

Indecomposable triple systems exist for all lambda

Dan Archdeacon* and Jeff Dinitz

*Department of Mathematics and Statistics, University of Vermont,
Burlington, VT 05405, USA*

Received 3 December 1990

Revised 21 May 1991

Abstract

Archdeacon, D. and J. Dinitz, Indecomposable triple systems exist for all lambda, *Discrete Mathematics* 113 (1993) 1–6.

A triple system $(v, 3, \lambda)$ is *indecomposable* if it is not the union of two triple systems $(v, 3, \lambda_1)$, $(v, 3, \lambda_2)$ with $\lambda = \lambda_1 + \lambda_2$. A triple system is *simple* if it has no repeated blocks. In this paper we show the existence of simple indecomposable triple systems for all v and λ satisfying the necessary conditions with v large. Specifically, for each λ we show that there is a $v_0(\lambda)$ (where $v_0(\lambda) = O(\lambda^4)$) such that there exists a simple indecomposable triple system $(v, 3, \lambda)$ for each $v \geq v_0(\lambda)$ with $\lambda v(v-1) \equiv 0 \pmod{6}$ and $\lambda(v-1) \equiv 0 \pmod{2}$. We then concentrate on the case of $\lambda = 5$ and show that $v_0(5) \leq 25$.

1. Introduction

We begin with some standard definitions from design theory. A (v, k, λ) *balanced incomplete block design* ((v, k, λ) -BIBD or simply a (v, k, λ) design) is a pair (V, \mathcal{B}) where V is a v -element set of *points* and \mathcal{B} is a collection of k -element subsets of V , called *blocks*, such that each pair of points appears together in exactly λ blocks. When $k = 3$ the design is a *triple system*. If in addition $\lambda = 1$, then it is a *Steiner triple system*. A design is *simple* if it contains no repeated blocks.

A common way to form a block design with larger λ is to union the sets of blocks of smaller designs sharing a common point-set. In particular, the union of a (v, k, λ_1) design with a (v, k, λ_2) design is a $(v, k, \lambda_1 + \lambda_2)$ design. Conversely, suppose that we can partition the blocks of a (v, k, λ) design so that each part induces a triple system with a strictly smaller λ . Then we say that the design is

Correspondence to: Jeff Dinitz, Department of Mathematics and Statistics, University of Vermont, Burlington, VT 05405, USA.

* Research partially supported by NSF grant number DMS-9007503.

decomposable; the partition is a *decomposition*. A design with no decompositions is *indecomposable*. Indecomposable block designs form the building blocks for general block designs under block unions.

The main problem. Construct indecomposable simple triple systems for all possible v and λ .

The well-known necessary conditions for the existence of a simple $(v, 3, \lambda)$ -BIBD are:

$$\lambda v(v-1) \equiv 0 \pmod{6}, \quad \lambda(v-1) \equiv 0 \pmod{2}, \quad \text{and} \quad \lambda \leq v-2.$$

In 1847, Kirkman [9] proved that these conditions are sufficient for Steiner triple systems, $(v, 3, 1)$ designs. Building on the work of others, Dehon [5] proved in 1983 that for every v and λ satisfying the above necessary conditions there exists a simple $(v, 3, \lambda)$ -BIBD (see also Sarvate [14]). Indecomposable simple designs were introduced in 1974 by Kramer [11] who showed that an indecomposable simple $(v, 3, 2)$ design exists for all $v \equiv 0, 1 \pmod{3}$, $v \geq 4$, except $v = 7$. He also showed that indecomposable simple $(v, 3, 3)$ designs exist for all $v \equiv 1 \pmod{2}$, $v \geq 5$. Colbourn and Rosa [4] proved that indecomposable simple $(v, 3, 4)$ designs exist for all $v \equiv 0, 1 \pmod{3}$, $v \geq 10$. In 1989, Dinitz [6] and independently Milici [12] showed that indecomposable simple $(v, 3, 6)$ designs exist for $v = 8, 14$ and all $v \geq 17$. Thus for $\lambda = 1, 2, 3, 4$ and 6 the necessary conditions for the existence of an indecomposable simple $(v, 3, \lambda)$ are sufficient, except for possibly a few small values of v .

For the general case of $\lambda > 6$, Colbourn and Colbourn [3] constructed a single indecomposable simple $(v, 3, \lambda)$ -BIBD for each odd λ . As they noted in their paper, their technique does not extend to even λ . Shen [15] used the Colbourn and Colbourn result and some recursive constructions to prove that the necessary conditions are asymptotically sufficient. Specifically, if λ is odd, then there exists a constant v_0 depending on λ with an indecomposable simple $(v, 3, \lambda)$ for all $v \geq v_0$ satisfying the necessary conditions. This result was proved using Wilson's Theorem and so the value of v_0 was not specified.

