Frame starters and adders were used to construct the following Room 3-frames. Note that the Room 3-frame of type 2^5 is given above.

Lemma 2.4. [5, 9] *There exist Room 3-frames of type 2^n for $n \in \{5, 12, 13, 16, 17, 20, 21\}$.*

We use orthogonal frame starters to construct several more Room 3-frames.

Lemma 2.5. *There exist Room 3-frames of type 2^n for $n \in \{8, 9, 24, 28, 32, 33, 44, 52\}$.*

Proof. For each of these orders we give 3 orthogonal frame starters S_1, S_1, S_3 of type 2^n in Appendix A. $\square$

Unfortunately, the following result tells us that we can not use orthogonal frame starters to find Room frames of type 2^n for certain congruence classes of n.

Theorem 2.6. [9] *There does not exist a frame starter for a Room frame of type 2^n for $n \equiv 2, 3 \pmod 4$.*

To cover these two congruence classes we use a variant of the frame starter called an intransitive frame starter, first described in [10, 18]. We give a different definition than in the original papers, but it is easy to show the definitions are equivalent. However, our definition will be easier to generalize to higher dimensions.

Let G be an abelian group of order g, and let H be a subgroup of order h where both g and h are even and let $n = g/h$. Let k be a positive integer. An $h^n(2k)-$*intransitive frame starter* in $G \setminus H$ is a pair (S, C) with

$$S = \{\{a_i, b_i\} : 1 \le i \le \frac{g-h}{2} - 2k\} \cup \{u_i : 1 \le i \le 2k\} \text{ and } C = \{\{p_i, q_i\} : 1 \le i \le k\}$$

where

(i) $S \cup C = G \setminus H$,

(ii) the elements $\{\pm(b_i - a_i)\}$ and $\{\pm(p_i - q_i)\}$ are all distinct and elements of $G \setminus H$,

(iii) for $1 \le i \le k$ each element $p_i - q_i$ has even order,

(iv) if $D = \{\{\pm(b_i - a_i)\} \bigcup \{\pm(p_i - q_i)\}\}$ are the differences in the intransitive frame starter, then the *missing differences*, namely $(G \setminus H) \setminus D$, all have even order in G.

We can now give the definition for two $h^n(2k)$-intransitive frame starters (S, C) and (T, R) to be *orthogonal $h^n(2k)$-intransitive frame starters*. Let
$S = \{\{a_i, b_i\} : 1 \le i \le \frac{g-h}{2} - 2k\} \cup \{u_i : 1 \le i \le 2k\}$, $C = \{\{p_i, q_i\} : 1 \le i \le k\}$,
$T = \{\{c_i, d_i\} : 1 \le i \le \frac{g-h}{2} - 2k\} \cup \{v_i : 1 \le i \le 2k\}$, and $R = \{\{p'_i, q'_i\} : 1 \le i \le k\}$.

Then (S, C) and (T, R) are orthogonal $h^n(2k)$-intransitive frame starters if the following conditions are satisfied.

1. The differences $\{\pm(p'_i - q'_i)\}$ are precisely the missing differences from (S, C) and $\{\pm(p_i - q_i)\}$ are the missing differences from (T, R).

2. If $b_i - a_i$ is not a difference missing from (T, R) and $d_i - c_i$ is not a missing difference from (S, C), then assume $b_i - a_i = d_i - c_i$. Then all of the elements $\{d_i - b_i\}$ and $\{v_i - u_i\}$ are distinct elements of $G \setminus H$.

As noted above, this definition is the same as given by Stinson in [18] hence the following theorem still follows.

Theorem 2.7. [18] *If there exist two orthogonal $h^n(2k)$-intransitive frame starters, then there exists a Room frame of type $h^n(2k)$.*

In this paper we are interested in constructing Room frames where all the holes have size 2. In this case the above definitions can be simplified significantly. We will always have that $G = \mathbb{Z}_{2n}$ and $H = \{0, n\}$. We will be constructing frames of type 2^{n+1} on the symbols $\mathbb{Z}_{2n} \cup \{\infty_1, \infty_2\}$ and an intransitive frame starter will have exactly one missing difference $d < n$. Let (S,C) be a $2^{n+1}-$intransitive frame starter where

$$S = \{\{a_i, b_i\} : 1 \le i \le n-1, i \notin \{d, q-p\}\} \cup \{u_1, u_2\} \text{ and } C = \{p, q\}.$$

We can assume that d and $q-p < n$ are odd. Also, if $i \neq d$, then $b_i - a_i = i$ for $1 \le i \le n-1$. Note that for the missing difference d we will have no pair $\{a_d, b_d\}$ in S. Let (T,R) be a second 2^{n+1}-intransitive frame starter (with missing difference d') where $T = \{\{c_i, d_i\} : 1 \le i \le n-1, i \notin \{d', q'-p'\}\} \cup \{v_1, v_2\}$ and $R = \{p', q'\}$. In order for (S,C) and (T,R) to be orthogonal $2^{n+1}-$intransitive frame starters the conditions in the definition above translate to the following.

1. The difference $q' - p' = d$ (the missing difference from (S,C)) and $q - p = d'$ (the missing difference from (T,R)), or equivalently

   $\{\pm(b_i - a_i)\} : 1 \le i \le n-1, i \neq d, d'\} \cup \{\pm d, \pm d'\} = \mathbb{Z}_{2n} \setminus \{0, n\}$ and

   $\{\pm(d_i - c_i)\} : 1 \le i \le n-1, i \neq d', d'\} \cup \{\pm d, \pm d'\} = \mathbb{Z}_{2n} \setminus \{0, n\}$.

2. All of the elements $\{(d_i - b_i) | 1 \le i \le n-1, i \neq d, d'\}$, $v_1 - u_1$ and $v_2 - u_2$ are distinct elements of $\mathbb{Z}_{2n} \setminus \{0, n\}$. (This is called the *adder*).

Corollary 2.8. *If there exist two orthogonal $2^{n+1}-$intransitive frame starters, then there is a Room frame of type 2^{n+1}. Furthermore, the two holey one-factorizations of $K_{2n+2} \setminus (n+1)K_2$ generated by the intransitive frame starters will be orthogonal holey one-factorizations.*

We present an example to illustrate the definitions above.

Example 2.9. *Two orthogonal $2^{10}-$intransitive frame starters.*

First note that the group is $G = \mathbb{Z}_{18}$ and the subgroup is $H = \{0, 9\}$. Let
$S = \{(3,4), (15,17), (8,12), (10,16), (7,14), (5,13)\} \cup \{1,2\}$, with $C = (6,11)$ and
$T = \{(7,8), (10,12), (15,1), (16,4), (17,6), (13,3)\} \cup \{2,5\}$ with $R = (11,14)$.

Note (S,C) is missing difference 3 and (T,R) is missing difference 5. The adder will be (4,13,7,6,10,8,1,3) and note that $S \cup C = T \cup R = \mathbb{Z}_{18} \setminus \{0, 9\}$. The first holey one-factorization is constructed from (S,C) by taking the 18 translates in $\mathbb{Z}_{18}$ to get the first 18 holey one-factors. The last two holey one-factors are constructed from the pairs with difference $d' = 3$, so they are $\{\{1+i, 4+i\} : 0 \le i \le 17, i \text{ even}\}$ and $\{\{1+i, 4+i\} : 0 \le i \le 17, i \text{ odd}\}$, respectively. Note that these last two holey one-factors miss the hole $\{\infty_1, \infty_2\}$.

¿From Theorem 2.7 we have that given two orthogonal 2^{n+1}-intransitive frame starters, there exists a Room frame of type 2^{n+1}. The following example gives the Room frame of type 2^{10} which results from this theorem using the starters given in the Example 2.9. One can see that the rows of this Room frame are essentially translates of (S,C) while the columns are translates of (T,R). The 2×2 "holes" down the diagonal are labeled for the reader's convenience. Also note that we have replaced ∞_1 by a and ∞_2 by b. The interested reader can see [18] for the details of this construction.

