# The Hamilton-Waterloo problem with triangle-factors and Hamilton cycles: The case $n \equiv 3 \pmod{18}$

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**Abstract**

The Hamilton-Waterloo problem in the case of triangle-factors and Hamilton cycles asks for a 2-factorization of $K_n$ in which each 2-factor is either a Hamilton cycle or a triangle-factor. Necessarily $n \equiv 3 \pmod{6}$. The case of $n \equiv 9 \pmod{18}$ was completely solved in 2004 by Horak, Nedela and Rosa. In this note we solve the problem when $n \equiv 3 \pmod{18}$ and there are at least two Hamilton cycles. A companion paper treats the case when there is exactly one Hamilton cycle and $n \equiv 3 \pmod{6}$ .

## 1 Definitions and Background

This paper is based on the paper by Horak, Nedela and Rosa [4]. We refer the reader to that paper and to [3] for background on the

Hamilton-Waterloo problem and also to [1] or [2] for general reference to terms which may not be defined. We begin by giving the necessary definitions from [4]. Let HW*($6k+3$) be the set of all integers $r$ such that there exists a 2-factorization of $K_{6k+3}$ in which $r$ of the 2-factors are Hamilton cycles, and the remaining $s = 3k + 1 - r$ two-factors are triangle-factors (spanning sets of $2k+1$ triangles). Clearly then HW*$(6k+3) \subseteq \{0, 1, \ldots 3k+1\}$. Define $I(6k+3) = \{0, 1, \ldots 3k+1\}$. It it is well-known that Hamilton factorizations of $K_{6k+3}$ exist as do triangle-factorizations (Kirkman triple systems), hence $\{0, 3k+1\} \subset$ HW*$(6k+3)$.

Let the vertex set of $K_{6k+3}$ be the set $\mathbb{Z}_{2k+1} \times \{0, 1, 2\}$, where $k \geq 1$. As was done in [4] the notation is simplified by letting $\mathbb{Z}_{2k+1} \times \{0\} = A = \{a_0, a_1, \ldots a_{2k}\}$, $\mathbb{Z}_{2k+1} \times \{1\} = B = \{b_0, b_1, \ldots b_{2k}\}$ and $\mathbb{Z}_{2k+1} \times \{2\} = C = \{c_0, c_1, \ldots c_{2k}\}$. All indices will be taken modulo $2k+1$. For $0 \leq d \leq 2k$, define the set of edges

$(AB)_d = \{\{a_i, b_{i+d} : i = 0, 1, \ldots 2k\}$,
$(BC)_d = \{\{b_i, c_{i+d} : i = 0, 1, \ldots 2k\}$, and
$(CA)_d = \{\{c_i, a_{i+d} : i = 0, 1, \ldots 2k\}$.

Hence the edge set $E$ of the complete graph $K_{6k+3}$ can be written as

$$E = E([A]) \cup E([B]) \cup E([C]) \cup \bigcup_{d=0}^{2k} \{(AB)_d \cup (BC)_d \cup (CA)_d\}.$$

where $[X]$ is the complete graph induced by the set of vertices $X$.

For each $0 \leq d \leq 2k$, let $F_d$ denote the subgraph induced by the set of edges $(AB)_d \cup (BC)_d \cup (CA)_{-2d}$. Then clearly, for each $0 \leq d \leq 2k$, $F_d$ is a triangle-factor of $K_{6k+3}$ where the edges of $F_d$ form the triangles $(a_i, b_{i+d}, c_{i+2d})$ for $i = 0, 1, \ldots 2k$.

The following three results are all from [4].

**Lemma 1.1** *[4] Let $-2k \leq p, q, r \leq 2k$ be integers such that $p+q+r$ and $2k+1$ are relatively prime. Then the set of edges $(AB)_p \cup (BC)_q \cup (CA)_r$ induce a Hamilton cycle of $K_{6k+3}$.*

**Corollary 1.2** *[4] For each $d = 0, 1, \ldots 2k-1$, the edges of $F_d \cup F_{d+1}$ can be decomposed into two Hamilton cycles.*

The following gives a partial result for HW*$(n)$ in the case when $n \equiv 3 \pmod{18}$.

**Theorem 1.3** *[4] Let $n = 6k+3$ with $k \equiv 0$ or $2 \pmod 3$. Then $\{(n+3)/6, (n+3)/6+2, (n+3)/6+3, \ldots (n-1)/2\} \subset HW^*(n)$.*

## 2 Main Result

We begin with a corollary to Lemma 1.1.

**Corollary 2.1** *The edges of* $F_1 \cup F_2 \cup F_3$ *can be decomposed into three Hamilton cycles.*

*Proof.* View the edge set of $F_1 \cup F_2 \cup F_3$ as the union of sets $G_1 \cup G_2 \cup G_3$ where $G_1 = (AB)_1 \cup (BC)_1 \cup (CA)_{-4}$, $G_2 = (AB)_2 \cup (BC)_2 \cup (CA)_{-6}$, and $G_3 = (AB)_3 \cup (BC)_3 \cup (CA)_{-2}$. As all three numbers $1+1-4, 2+2-6$ and $3+3-2$ are relatively prime to $2k+1$, by Lemma 1.1 each of the three graphs $G_1, G_2$ and $G_3$ is a Hamilton cycle. $\Box$

An important ingredient in the main proof is a Hanani triple system. Assume that $v \equiv 1 \pmod 6$, a *Hanani triple system*, HTS($v$), is a Steiner triple system on $v$ points with a partition of its blocks into $(v-1)/2$ maximum parallel classes, and a single partial parallel class with $(v-1)/6$ blocks. If we let $v = 2k+1$, then we see that there are k classes each missing exactly one point and there is one class which misses $k+1$ points. It follows since the replication number of an STS($2k+1$) is $k$, that each point is missing from exactly one class of blocks. The following theorem gives the necessary and sufficient conditions for the existence of a Hanani triple system of order $v$.