In this paper we show that for each λ the necessary conditions are sufficient except for finitely many small values of v . In fact, in Section 2 we give a specific upper bound for $v_0(\lambda)$ such that if $v > v_0(\lambda)$ and v satisfies the necessary conditions, then there exists a simple indecomposable triple system $(v, 3, \lambda)$. In Section 3 we will consider the remaining small case $\lambda = 5$. We show that there exists an indecomposable simple $(v, 3, 5)$ for all $v \geq 25$, $v \equiv 1, 3 \pmod{6}$.

2. Main results

A $(v, 3, \lambda)$ *partial triple system* (denoted $(v, 3, \lambda)$ -PTS) is a pair (V, \mathcal{B}) , where \mathcal{B} is a collection of triples from a point set V such that each pair occurs at most λ

times. A subset of the blocks in a triple system forms a partial triple system, but it is not the case that every $(v, 3, \lambda)$ -PTS completes to a triple system $(v, 3, \lambda)$.

A partial triple system (V, \mathcal{B}) is λ -indecomposable if there is no partitioning of the triples \mathcal{B} into sets $\mathcal{B}_1, \mathcal{B}_2$ such that for $i = 1, 2$, (V, \mathcal{B}_i) is a $(v, 3, \lambda_i)$ -PTS with $\lambda_i > 0$ and $\lambda_1 + \lambda_2 = \lambda$. Note that indecomposability depends on the value of λ for partial triple systems. For suppose that (V, \mathcal{B}) is a partial triple system in which no pair occurs more than k times. Then (V, \mathcal{B}) is also a partial triple system for any value of $\lambda' \geq k$. But for $\lambda' \geq k + 1$ it always decomposes into a partial triple system with $\lambda = k$ (containing all of the blocks) and a partial triple system with $\lambda = \lambda' - k$ (containing no blocks).

If a partial triple system P is contained in a triple system T , then a decomposition of T induces a decomposition of P . The contrapositive is stated in the following lemma.

Lemma 2.1. *A triple system $(v, 3, \lambda)$ which contains a λ -indecomposable $[w, 3, \lambda]$ -PTS is itself indecomposable.*

We next construct a single λ -indecomposable partial triple system for each λ .

Theorem 2.2. *For each λ there exists a simple λ -indecomposable partial triple system $(v, 3, \lambda)$ -PTS with $v = \lambda(\lambda + 2)$.*

Proof. We begin with a chain of triples shown as triangles in Fig. 1. Specifically, our point set is $V = \{1, 2, \dots, \lambda\} \cup \{a, b\}$. The triples are $\{i, a, b\}$ for $1 \leq i \leq \lambda$, $\{i, i + 1, b\}$ for odd i 's between 1 and $\lambda - 1$, and $\{i, i + 1, a\}$ for even i 's in this range. The triples with both a and b are called *white*, the remaining triples are called *black*.

We modify this partial configuration by ‘blowing up’ each point λ times. Our new point set is $V \times I_\lambda$, where $I_\lambda = \{1, 2, \dots, \lambda\}$ and we will say that a point $(v, i) \in V \times I_\lambda$ lies *above* the point $v \in V$. Similarly, the triangles in our derived partial triple system lie *above* triangles in the base PTS. Specifically, above each black triple $\{i, i + 1, x\}$ (where $x \in \{a, b\}$) we put every triple $\{(i, k_1), (i + 1, k_2), (x, k_3)\}$, except the one with $k_1 = k_2 = k_3 = 1$. Above each white triple we put in a single triple $\{(i, 1), (a, 1), (b, 1)\}$. These new triples are colored black and white, depending on the color of the corresponding triples in the base design. Let P denote the resulting partial triple system.

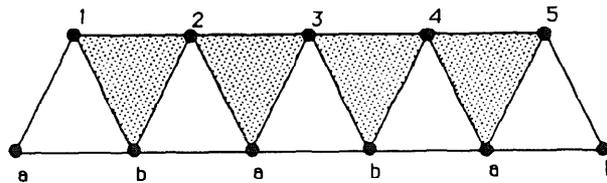


Fig. 1.

It is straightforward to show that P is simple, that no pair occurs more than λ times, and that λ is the smallest such value. We need only show that P is λ -indecomposable.