Example 2.10. *A 2^{10} Room frame constructed from the intransitive starters in Example 2.9.*

0,9			5,13		8,12		10,16			15,17	3,4		2,b			7,14	1,a	6,11	
		14,4		17,3		1,7				12,13	6,8	11,b				10,a	16,5		15,2
2,a	8,15	1,10			6,14		9,13		11,17			16,0	4,5		3,b				7,12
17,6	11,a			15,5		0,4		2,8				13,14	7,9	12,b				16,3	
		3,a	9,16	2,11			7,15		10,14		12,0			17,1	5,6		4,b	8,13	
		0,7	12,a			16,6		1,5		3,9				14,15	8,10	13,b			17,4
5,b				4,a	10,17	3,12			8,16		11,15		13,1			0,2	6,7		9,14
	14,b			1,8	13,a			17,7		2,6		4,10				15,16	9,11	0,5	
7,8	1,3	6,b				5,a	11,0	4,13			9,17		12,16		14,2			10,15	
10,12	16,17		15,b			2,9	14,a			0,8		3,7		5,11					1,6
		8,9	2,4	7,b				6,a	12,1	5,14			10,0		13,17		15,3		11,16
		11,13	17,0		16,b			3,10	15,a			1,9		4,8		6,12		2,7	
16,4				9,10	3,5	8,b				7,a	13,2	6,15			11,1		14,0	12,17	
	7,13			12,14	0,1		17,b			4,11	16,a			2,10		5,9			3,8
15,1		17,5				10,11	4,6	9,b				8,a	14,3	7,16			12,2		13,18
	6,10		8,14			13,15	1,2		0,b			5,12	17,a			3,11		4,9	
13,3		16,2		0,6				11,12	5,7	10,b				9,a	15,4	8,17		14,1	
	4,12		7,11		9,15			14,16	2,3		1,b			6,13	0,a				5,10
11,14			3,6	13,16			5,8	15,0			7,10	17,2			9,12	1,4		a, b	
	2,5	12,15			4,7	14,17			6,9	16,1			8,11	0,3			10,13		

We are now in a position to discuss the manner in which three intransitive frame starters could be mutually orthogonal. As before we will restrict to the case where all the holes are size 2. We again assume $G = Z_{2n}$ and $H = \{0, n\}$. Let $1 \leq d_1, d_2, d_3 < n$ all be odd. Assume $q_1 - p_1 = s_3 - r_3 = d_1$, $q_2 - p_2 = r_1 - s_1 = d_2$ and $q_3 - p_3 = s_2 - r_2 = d_3$ and that if $1 \leq i \leq n-1$ with $i \neq d_1, d_2, d_3$, then $b_i - a_i = d_i - c_i = f_i - e_i = i$. Let

$$S_1 = \{\{a_i, b_i\} : 1 \leq i \leq n-1, i \neq d_1, d_2, d_3\} \cup \{u_1, u_2\}, P_1 = \{p_1, q_1\}, R_1 = \{r_1, s_1\},$$

$$S_2 = \{\{c_i, d_i\} : 1 \leq i \leq n-1, i \neq d_1, d_2, d_3\} \cup \{v_1, v_2\}, P_2 = \{p_2, q_2\}, R_2 = \{r_2, s_2\},$$

$$S_3 = \{\{e_i, f_i\} : 1 \leq i \leq n-1, i \neq d_1, d_2, d_3\} \cup \{w_1, w_2\} P_3 = \{p_3, q_3\}, R_3 = \{r_3, s_3\},$$

satisfy the following conditions:

1. For $i = 1, 2, 3$, the elements in $S_i \cup P_i \cup R_i$ are precisely $Z_{2n} \setminus \{0, n\}$.

2. Let $\{a_{d_2}, b_{d_2}\} = \{r_1, s_1\}$ and $\{c_{d_2}, d_{d_2}\} = \{p_2, q_2\}$, then (S_1, P_1) and (S_2, R_2) are orthogonal $2^{n+1}-$intransitive frame starters with (S_1, P_1) missing difference d_3 and (S_2, R_2) missing difference d_1.

3. Let $\{a_{d_1}, b_{d_1}\} = \{p_1, q_1\}$ and $\{e_{d_1}, f_{d_1}\} = \{r_3, s_3\}$, then (S_1, R_1) and (S_3, P_3) are orthogonal $2^{n+1}-$intransitive frame starters with (S_1, R_1) missing difference d_3 and (S_3, P_3) missing difference d_2.

4. Let $\{c_{d_3}, d_{d_3}\} = \{r_2, s_2\}$ and $\{e_{d_3}, f_{d_3}\} = \{p_3, q_3\}$, then (S_2, P_2) and (S_3, R_3) are orthogonal $2^{n+1}-$intransitive frame starters with (S_2, P_2) missing difference d_1 and (S_3, R_3) missing difference d_2.

We will refer to the above sets (S_1, P_1, R_1), (S_2, P_2, R_2) and (S_3, P_3, R_3) as 3 mutually orthogonal $2^{n+1}-$intransitive frame starters. Since the holey one-factorizations generated by each of the three intransitive starters will be orthogonal we will have three mutually orthogonal one-factorizations of the graph $K_{2n+2} - (n+1)K_2$ or a Room 3-frame of type 2^{n+1} as desired.

Example 2.11. *3 mutually orthogonal $2^{10}-$intransitive frame starters.*

Again we note that the group is $G = \mathbb{Z}_{18}$ and the subgroup is $H = \{0, 9\}$. Let

$S_1 = \{(15, 17), (8, 12), (10, 16), (7, 14), (5, 13)\} \cup \{1, 2\}$, $P_1 = (6, 11)$ and $R_1 = (3, 4)$.

$S_2 = \{(10, 12), (15, 1), (16, 4), (17, 6), (13, 3)\} \cup \{2, 5\}$, $P_2 = (7, 8)$ and $R_2 = (11, 14)$.

$S_3 = \{(4, 6), (11, 15), (14, 2), (12, 1)(17, 7)\} \cup \{3, 8\}$, $P_3 = (13, 16)$ and $R_3 = (5, 10)$.

Note here that $d_1 = 5, d_2 = 1$ and $d_3 = 3$. Also see that $(S_1 \cup R_1, P_1) = (S, C)$ from Example 2.9 while the set $(S_2 \cup P_2, R_2) = (T, R)$ in that example. Also, looking at the Room frame in Example 2.10 note that the cells containing each translate in Z_{18} of the third starter have the property that no two are in the same row or column and that each translate contains all of the symbols in $Z_{18} \cup \{\infty_1, \infty_2\}$, i.e. each is a *transversal* of the Room frame. The final two one-factors obtained from that third starter (for the hole $\{\infty_1, \infty_2\}$) are $\{\{1 + i, 2 + i\} : 0 \leq i \leq 17, i \text{ even}\}$ and $\{\{1 + i, 2 + i\} : 0 \leq i \leq 17, i \text{ odd}\}$ – note the cells containing the pairs in each of these sets is also transversal of the Room frame. Hence we have a Room 3-frame of type 2^{10}.

Theorem 2.12. *There exist Room 3-frames of type 2^n for $n \in \{10, 14, 15, 18, 22, 26, 30, 34\}$.*

Proof. Example 2.11 gives 3 mutually orthogonal $2^{10}-$intransitive frame starters. In appendix B we give 3 mutually orthogonal 2^n-intransitive frame starters for each $n \in \{14, 15, 18, 22, 26, 30, 34\}$. □

Skew strong starters can be used to construct 3-dimensional Room frames. The following construction is a slight extension of a construction found in [10]. We need the following definition. Let $P = \{S_1, S_2, \ldots, S_m\}$ be a partition of a set S where $|S_i| = h$ for $1 \leq i \leq m$. A *partitioned incomplete latin square of type h^m*, is an $|S| \times |S|$ array L, indexed by the elements of S, satisfying the following properties: (1) A cell of L either contains an element of S or is empty; (2) The subarrays indexed by $S_i \times S_i$ are empty for $1 \leq i \leq m$. (3) Let $j \in S_i$, then row j of L contains each element of $S \setminus S_i$ precisely once and column j of L contains each element of $S \setminus S_i$ precisely once. Note that a partitioned incomplete latin square of type 1^m is equivalent to an idempotent latin square of side m. Orthogonality of partitioned latin squares is defined in the obvious way.