**Theorem 2.2** *[5] A Hanani triple system of order* $v$ *exists if and only if* $v \equiv 1 \pmod 6$ *and* $v \notin \{7, 13\}$.

We are now ready to prove our main result.

**Theorem 2.3** *Let* $n = 6k+3$ *with* $k \equiv 0$ *(mod 3) and* $k \neq 3, 6$. *Then* $\{2, 3, \ldots, (n-1)/2\} \subset HW^*(n)$.

*Proof.* Since $n = 6k+3$ with $k \equiv 0 \pmod 3$ we have that $|A| = |B| = |C| \equiv 1 \pmod 6$. Hence from Theorem 2.2 there exists a Hanani triple system of order $|A| = 2k+1$. Put the blocks of a HTS($2k+1$) on the points of $A$ such that for $0 \leq i \leq k-1$ the point $a_i$ is missing from the $i$th maximum parallel class (call this $A_i$) and the points $a_k, a_{k+1}, \ldots a_{2k}$ are missing from the last class, $A_k$. Do the same on the points of $B$ and $C$. Now for $0 \leq i \leq k-1$, let $T_i = A_i \cup B_i \cup C_i \cup \{\{a_i, b_i, c_i\}\}$ and let $T_k = A_k \cup B_k \cup C_k \cup \bigcup_{j=k}^{2k} \{a_j, b_j, c_j\}$. There are $k+1$ of these $T_i$'s each of which is a triangle-factor of $K_n$ and $\bigcup_{i=0}^{k} T_i = E([A]) \cup E([B]) \cup E([C]) \cup F_0$.

To show that $2m \in \mathrm{HW}^*(n)$ for $2m \leq 2k$ apply Corollary 1.2 to $\{F_{2i-1}, F_{2i}\}$ for each $i = 1, 2, \ldots, m$. The triangle-factors are formed by $F_j$, $j = 2m+1, 2m+2, \ldots, 2k$ and by $T_j$, $j = 0, 1, \ldots k$. To show that $2m+1 \in \mathrm{HW}^*(n)$ for $3 \leq 2m+1 \leq 2k-1$ first apply Corollary 2.1 to $\{F_1, F_2, F_3\}$ to obtain 3 Hamilton cycles. If $2m+1 \geq 5$ apply Corollary 1.2 to $\{F_{2i}, F_{2i+1}\}$ for $i = 2, 3, \ldots, m$. The $T_i$'s and the remaining $F_i$'s form the triangle-factors in the 2-factorization.

Hence we have that $\{2, 3, \ldots 2k-1, 2k\} = \{2, 3, \ldots (n-3)/3\} \subset \mathrm{HW}^*(n)$. Combining these values with those from Theorem 1.3 completes the proof. ∎

Combining Theorem 2.3 with some other known results give us the (almost) complete spectrum for $\mathrm{HW}(n)$ in the case where $n \equiv 3 \pmod{18}$. We state this as our final theorem.

**Theorem 2.4** *Let* $n \equiv 3$ *(mod 18), then HW*(n)* $= I(n) = \{0, 1, \ldots, (n-1)/2\}$ *except possibly that* $1 \notin HW^*(n)$ *if* $n = 93, 111, 129, 183,$ *or* $201$.

*Proof.* When $n = 21$ and $n = 39$ the result is proven in [4]. The existence of Hamilton cycle decompositions for all $K_v$ with $v$ odd proves that $0 \in \mathrm{HW}^*(n)$ for $n \equiv 3 \pmod{18}$ and hence in conjunction with Theorem 2.3 we have that $I(n) \setminus \{1\} \subset \mathrm{HW}^*(n)$ for all $n \equiv 3 \pmod{18}$. In [3] it is shown that $1 \in \mathrm{HW}^*(n)$ for all but 13 values of $n$. These exceptional cases when $n \equiv 3 \pmod{18}$ are $n = 93, 111, 129, 183,$ and 201.

# 3 Conclusion

We conclude by summarizing what is known about the Hamilton-Waterloo problem on $n$ vertices with triangle-factors and Hamilton cycles. The necessary condition is that $n = 3 \bmod 6$. The problem is completely solved when $n = 3$ or 9 (mod 18) and the number of Hamilton cycles is at least 2 (from [4] and this paper). The problem is also solved when $n = 15$ (mod 18) and the number of Hamilton cycles is in the set $\{(n+3)/6, (n+3)/6+2, (n+3)/6+3, \ldots (n-1)/2\}$ [4] (this is the top 2/3 of the possible spectrum).

In the case where there is exactly one Hamilton cycle and all the other cycles are triangle factors, there is a solution to the Hamilton-Waterloo problem for all $n = 3$ (mod 6) except possibly for 14 cases, namely when $n \in \{93, 111, 123, 129, 141, 153, 159, 177, 183, 201, 207, 213, 237, 249\}$ [3].

# References


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