Let t_i denote the i th white triple, $\{(i, q), (a, 1), (b, 1)\}$. Color the edges lying above $(1, b)$ *red*, and those above $(2, b)$ *blue*. Note that every black triple above $(1, 2, b)$ contains one red and one blue edge. Moreover t_1 is the only triple with a red but not a blue edge, and t_2 is the only triple with a blue but not a red edge.

Suppose that P were decomposable into partial triple systems P_1, P_2 with multiplicities λ_1, λ_2 , where $\lambda_1 + \lambda_2 = \lambda$. Each red edge (and similarly each blue edge) appears in exactly λ triples in P . Hence it appears in exactly λ_i triples in P_i . It follows that the total number of red edges appearing in triples of P_i is the same as the total number of blue edges in triples of P_i , namely $\lambda_i \lambda^2$. Each black triple of P either adds 1 to both sums or adds 0 to both sums. Suppose without loss of generality that $t_1 \in P_1$. Then there is one more red edge than blue edge in P_1 . For the sums to equate there must be a triple with a blue but not a red edge. The only such triple is t_2 . Summarizing, in any λ -decomposition the part with the triple t_1 must contain the triple t_2 .

A similar argument shows that the part with triple t_2 also contains t_3 . Inductively, we can conclude that one part contains each t_i , $1 \leq i \leq \lambda$. But these white triples contain the edge $\{(a, 1), (b, 1)\}$ a total of λ times, so $\lambda_1 = \lambda$. It follows that the decomposition is trivial, and that P is λ -indecomposable. \square

We next complete this λ -indecomposable partial triple system to a triple system using a theorem of Rodger [13]. Our systems are simple, so here $\alpha = 0$.

Theorem 2.4 (Rodger). *A partial triple system $(n, 3, \lambda)$ -PTS with α repeated triples can be embedded in a $(v, 3, \lambda)$ triple system with α repeated triples, for some $v \leq 3(2\lfloor \lambda/2 \rfloor + 1)((3\lambda + 2)n + 1)$.*

Combining Lemma 2.1, Theorem 2.2 and Theorem 2.4 we get the following.

Theorem 2.5. *For each $\lambda \geq 1$ there exists an indecomposable simple $(v, 3, \lambda)$ triple system. Furthermore $v \leq 3(2\lfloor \lambda/2 \rfloor + 1)((3\lambda + 2)\lambda(\lambda + 2) + 1)$.*

To prove our main theorem we will now embed this indecomposable simple triple system of order v into a simple triple system of order w for every $w \geq 2v + 1$. This will produce an indecomposable simple triple system $(w, 3, \lambda)$ for all w larger than some specified value (roughly $18\lambda^4$). We need only apply a new and powerful result of Shen [16] which is an analogue of the Doyen–Wilson theorem [7] for higher λ .

Theorem 2.6 (Shen). *A simple triple system $(v, 3, \lambda)$ can be embedded in a simple triple system $(w, 3, \lambda)$ for every $w \geq 2v + 1$ satisfying the necessary conditions.*

Using Theorems 2.5, 2.6 and Lemma 2.1 we have our main theorem.

Theorem 2.7. *An indecomposable simple triple system $(v, 3, \lambda)$ exists for all $v \geq v_0(\lambda)$ satisfying the necessary conditions where $v_0(\lambda) \leq 6(2\lfloor \lambda/2 \rfloor + 1)((3\lambda + 2)\lambda(\lambda + 2) + 1) + 1$.*

The bound in Theorem 2.7 can be improved for odd λ by roughly a factor of $9\lambda^2$ if the triple systems constructed are not necessarily simple. König [10] gives examples of λ -regular graphs on $\lambda(\lambda + 2) + 1$ vertices without any regular factors (Hoffman, Rodger, and Rosa [8] show that these graphs are the smallest possible). Colbourn [2] shows that these graphs are the neighborhood of a point in a (not necessarily simple) triple system on $\lambda(\lambda + 2) + 2$ points. We now use the theorem of Stern [17] to embed this triple system in one of order v for all $v \geq 2\lambda(\lambda + 2) + 5$ which satisfies the necessary conditions.

3. Indecomposable triple systems with $\lambda = 5$

In this section we consider the case $\lambda = 5$. The necessary conditions for the existence of a $(v, 3, 5)$ triple system are $v \equiv 1, 3 \pmod{6}$. We use the computer to construct several small examples of indecomposable simple $(v, 3, 5)$ designs and then again appeal to Shen's Theorem (Theorem 2.6) to complete the spectrum.