Lemma 2.13. *If there exists a skew strong starter of order q and a set of 3 orthogonal partitioned incomplete latin squares of type 1^q, then there is a Room 3-frame of type 2^q.*

Proof. Let $|V| = |W| = q$ where $V = \{v_1, v_2, \ldots, v_q\}$ and $W = \{w_1, w_2, \ldots, w_q\}$, and suppose S is a skew strong starter of order q. F will denote the Room frame of type 1^q defined on V and generated by S where the *ith* hole is associated with the element v_i of V. F' will denote the skew complement of F defined on W. (So F' is a Room frame of type 1^q defined on W where the w_i is the element associated with the *ith* hole.) Since F is generated by a skew strong starter, the patterned starter can be used to construct a set of q skew transversals. Let A be the array of pairs formed by the superposition of F and F', $A = F \circ F'$. Let $T_1, T_2, \ldots, T_q$ denote the q skew transversals of A. So every element of $V \cup W \setminus \{v_i, w_i\}$ occurs precisely once in T_i, $i = 1, 2, \ldots, q$.

Let L_1 be a partitioned incomplete latin square of type 1^q defined on V where v_i is associated with cell (i, i). Similarly, let L_2 be an orthogonal partitioned incomplete latin square of type 1^q defined on W where w_i is associated with cell (i, i). Let L be the array of pairs formed by the superposition of L_1 and L_2, $L = L_1 \circ L_2$. L contains every pair in $V \times W \setminus \{(v_i, w_i) : i = 1, 2, \ldots, q\}$. We use the third partitioned incomplete latin square of type 1^q to construct a set of q transversals of L, namely $N_1, N_2, \ldots, N_q$, where every element of $V \cup W \setminus \{v_i, w_i\}$ occurs once in N_i. (Note that cell (i, i) which is empty is in N_i.)

We construct a $2q \times 2q$ array as follows:

$$B = \begin{pmatrix} A & \\ & L \end{pmatrix}$$

It is straightforward to verify that B is a Room frame of type 2^q defined on $V \cup W$ with holes $B_i = \{v_i, w_i\}$. B has a third orthogonal resolution. For hole B_i, the two (partial) resolution classes which contain every element of $V \cup W \setminus \{v_i, w_i\}$ are T_i and N_i. □

It is easy to see that in Lemma 2.13 one can replace the ingredient of a skew strong starter of order q, with a skew frame of type 1^n which has a set of n skew transversals.

Corollary 2.14. *There exists a Room 3-frame of type* $2^7, 2^{11}, 2^{19}, 2^{23}, 2^{27}, 2^{29}$ *and* 2^{31}.

Proof. Skew strong starters for the necessary orders can be found in [15]. Since each q is a prime power, there exits $q - 1$ mutually orthogonal latin squares, hence there exist three orthogonal idempotent latin square of order q. □

The next Theorem provides Room 3-frames of type 2^n for many odd values of n.

Theorem 2.15. *Let* $n \equiv 1 \pmod 2$, $n \geq 7$ *and* $n \neq 3m$ *where* $m \geq 5$ *and* $(m, 3) = 1$. *Then there is a Room* 3*-frame of type* 2^n.

Proof. Given n, Theorem 2.8 in [14] provides a skew Room frame of type 1^n with a set of n skew transversals. When n is odd and $n \geq 7$, there exists three idempotent MOLS(n) or equivalently 3 orthogonal partitioned incomplete latin squares of type 1^q (See [3], Table III.3.83). The result now follows from Lemma 2.13 and the comment that follows that lemma. □

Our main recursive construction uses group divisible designs. A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties.

(1) $\mathcal{G}$ is a partition of X into subsets called groups; $\mathcal{G} = \{G_1, G_2, \ldots, G_m\}$.

(2) $\mathcal{B}$ is a collection of subsets of X, called blocks, such that a group and a block contain at most one element in common.

(3) Every pair of elements from distinct groups occurs in precisely λ blocks.

Let K be a set of positive integers. A GDD$(v; K; G_1, G_2, \ldots, G_m; 0, \lambda)$ G is a group divisible design of index λ with $|X| = v$, $|B| \in K$ for every $B \in \mathcal{B}$, and $\mathcal{G} = \{G_1, G_2, \ldots, G_m\}$. The type of G is the multiset $\{|G_1|, |G_2|, \ldots, |G_m|\}$. We usually use exponential notation to describe the type; G has type $t_1^{u_1} t_2^{u_2} \ldots t_\ell^{u_\ell}$ if there are u_i G_j's of cardinality t_i, $1 \le i \le m$. A GDD$(v; K; G_1, \ldots, G_m; 0, \lambda)$ is often denoted as a (K, λ)-GDD of type $t_1^{u_1} t_2^{u_2} \ldots t_\ell^{u_\ell}$. If $\lambda = 1$, the design is usually denoted as a K-GDD of type $t_1^{u_1} t_2^{u_2} \ldots t_\ell^{u_\ell}$. For convenience, if $K = \{k\}$, the designs are denoted as a (k, λ)-GDD or simply as a k-GDD if the index $\lambda = 1$. A K-GDD of type 1^v is also known as a pairwise balanced design and denoted by PBD(v, K). A transversal design $TD(k, n)$ is a k-GDD of type n^k. It is well known that a $TD(k, n)$ is equivalent to the existence of a set of $k - 2$ mutually orthogonal latin squares of side n.

The next two constructions are standard recursive constructions for Room frames (see [18]). It is straightforward to extend the proofs to d dimensions. The first construction expands the hole size and the second fills in the holes with smaller Room frames.

Theorem 2.16. *If there exists a d-dimensional Room frame of type $t_1^{u_1} t_2^{u_2} \cdots t_n^{u_n}$ and a set of d mutually orthogonal latin squares of side m, then there is a d-dimensional Room frame of type $(mt_1)^{u_1} (mt_2)^{u_2} \cdots (mt_n)^{u_n}$.*

Theorem 2.17. *Let F be a d-dimensional $\{S_1, S_2, \ldots, S_n\}$-Room frame where $|S_i| = 2t_i$ for $i = 1, 2, \ldots, n$, and let $t = \sum_{i=1}^{n} t_i$.*

(i) If there exists a Room d-frame of type 2^{t_i} for each i, $1 \le i \le n$, then there is Room d-frame of type 2^t.

(ii) If there exists a Room d-frame of type 2^{t_i+1} for each i, $1 \le i \le n$, then there is a Room d-frame of type 2^{t+1}.

Our main recursive construction is a special case of Wilson's Fundamental Construction, [21], for frames. The proof in [19] is for the two dimensional case and also doesn't fill in the holes, but this extension is immediate.

Theorem 2.18. *(a) If there exists a GDD$(v; K; G_1, G_2, \ldots, G_m; 0, 1)$ such that (1) there exists a Room d-frame of type 2^k for each $k \in K$ and (2) there exists a Room d-frame of type $2^{|G_i|+1}$ for $i = 1, \ldots, m$, then there exists a Room d-frame of type 2^{v+1}.*

(b) If there exists a GDD$(v; K; G_1, G_2, \ldots, G_m; 0, 1)$ such that (1) there exists a Room d-frame of type 2^k for each $k \in K$ and (2) there exists a Room d-frame of type $2^{|G_i|}$ for $i = 1, \ldots, m$, then there exists a Room d-frame of type 2^v.

We are now in a position to construct Room 3-frames of type 2^n with only a few possible exceptions. We first gather the necessary group divisible designs in order to employ Theorem 2.18.

Theorem 2.19. [1] *There exists a PBD*$(v, \{5, 7, 8, 9\})$ *for* v *a positive integer* $v \geq 5$ *and* $v \notin N_1 = \{6, 10-20, 22, 23, 24, 26-34, 38, 39\}$ *except possibly for* $v \in N_2 = \{42, 43, 44, 46, 51, 52, 60, 94, 95, 96, 98, 99, 100, 102, 104, 106, 107, 108, 110, 111, 116, 138, 140, 142, 146, 150, 154, 156, 158, 162, 166, 170, 172, 174, 206\}$.

We use this PBD result to construct group divisible designs.

Theorem 2.20. *Let* v *be a positive integer,* $v \geq 5$ *and* $v \notin N_1 \cup N_2$. *There exists a GDD*$(v-1; \{5, 7, 8, 9\}; \{4, 6, 7, 8\}; 0, 1)$.

Proof. Delete one element from the pairwise balanced designs constructed in Theorem 2.19. The resulting GDDs have blocks of sizes 5, 7, 8, and 9 and groups of sizes 4, 6, 7, and 8. □

Theorem 2.21. *Let* v *be a positive integer,* $v \geq 5$ *and* $v \notin N_1 \cup N_2$. *There exists a Room* 3*-frame of type* 2^v.

Proof. We apply Theorem 2.18(a) using the group divisible designs constructed in Theorem 2.20 and the existence of the Room 3-frames of type 2^k for $k = 5, 7, 8, 9$. The Room 3-frames of type $2^5, 2^8$, and 2^9 are from Lemma 2.5 while a Room 3-frame of type 2^7 exists from Corollary 2.14. □

For most of the values of $n \in N_1$, Room 3-frames of type 2^n have been constructed. The following lemma gives pointers to proofs of their existence.

Lemma 2.22. *Let* $n \in N_1 \setminus \{6, 38, 39\}$*, then there is a Room* 3*-frames of type* 2^n.

Proof. Room 3-frames of type 2^n exist for $n \in \{10, 14, 15, 18, 22, 26, 30, 34\}$ by Theorem 2.12.

Room 3-frames of type 2^n exist for $n \in \{11, 19, 23, 27, 29, 31\}$ by Corollary 2.14.

Room 3-frames of type 2^n for $n \in \{12, 16, 20\}$ are given in [5].

Room 3-frames of type 2^n for $n \in \{13, 17\}$ are given in [9].

Room 3-frames of type 2^n exist for $n \in \{24, 28, 32, 33\}$ by Lemma 2.5. □

We now construct most of the Room 3-frames of type 2^n for the values of $n \in N_2$. These Room 3-frames are made via Theorem 2.18 using group divisible designs which are constructed by truncating transversal designs. The following lemma describes this process in more detail.

Lemma 2.23. *If there exists a TD*$(9, m)$ *and there exist Room* $3-$*frames of type* $2^x, 2^y$ *and* 2^m *where* $0 \leq x, y \leq m$*, then there is a Room* $3-$*frames of type* 2^{7m+x+y}.

Proof. Delete $m-x$ elements from one group and $m-y$ elements from a second group of the TD$(9, m)$. The resulting design is a GDD with blocks of size 7,8, and 9 and groups of sizes m, x, and y, i.e. a GDD$(7m+x+y; \{7, 8, 9\}; \{m, x, y\})$. Now use Theorem 2.18(b) to construct Room $3-$frames of type 2^{7m+x+y}. □

We are now in position to construct Room 3−frames of type 2^n for nearly all of the values in N_2.

Lemma 2.24. *Let $n \in N_2 - \{42, 46\}$. Then there exists a Room 3-frame of type 2^v.*

Proof. First note that 3-frames of types 2^{44} and 2^{52} exist by Lemma 2.5 and there exist 3-frames of type 2^n for $n = 43, 95$ and 107 from Theorem 2.15.

We use the direct product construction, Theorem 2.16, combined with Theorem 2.17 to construct 2 more cases. For type 2^{51}, we start with a Room 3-frame of type 4^5 [14] and expand by 5, then fill in the holes of the resulting frame with Room 3-frames of type 2^{11} (via Theorem 2.17(ii)). For type 2^{60}, we start with a Room 3-frame of type 2^{12} and expand by 5, then we fill in the holes with Room 3-frames of type 2^5 (via Theorem 2.17(i)).

We construct all of the remaining cases using Lemma 2.23 with $m = 11, 13, 19, 23,$ and 25 and $x, y \in \{5, 7, 8, 9, 11, 12, 13, 16, 17, 19, 23\}$. There exist Room 3-frames of type 2^k for $k \in \{5, 7, 8, 9, 11, 12, 13, 16, 17, 19, 23, 25\}$ from Theorem 2.21 or Lemma 2.22. The following table contains the details of the construction from Lemma 2.23.

n	Construction	n	Construction
94	$7 \cdot 11 + 9 + 8$	96	$7 \cdot 13 + 5$
98	$7 \cdot 13 + 7$	99	$7 \cdot 13 + 8$
100	$7 \cdot 13 + 9$	102	$7 \cdot 13 + 11$
104	$7 \cdot 13 + 5 + 8$	106	$7 \cdot 13 + 8 + 7$
108	$7 \cdot 13 + 9 + 8$	110	$7 \cdot 13 + 8 + 11$
111	$8 \cdot 13 + 12$	116	$8 \cdot 13 + 12$
138	$7 \cdot 19 + 5$	140	$7 \cdot 19 + 7$
142	$7 \cdot 19 + 9$	146	$7 \cdot 19 + 13$
150	$7 \cdot 19 + 17$	154	$7 \cdot 19 + 12 + 9$
156	$7 \cdot 19 + 12 + 11$	158	$7 \cdot 19 + 12 + 13$
162	$7 \cdot 19 + 16 + 13$	166	$7 \cdot 19 + 16 + 17$
170	$7 \cdot 23 + 9$	172	$7 \cdot 23 + 11$
174	$7 \cdot 23 + 13$	206	$7 \cdot 25 + 23 + 8$

□

Combining the existence results in Theorem 2.21 and Lemmas 2.22 and 2.24 above, we have the following.

Theorem 2.25. *Let v be a positive integer, $v \geq 5$. Then there exists a Room 3-frame of type 2^v except possibly for $v \in \{6, 38, 39, 42, 46\}$.*

We do not hesitate to conjecture that Room 3-frame of type 2^v exist for $v \in \{38, 39, 42, 46\}$. We are not as certain about the existence of a Room 3-frame of type 2^6.

3 Howell cubes

In this section, we use the Room 3-frames from Section 2 and a combination of recursive and direct constructions to establish the existence of Howell cubes, $H_3(2n, 2n+\alpha)$, for all $n \geq 4$ and $\alpha \in \{2, 4, 6, 8\}$.

We begin with a direct construction for Howell cubes which we will use for the small values. As with Room frames, we again use a modified type of starter, termed a Howell n-starter, for finding small Howell cubes. Suppose G is an additive abelian group of order s, and $s + 1 \leq 2n \leq 2s$. A *Howell n-starter* in G is a set

$$S = \{\{s_i, t_i\} : 1 \leq i \leq s - n\} \cup \{\{s_i\} : s - n + 1 \leq i \leq n\}$$

that satisfies the two properties:

1. $\{s_i : 1 \leq i \leq n\} \cup \{t_i : 1 \leq i \leq s - n\} = G$;
2. $(s_i - t_i) \neq \pm(s_j - t_j)$ if $i \neq j$.

If S is a Howell n-starter, then a set $A = \{\{a_i\} : 1 \leq i \leq n\}$ is an *adder* for S if the elements in A are distinct, and the set

$$S + A = \{\{s_i + a_i, t_i + a_i\} : 1 \leq i \leq s - n\} \cup \{\{s_i + a_i\} : s - n + 1 \leq i \leq n\}$$

is again a Howell n-starter. In this case the two Howell n-starters are called *orthogonal Howell n-starters*.

Example 3.1. *$S = \{\{5, 6\}, \{2, 0\}, \{1\}, \{3\}, \{4\}\}$ is a Howell 5-starter in the group $G = \mathbb{Z}_7$. $A = \{4, 5, 0, 1, 2\}$ is an adder for S and the orthogonal Howell 5-starter is $S + A = \{\{2, 3\}, \{0, 5\}, \{1\}, \{4\}, \{6\}\}$.*

The existence of two orthogonal Howell n-starters in a group of order s implies the existence of a $H^*(s, 2n)$ (see [6]) and hence the existence of a pair of orthogonal one-factorizations of an underlying s-regular graph on $2n$ vertices. Hence a set of d mutually orthogonal Howell n-starters in a group of order s implies the existence of a $H^*_d(s, 2n)$. We record this in the following theorem for the case of $d = 3$.