In [18], an extremely effective hill-climbing algorithm for finding Steiner triple system is discussed. It is straightforward to modify that algorithm to find simple triple system with higher λ . It is also an easy modification to the algorithm to fix a set of blocks that must occur in the final triple system. This is done by beginning with this set of blocks and then hill-climbing, never allowing any block from the initial set to be deleted.

Using these two modifications of Stinson's algorithm we found triple systems T with $\lambda = 5$ which contain the simple 5-indecomposable partial triple system P described in Theorem 2.2, so by Lemma 2.1 we have constructed a simple indecomposable triple system $(v, 3, 5)$. Note that P contains 35 points, so that necessarily T must contain at least 35 points. In fact, we were surprised to find a $(39, 3, 5)$ simple indecomposable triple system which contained P . Using this modified algorithm, we also found $(v, 3, 5)$ simple indecomposable triple system for all $v \equiv 1, 3 \pmod{6}$, $39 \leq v \leq 49$. These triple systems are given in [1].

Using a smaller simple 5-indecomposable partial triple system and the aforementioned modification to Stinson's algorithm, Colbourn (personal communication) has constructed simple indecomposable triple systems $(v, 3, 5)$ for $v = 25, 27, 31, 33$ and 37 .

We have the following theorem for $\lambda = 5$.

Theorem 3.1. *A simple indecomposable triple system $(v, 3, 5)$ exists for all $v \equiv 1, 3 \pmod{6}$, $v \geq 25$.*

Proof. Assume $v \equiv 1, 3 \pmod{6}$. If $25 \leq v \leq 49$, then a simple indecomposable triple system $(v, 3, 5)$ exists by the above computer constructions. If $v \geq 51$, then a simple indecomposable triple system $(v, 3, 5)$ exists by Theorem 2.6 and the existence of a simple indecomposable $(25, 3, 5)$ triple system. \square

The values of v for which the existence of a simple indecomposable $(v, 3, 5)$ design remains open are $v = 13, 15, 19$, and 21 .

Acknowledgment

The authors thank Charles Colbourn for helpful discussions on this material.

References

- [1] D.S. Archdeacon and J.H. Dinitz, Small indecomposable triple systems with $\lambda = 5$, Department of Mathematics, Research Report 90–06.
- [2] C.J. Colbourn, Simple neighbourhoods in triple systems, *J. Combin. Theory Ser. A* 52; 10–19.
- [3] C.J. Colbourn and M.J. Colbourn, The computational complexity of decomposing block designs, *Ann. Discrete Math* 26 (1985) 345–350.
- [4] C.J. Colbourn and A. Rosa, Indecomposable triple systems with $\lambda = 4$, *Studia Scientiarum Mathematicarum Hungarica* 20 (1985) 139–144.
- [5] M. Dehon, On the existence of 2-designs $S_\lambda(2, 3, v)$ without repeated blocks, *Discrete Math.* 43 (1983) 155–171.
- [6] J.H. Dinitz, Indecomposable triple systems with $\lambda = 6$, *J. Combin. Math. Combin. Comput.* 5 (1989) 139–142.
- [7] J. Doyen and R.M. Wilson, Embeddings of Steiner triple systems, *Discrete Math.* 5 (1973) 229–239.
- [8] D.G. Hoffman, C.A. Rodger and A. Rosa, Maximal sets of 2-factors and Hamiltonian cycles, *J. Combin. Theory Ser. B*, to appear.
- [9] T.P. Kirkman, On a problem in combinations, *Cambridge and Dublin Math. J.* 2 (1847) 191–204.
- [10] D. König, *Theorie der endlichen und unendlichen Graphen; Kombinatorische Topologie der Streck-enkomplexe.* (Chelsea Pub. Co., New York, 1950).
- [11] E.S. Kramer, Indecomposable triple systems, *Discrete Math.* 8 (1974) 177–180.
- [12] S. Milici, Indecomposable $S_\lambda(2, 3, v)$'s, *J. Combin. Theory Ser. A* 25 (1991) 15–26.
- [13] C.A. Rodger, Triple systems with a fixed number of repeated triples, *J. Austral Math Soc. Ser. A* 41 (1986) 180–187.
- [14] D.G. Sarvate, All simple BIBD's with block size 3 exist, *Am. Combin.* 21-A (1986) 157–270.
- [15] H. Shen, Indecomposable triple systems without repeated blocks, preprint.
- [16] H. Shen, Embeddings of simple triple systems, preprint.
- [17] G. Stern, Triplesysteme mit untersystemen, *Arch. Math (Basel)* 33 (1979) 204–208.
- [18] D.R. Stinson, Efficient algorithms for the construction of combinatorial designs, *Ann. Discrete Math.* 26 (1985) 321–334.