Theorem 3.2. *The existence of three orthogonal Howell n-starters in the group $\mathbb{Z}_s$ implies the existence of an $H^*_3(s, 2n)$*

Our main recursive construction is a easy generalization of Theorem 7.1 in [9] to higher dimension and frames of general type.

Theorem 3.3. *If there exists a GDD$(v; K; M)$ such that (1) there exists a Room d-frame of type 2^k for each $k \in K$ and (2) there exists an $H^*_d(2m, 2m + \alpha)$ for each $m \in M$, then there exists an $H^*_d(2v, 2v + \alpha)$.*

We will also use the standard direct product construction for a few small cases.

Theorem 3.4. *If there exists a d-dimensional $\{S_1, S_2, \ldots, S_n\}$-Room frame, d mutually orthogonal latin squares of side u, and $H^*_d(u|S_i|, u|S_i| + \alpha)$ for $i = 1, 2, \ldots, n$, then there is an $H^*_d(u\sum_{i=1}^{n} |S_i|, u(\sum_{i=1}^{n} |S_i|) + \alpha)$.*

We will apply our main recursive construction Theorem 3.3 using the same group divisible designs that were used in Theorem 2.21 for Room 3-frames. Recall that $N_1 = \{6, 10{-}20, 22, 23, 24, 26{-}34, 38, 39\}$ and $N_2 = \{42, 43, 44, 46, 51, 52, 60, 94, 95, 96, 98, 99, 100, 102, 104, 106, 107, 108, 110, 111, 116, 138, 140, 142, 146, 150, 154, 156, 158, 162, 166, 170, 172, 174, 206\}$.

We will need to construct some of the small $H_3^*(2v, 2v+\alpha)$ directly. The following lemma describes their construction.

Lemma 3.5. *There exist $H_3^*(2n, 2n+\alpha)$ for $\alpha = 2, 4, 6, 8$ and $2n \in \{8, 10, 12, 14, 16\}$ and also all n where $n+1 \in N_1 \cup \{42, 43, 44, 46, 51, 52, 60\}$.*

Proof. An $H_3(12, 14)$ is given in Example 1.2. An $H_3(8, 16)$ is equivalent to three MOLS(8) (which clearly exist). Two of these designs can be constructed using finite geometry; we apply Theorem 1.6 with $m = 3, 5$, to construct an $H_4(8, 10)$ and an $H_{16}(32, 34)$.

Three orthogonal Howell starters for the $H_3^*(12, 14)$, $H_3^*(12, 16)$, $H_3^*(12, 18)$ and $H_3^*(12, 20)$ are given in Appendix C. We use these Howell cubes and apply Theorem 3.4 using a Room 3-frame of type 1^7 and $u = 12$ to construct an $H_3(84, 84+\alpha)$. Three orthogonal Howell starters for the $H_3^*(10, 12)$, $H_3^*(10, 14)$, $H_3^*(10, 16)$ and $H_3^*(10, 18)$ are also given in Appendix C. Using these Howell cubes we apply Theorem 3.4 with $u = 5$ and a Room 3-frames of type 2^9 to construct an $H_3(90, 90+\alpha)$. We again apply Theorem 3.4 with $u = 5$ and Room 3-frames of type 4^5 to construct an $H_3(100, 100+\alpha)$: the three orthogonal Howell starters for the $H_3^*(20, 22)$, $H_3^*(20, 24)$, $H_3^*(20, 26)$ and $H_3^*(20, 28)$ are given at [8].

All of the remaining Howell cubes $H_3(2n, 2n+\alpha)$ listed above are constructed directly via Theorem 3.2 using computer searches for 3 orthogonal Howell n-starters. For $2n \in \{8, 10, 12, 14, 16\}$ these starters are given in Appendix C, the larger ones can all be found at [8]. $\square$

Theorem 3.6. *Let $v \geq 5$ be a positive integer with $v \notin N_2$. Then there exists an $H_3^*(2(v-1), 2(v-1)+\alpha)$ for $\alpha = 2, 4, 6, 8$.*

Proof. Let $v \geq 5$ be a positive integer, $v \notin N_1 \cup N_2$. Since there exist 3-dimensional Room frames of type 2^k for $k \in \{5, 7, 8, 9\}$ and $H_3^*(2m, 2m+\alpha)$ for $m \in \{4, 6, 7, 8\}$ by Lemma 3.5, we can apply Theorem 2.20 and Theorem 3.3 to construct $H_3^*(2(v-1), 2(v-1)+\alpha)$ for $\alpha = 2, 4, 6, 8$. If $v \in N_1$, then there is a $H_3^*(2(v-1), 2(v-1)+\alpha)$ for $\alpha = 2, 4, 6, 8$ by Lemma 3.5. $\square$

¿From Lemma 3.5 and Theorem 3.6 we see that the only remaining cases are for $H_3(2(v-1), 2(v-1)+\alpha)$ where $v \geq 94$ and $v \in N_2$. These remaining cases are done using group divisible designs which are constructed by truncating transversal designs and applying Theorem 3.3. The next result covers all of the remaining cases for $v-1$ where $v \in N_2$.

Lemma 3.7. *There exist $H_3^*(2n, 2n+\alpha)$ for $\alpha = 2, 4, 6, 8$ and $n \in \{93, 94, 95, 97, 98, 99, 101, 103, 105, 106, 107, 109, 110, 115, 137, 139, 141, 145, 149, 153, 155, 157, 161, 165, 169, 171, 173, 205\}$.*

Proof. As with Lemma 2.23 we again begin with a TD$(9, m)$ to construct a GDD$(7m+x+y; \{7,8,9\}; \{m,x,y\})$. We then apply Theorem 3.3 to obtain an $H_3^*(2n, 2n+\alpha)$ where $n = 7m + x + y$. For ingredients we will use $m = 11, 13, 17$, and 23 and $4 \le x, y \le m$. There exist Room 3-frames of types $2^7, 2^8, 2^9$ and the required Howell cubes are constructed in Lemma 3.5. The following table contains the details of the GDD constructions.

n	Construction	n	Construction	n	Construction
93	$7 \cdot 11 + 8 + 8$	94	$7 \cdot 11 + 9 + 8$	95	$7 \cdot 13 + 4$
97	$7 \cdot 13 + 6$	98	$7 \cdot 13 + 7$	99	$7 \cdot 13 + 8$
101	$7 \cdot 13 + 5 + 5$	103	$7 \cdot 13 + 7 + 5$	105	$7 \cdot 13 + 7 + 7$
106	$7 \cdot 13 + 8 + 7$	107	$7 \cdot 13 + 8 + 8$	109	$7 \cdot 13 + 10 + 8$
110	$7 \cdot 13 + 10 + 9$	115	$7 \cdot 13 + 12 + 12$	137	$7 \cdot 19 + 4$
139	$7 \cdot 19 + 6$	141	$7 \cdot 19 + 8$	145	$7 \cdot 19 + 12$
149	$7 \cdot 19 + 8 + 8$	153	$7 \cdot 19 + 8 + 12$	155	$7 \cdot 19 + 10 + 12$
157	$7 \cdot 19 + 12 + 12$	165	$7 \cdot 23 + 4$	169	$7 \cdot 23 + 8$
171	$7 \cdot 23 + 5 + 5$	173	$7 \cdot 23 + 5 + 7$	205	$7 \cdot 23 + 22 + 22$

$\square$

Combining Lemma 3.5, Theorem 3.6 and Lemma 3.7 above gives us the following result.

Theorem 3.8. *Let n be a positive integer, $n \ge 4$, and let $\alpha \in \{2, 4, 6, 8\}$. Then there exists an $H_3^*(2n, 2n+\alpha)$.*

4 Conclusion

Room 3-frames and Howell cubes are natural generalizations of Room frames and Howell designs to three dimensions. The two dimensional problems were solved in the 80's, but only sporadic examples were known for higher dimensions. In this paper we have begun the systematic study of Howell cubes and Room 3-frames. We introduced the notion of three orthogonal intransitive frame starters which provided some small examples of Room 3-frames. We then proved the existence of Room 3-frames of type 2^n (with at present only 5 possible exceptions). We are continuing our study of Room 3-frames of other types; the natural case to focus on next is type h^n, for all h. Similarly, we have proven the existence of all 3 dimensional Howell designs, $H_3^*(2n, 2n+\alpha)$ for $n \ge 4$ and $\alpha \in \{2, 4, 6, 8\}$. In addition to Room 3-frames, the recursive constructions that we would like to use to determine the spectrum of Howell cubes in general also require information about the spectrum of $H_3(s, s+k)$ for k small and odd. This is the focus of our current work as well as constructing further cases of $H_3^*(s, 2n+\alpha)$ where $\alpha > 8$.

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Appendix A

In this appendix, we give three mutually orthogonal frame starters S_1, S_2, S_3 of type 2^n for each $n \in \{8, 9, 24, 28, 32, 33, 44, 52\}$.

$n = 8$

S_1: (4,5) (13,15) (6,9) (10,14) (2,7) (11,1) (12,3)

S_2: (5,6) (9,11) (12,15) (3,7) (13,2) (14,4) (10,1)

S_3: (10,11) (2,4) (3,6) (9,13) (12,1) (15,5) (7,14)

$n = 9$

S_1: (2,3) (4,6) (11,14) (13,17) (7,12) (10,16) (1,8) (15,5)

S_2: (12,13) (3,5) (1,4) (2,6) (10,15) (11,17) (7,14) (8,16)

S_3: (1,2) (11,13) (14,17) (6,10) (3,8) (16,4) (5,12) (7,15)

$n = 24$

S_1: (32,33) (6,8) (41,44) (5,9) (22,27) (19,25) (3,10) (15,23) (29,38) (42,4) (35,46) (31,43) (26,39) (47,13) (1,16) (20,36) (11,28) (12,30) (2,21) (45,17) (34,7) (18,40) (14,37)

S_2: (6,7) (18,20) (32,35) (41,45) (37,42) (8,14) (23,30) (28,36) (10,19) (2,12) (4,15) (17,29) (25,38) (43,9) (34,1) (31,47) (16,33) (26,44) (3,22) (39,11) (40,13) (5,27) (46,21)

S_3: (45,46) (2,4) (29,32) (3,7) (42,47) (6,12) (30,37) (27,35) (11,20) (33,43) (15,26) (9,21) (10,23) (14,28) (38,5) (1,17) (22,39) (13,31) (25,44) (16,36) (19,40) (34,8) (18,41)

$n = 28$

S_1: (10,11) (32,34) (15,18) (25,29) (54,3) (49,55) (33,40) (50,2) (53,6) (27,37) (5,16) (9,21) (39,52) (22,36) (31,46) (41,1) (13,30) (17,35) (44,7) (4,24) (26,47) (20,42) (45,12) (51,19) (23,48) (38,8) (43,14)

S_2: (31,32) (12,14) (38,41) (13,17) (53,2) (9,15) (44,51) (26,34) (27,36) (49,3) (46,1) (42,54) (35,48) (11,25) (6,21) (4,20) (7,24) (22,40) (47,10) (30,50) (8,29) (33,55) (16,39) (37,5) (18,43) (19,45) (52,23)

S_3: (21,22) (31,33) (3,6) (12,16) (4,9) (13,19) (40,47) (7,15) (2,11) (35,45) (37,48) (43,55) (51,8) (20,34) (23,38) (36,52) (29,46) (32,50) (25,44) (54,18) (5,26) (17,39) (30,53) (42,10) (24,49) (1,27) (14,41)

$n = 32$

S_1: (53,54) (46,48) (37,40) (24,28) (56,61) (11,17) (62,5) (7,15) (51,60) (25,35) (12,23) (8,20) (26,39) (16,30) (18,33) (42,58) (21,38) (27,45) (44,63) (47,3) (10,31) (55,13) (34,57) (49,9) (43,4) (52,14) (59,22) (1,29) (41,6) (36,2) (19,50)

S_2: (26,27) (2,4) (7,10) (37,41) (46,51) (15,21) (43,50) (1,9) (35,44) (52,62) (34,45) (55,3) (47,60) (24,38) (5,20) (61,13) (6,23) (22,40) (11,30) (58,14) (28,49) (54,12) (16,39) (18,42) (8,33) (31,57) (36,63) (53,17) (19,48) (29,59) (25,56)

S_3: (48,49) (37,39) (23,26) (20,24) (36,41) (29,35) (61,4) (11,19) (63,8) (40,50) (14,25) (46,58) (43,56) (17,31) (54,5) (2,18) (62,15) (3,21) (34,53) (13,33) (59,16) (52,10) (22,45) (47,7) (51,12) (44,6) (38,1) (27,55) (28,57) (30,60) (42,9)

$n = 33$

S_1: (50,51) (35,37) (38,41) (5,9) (52,57) (15,21) (24,31) (22,30) (4,13) (2,12) (54,65) (62,8) (27,40) (55,3) (49,64) (29,45) (17,34) (10,28) (23,42) (43,63) (18,39) (36,58) (59,16) (1,25) (60,19) (47,7) (26,53) (44,6) (32,61) (56,20) (46,11) (48,14)

S_2: (36,37) (30,32) (39,42) (31,35) (15,20) (17,23) (1,8) (26,34) (46,55) (61,5) (27,38) (52,64) (57,4) (62,10) (14,29) (43,59) (2,19) (47,65) (44,63) (21,41) (54,9) (56,12) (50,7) (25,49) (3,28) (53,13) (45,6) (60,22) (11,40) (18,48) (51,16) (58,24)

S_3: (61,62) (3,5) (10,13) (55,59) (34,39) (43,49) (4,11) (7,15) (63,6) (17,27) (31,42) (9,21) (45,58) (26,40) (50,65) (8,24) (12,29) (30,48) (37,56) (32,52) (23,44) (19,41) (57,14) (22,46) (35,60) (25,51) (20,47) (54,16) (38,1) (64,28) (53,18) (36,2)

$n = 44$

S_1: (31,32) (62,64) (2,5) (82,86) (79,84) (63,69) (26,33) (27,35) (51,60) (58,68) (41,52) (3,15) (85,10) (67,81) (39,54) (34,50) (25,42) (12,30) (17,36) (77,9) (75,8) (80,14) (48,71) (22,46) (45,70) (57,83) (74,13) (28,56) (24,53) (65,7) (76,19) (11,43) (4,37) (38,72) (73,20) (23,59) (18,55) (66,16) (78,29) (21,61) (87,40) (47,1) (6,49)

S_2: (23,24) (84,86) (29,32) (8,12) (52,57) (75,81) (66,73) (83,3) (61,70) (20,30) (71,82) (67,79) (49,62) (5,19) (48,63) (87,15) (68,85) (13,31) (34,53) (17,37) (55,76) (28,50) (18,41) (35,59) (22,47) (38,64) (45,72) (14,42) (65,6) (10,40) (58,1) (77,21) (27,60) (56,2) (4,39) (7,43) (9,46) (16,54) (74,25) (11,51) (80,33) (36,78) (26,69)

S_3: (86,87) (25,27) (72,75) (63,67) (17,22) (4,10) (46,53) (15,23) (42,51) (71,81) (62,73) (1,13) (57,70) (54,68) (33,48) (24,40) (77,6) (79,9) (64,83) (38,58) (26,47) (30,52) (37,60) (76,12) (20,45) (80,18) (82,21) (11,39) (66,7) (5,35) (34,65) (29,61) (41,74) (16,50) (84,31) (19,55) (32,69) (78,28) (85,36) (56,8) (2,43) (49,3) (59,14)

$n = 52$

S_1: (21,22) (71,73) (3,6) (4,8) (78,83) (100,2) (27,34) (60,68) (82,91) (57,67) (50,61) (77,89) (56,69) (30,44) (32,47) (85,101) (45,62) (41,59) (29,48) (26,46) (10,31) (80,102) (63,86) (92,12) (84,5) (11,37) (1,28) (96,20) (94,19) (51,81) (97,24) (40,72) (66,99) (54,88) (18,53) (7,43) (76,9) (17,55) (98,33) (103,39) (49,90) (23,65) (36,79) (14,58) (74,15) (93,35) (70,13) (16,64) (38,87) (25,75) (95,42)

S_2: (4,5) (29,31) (70,73) (75,79) (15,20) (87,93) (91,98) (36,44) (97,2) (37,47) (99,6) (68,80) (45,58) (9,23) (102,13) (25,41) (72,89) (35,53) (76,95) (103,19) (63,84) (21,43) (67,90) (27,51) (34,59) (100,22) (12,39) (66,94) (56,85) (32,62) (18,49) (69,101) (48,81) (8,42) (11,46) (28,64) (77,10) (50,88) (57,96) (71,7) (24,65) (92,30) (78,17) (16,60) (38,83) (40,86) (14,61) (26,74) (33,82) (55,1) (3,54)

S_3: (83,84) (36,38) (31,34) (88,92) (57,62) (44,50) (102,5) (29,37) (39,48) (17,27) (11,22) (13,25) (67,80) (64,78) (58,73) (24,40) (55,72) (3,21) (86,1) (81,101) (49,70) (6,28) (85,4) (32,56) (66,91) (16,42) (18,45) (33,61) (87,12) (69,99) (43,74) (65,97) (63,96) (77,7) (89,20) (98,30) (75,8) (9,47) (100,35) (53,93) (82,19) (76,14) (51,94) (46,90) (15,60) (68,10) (59,2) (23,71) (54,103) (95,41) (79,26)

Appendix B

In this appendix, we give three mutually orthognal 2^n-intransitive frame starters S_1, S_2, S_3 for each $n \in \{14, 15, 18, 22, 26, 30, 34\}$.

$n = 14$

$$
\begin{aligned}
S_1 &= \{(21,19),(15,11),(7,12),(17,23),(14,22),(25,8),(10,20),(24,9),(4,18)\} \cup \{5,16\},\\
&\quad P_1 = (3,6), \quad R_1 = (1,2)
\end{aligned}
$$

$S_2 = \{(7,5),(19,15),(22,1),(14,20),(16,24),(23,6),(18,2),(25,10),(3,17)\} \cup \{21,12\}$,
$P_2 = (8,9)$, $R_2 = (4,11)$

$S_3 = \{(9,7),(5,1),(12,17),(24,4),(8,16),(14,23),(19,3),(21,6),(22,10)\} \cup \{15,20\}$
$P_3 = (11,18)$, $R_2 = (25,2)$

$n = 15$

$S_1 = \{(24,26),(8,11),(16,20),(13,18),(17,23),(27,7),(15,25),(10,21),(22,6),(19,4)\} \cup \{5,9\}$, $P_1 = (3,12)$, $R_1 = (1,2)$

$S_2 = \{(17,19),(4,7),(22,26),(15,20),(5,11),(2,10),(13,23),(1,12),(9,21),(3,16)\} \cup \{18,8\}$,
$P_2 = (24,25)$, $R_2 = (27,6)$

$S_3 = \{(9,11),(20,23),(6,10),(16,21),(27,5),(4,12),(19,1),(25,8),(18,2),(13,26)\} \cup \{7,15\}$,
$P_3 = (17,24)$, $R_3 = (22,3)$

$n = 18$

$S_1 = \{(10,12),(30,33),(14,18),(11,16),(26,32),(22,29),(20,28),(15,24),(31,7),(8,19),(27,5),(23,3),(9,25)\} \cup \{4,13\}$, $P_1 = (6,21)$, $R_1 = (1,2)$

$S_2 = \{(2,4),(18,21),(26,30),(32,3),(29,1),(24,31),(15,23),(16,25),(10,20),(28,5),(7,19),(33,13),(6,22)\} \cup \{12,11\}$, $P_2 = (8,9)$, $R_2 = (14,27)$

$S_3 = \{(5,7),(15,18),(28,32),(3,8),(14,20),(16,23),(30,4),(24,33),(21,31),(29,6),(1,13),(22,2),(27,9)\} \cup \{19,10\}$, $P_3 = (12,25)$, $R_3 = (11,26)$

$n = 22$

$S_1 = \{(32,33),(18,20),(9,13),(31,37),(29,36),(27,35),(39,7),(17,28),(4,16),(41,12),(11,25),(23,38),(8,24),(2,19),(22,40),(15,34),(10,30)\} \cup \{3,26\}$, $P_1 = (5,14)$,
$R_1 = (1,6)$

$S_2 = \{(26,27),(38,40),(20,24),(23,29),(10,17),(31,39),(2,12),(5,16),(41,11),(15,28),(35,7),(30,3),(18,34),(33,8),(37,13),(6,25),(36,14)\} \cup \{1,32\}$, $P_2 = (4,9)$,
$R_2 = (19,22)$

$S_3 = \{(4,5),(15,17),(36,40),(38,2),(6,13),(25,33),(31,41),(7,18),(10,22),(24,37),(39,11),(1,16),(19,35),(3,20),(32,8),(9,28),(34,12)\} \cup \{26,29\}$, $P_3 = (27,30)$,
$R_3 = (14,23)$

$n = 26$

$S_1 = \{(13,14),(44,46),(5,8),(47,1),(31,36),(17,23),(30,37),(40,48),(24,33),(2,12),(18,29),(4,16),(15,28),(42,6),(19,35),(26,43),(20,38),(22,41),(7,27),(39,11),(21,45)\} \cup \{3,10\}$,
$P_1 = (34,49))$, $R_1 = (9,32))$

$S_2 = \{(6,7),(40,42),(34,37),(24,28),(17,22),(8,14),(38,45),(47,5),(11,20),(3,13),(15,26),(39,1),(36,49),(46,10),(32,48),(2,19),(23,41),(16,35),(9,29),(21,43),(44,18)\} \cup \{27,30\}$,
$P_2 = (31,4)$, $R_2 = (12,33)$

$S_3 = \{(26,27),(32,34),(20,23),(15,19),(40,45),(48,4),(42,49),(44,2),(38,47),(18,28),(10,21),(24,36),(43,6),(39,3),(41,7),(13,30),(11,29),(12,31),(17,37),(33,5),(22,46)\}\cup\{8,9\}$, $P_3 = (14,35)$, $R_3 = (1,16)$

$n = 30$

$S_1 = \{(18,19),(22,24),(4,7),(2,6),(20,25),(44,50),(42,49),(48,56),(57,8),(37,47),(32,43),(28,40),(21,35),(38,54),(10,27),(23,41),(17,36),(53,15),(30,51),(45,9),(11,34),(31,55),(46,13),(26,52),(5,33)\} \cup \{3,14\}$, $P_1 = (12,39)$, $R_1 = (1,16)$

$S_2 = \{(16,17),(49,51),(3,6),(11,15),(7,12),(41,47),(55,4),(28,36),(39,48),(13,23),(20,31),(45,57),(24,38),(10,26),(33,50),(19,37),(2,21),(32,52),(25,46),(18,40),(30,53),(35,1),(9,34),(54,22),(44,14)\} \cup \{5,8\}$, $P_2 = (27,42)$, $R_2 = (43,56)$

$S_3 = \{(1,2),(26,28),(43,46),(5,9),(6,11),(32,38),(3,10),(14,22),(12,21),(39,49),(25,36),(15,27),(41,55),(34,50),(40,57),(35,53),(23,42),(51,13),(45,8),(52,16),(33,56),(7,31),(19,44),(4,30),(48,18)\} \cup \{17,54\}$, $P_3 = (24,37)$, $R_3 = (20,47)$

$n = 34$

$S_1 = \{(7,8),(52,54),(36,39),(5,9),(16,21),(35,41),(22,29),(20,28),(10,19),(47,57),(48,59),(56,2),(65,12),(37,51),(55,4),(44,60),(14,32),(63,17),(40,61),(24,46),(25,49),(64,23),(43,3),(26,53),(6,34),(13,42),(15,45),(27,58),(30,62)\} \cup \{11,38\}$, $P_1 = (31,50)$, $R_1 = (1,18)$

$S_2 = \{(3,4),(41,43),(27,30),(54,58),(48,53),(61,1),(52,59),(8,16),(15,24),(34,44),(46,57),(37,49),(6,19),(21,35),(2,17),(7,23),(13,31),(36,56),(63,18),(28,50),(47,5),(26,51),(60,20),(11,38),(14,42),(10,39),(25,55),(9,40),(32,64)\} \cup \{45,62\}$, $P_2 = (12,29)$, $R_2 = (65,22)$

$S_3 = \{(18,19),(34,36),(9,12),(58,62),(42,47),(32,38),(65,6),(14,22),(1,10),(45,55),(43,54),(59,5),(40,53),(13,27),(16,31),(28,44),(63,15),(26,46),(2,23),(17,39),(24,48),(49,8),(51,11),(30,57),(41,3),(35,64),(61,25),(29,60),(20,52)\}\cup\{21,4\}$, $P_3 = (50,7)$, $R_3 = (37,56)$

Appendix C

In the following, three orthogonal Howell starters are given for $2n = 8, 10, 12, 14$ and 16. These can be used to construct $H_3(2n, 2n+\alpha)$ for $\alpha = 2, 4, 6, 8$. Note that if $x = 2n+i$, then x represents ∞_i for $1 \le i \le \alpha$.

$2n = 8$

$H_3(8,12)$

S_1: (4,5) (2,12) (3,6) (8,9) (1,11) (7,10)

S_2: (2,3) (7,12) (5,8) (4,9) (1,11) (6,10)

S_3: (3,4) (5,12) (7,2) (8,9) (6,11) (1,10)

$H_3(8,14)$

S_1: (8,1) (7,12) (5,9) (2,13) (4,11) (3,10) (6,14)

S_2: (2,3) (6,12) (5,9) (7,13) (8,11) (4,10) (1,14)

S_3: (5,6) (3,12) (7,9) (8,13) (4,11) (2,10) (1,14)

$2n = 10$

$H_3(10, 12)$

S_1: (5,6) (10,2) (1,4) (3,7) (8,11) (9,12)
S_2: (9,10) (6,8) (2,5) (3,7) (1,11) (4,12)
S_3: (3,4) (7,9) (5,8) (2,6) (1,11) (10,12)

$H_3(10, 14)$

S_1: (7,8) (4,6) (9,2) (5,13) (1,12) (10,11) (3,14)
S_2: (9,10) (4,6) (8,1) (3,13) (2,12) (5,11) (7,14)
S_3: (5,6) (7,9) (8,1) (2,13) (3,12) (10,11) (4,14)

$H_3(10, 16)$

S_1: (7,13) (3,15) (9,2) (1,5) (4,14) (10,16) (6,11) (8,12)
S_2: (6,13) (4,15) (2,5) (9,3) (10,14) (7,16) (1,11) (8,12)
S_3: (5,13) (7,15) (8,1) (6,10) (4,14) (2,16) (3,11) (9,12)

$H_3(10, 18)$

S_1: (8,17) (10,18) (4,7) (6,16) (2,14) (3,15) (1,11) (9,13) (5,12)
S_2: (4,17) (1,18) (9,2) (3,16) (10,14) (6,15) (5,11) (8,13) (7,12)
S_3: (4,17) (2,18) (3,6) (9,16) (7,14) (1,15) (8,11) (10,13) (5,12)

$2n = 12$

$H_3(12, 14)$

S_1: (11,12) (8,10) (6,9) (1,5) (2,7) (3,14) (4,13)
S_2: (1,2) (5,7) (9,12) (6,10) (3,8) (11,14) (4,13)
S_3: (8,9) (3,5) (10,1) (12,4) (2,7) (11,14) (6,13)

$H_3(12, 16)$

S_1: (12,1) (4,6) (11,2) (8,13) (5,10) (7,14) (3,15) (9,16)
S_2: (6,7) (1,3) (2,5) (8,13) (4,9) (12,14) (11,15) (10,16)
S_3: (8,9) (11,1) (2,5) (12,13) (10,3) (7,14) (4,15) (6,16)

$H_3(12, 18)$

S_1: (6,7) (2,4) (1,17) (5,9) (10,15) (3,16) (8,18) (12,14) (11,13)
S_2: (7,8) (11,1) (12,17) (5,9) (4,15) (10,16) (6,18) (2,14) (3,13)
S_3: (7,8) (12,2) (3,17) (9,1) (10,15) (6,16) (4,18) (11,14) (5,13)

$H_3(12, 20)$

S_1: (7,16) (8,20) (3,6) (9,18) (11,4) (12,14) (2,19) (1,17) (10,13) (5,15)
S_2: (12,16) (5,20) (7,10) (11,18) (9,2) (6,14) (3,19) (8,17) (1,13) (4,15)
S_3: (8,16) (10,20) (11,2) (12,18) (4,9) (6,14) (1,19) (5,17) (7,13) (3,15)

$2n = 14$

$H_3(14, 16)$

S_1: (9,10) (6,8) (13,2) (14,4) (7,12) (11,3) (5,15) (1,16)
S_2: (7,8) (10,12) (1,4) (9,13) (14,5) (11,3) (2,15) (6,16)
S_3: (6,7) (13,1) (9,12) (4,8) (5,10) (11,3) (14,15) (2,16)

$H_3(14, 18)$

S_1: (6,7) (8,10) (12,1) (14,4) (9,18) (11,3) (13,15) (5,17) (2,16)

S_2: (7,8) (12,14) (6,9) (13,3) (1,18) (4,10) (11,15) (2,17) (5,16)
S_3: (12,13) (7,9) (5,8) (10,14) (4,18) (11,3) (1,15) (2,17) (6,16)

$H_3(14, 20)$
S_1: (8,9) (11,13) (3,18) (6,10) (14,16) (1,7) (5,20) (4,19) (2,15) (12,17)
S_2: (4,5) (14,2) (1,18) (6,10) (9,16) (7,13) (12,20) (8,19) (3,15) (11,17)
S_3: (11,12) (6,8) (3,18) (10,14) (5,16) (7,13) (1,20) (2,19) (4,15) (9,17)

$H_3(14, 22)$
S_1: (3,4) (2,19) (12,1) (10,14) (7,17) (5,20) (8,16) (11,15) (6,22) (13,18) (9,21)
S_2: (5,6) (8,19) (10,13) (14,4) (3,17) (2,20) (1,16) (12,15) (11,22) (7,18) (9,21)
S_3: (9,10) (14,19) (5,8) (13,3) (1,17) (2,20) (4,16) (6,15) (7,22) (12,18) (11,21)

$2n = 16$
$H_3(16, 18)$
S_1: (5,6) (10,12) (11,14) (16,4) (3,8) (1,7) (2,9) (15,18) (13,17)
S_2: (13,14) (4,6) (8,11) (1,5) (10,15) (3,9) (16,7) (2,18) (12,17)
S_3: (10,11) (12,14) (4,7) (15,3) (1,6) (2,8) (9,16) (5,18) (13,17)

$H_3(16, 20)$
S_1: (13,14) (2,4) (7,10) (5,9) (12,1) (11,17) (8,15) (3,18) (16,19) (6,20)
S_2: (5,6) (7,9) (1,4) (8,12) (13,2) (11,17) (3,10) (16,18) (14,19) (15,20)
S_3: (13,14) (16,2) (6,9) (7,11) (15,4) (12,17) (1,8) (10,18) (5,19) (3,20)

$H_3(16, 22)$
S_1: (16,20) (5,7) (3,6) (10,14) (12,18) (11,1) (2,9) (13,19) (15,17) (8,22) (4,21)
S_2: (8,20) (2,4) (6,9) (11,15) (3,18) (10,16) (14,5) (7,19) (1,17) (12,22) (13,21)
S_3: (13,20) (14,16) (4,7) (15,3) (12,18) (5,11) (1,8) (9,19) (2,17) (6,22) (10,21)

$H_3(16, 24)$
S_1: (1,17) (14,23) (16,3) (2,18) (7,12) (5,11) (15,6) (10,24) (9,22) (13,19) (8,20) (4,21)
S_2: (7,17) (13,23) (5,8) (9,18) (10,15) (16,6) (12,3) (2,24) (11,22) (14,19) (1,20) (4,21)
S_3: (16,17) (5,23) (10,13) (8,18) (12,1) (3,9) (7,14) (6,24) (2,22) (15,19) (11,20) (4,21)