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Constructing indecomposable 1-factorizations of the complete multigraph

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Abstract

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The complete multigraph λK_{2n} has $2n$ vertices and λ edges joining each pair of vertices. In this paper we examine 1-factorizations of λK_{2n}. Observe that a 1-factorization of λK_{2n} is equivalent to a collection of 1-factors of K_{2n} such that every edge of K_{2n} appears in exactly λ 1-factors. This collection of 1-factors is *simple* if no 1-factor is repeated. It is *indecomposable* if no subset of 1-factors contains each edge λ' times where $1 \leqslant \lambda' \leqslant \lambda - 1$. An indecomposable 1-factorization of λK_{2n} is denoted IOF$(2n, \lambda)$. In Section 1 we construct an IOF$(2(\lambda + p), \lambda)$ for all λ where p is the smallest prime not dividing λ. As a corollary we get a simple indecomposable 1-factorization of λK_{2n} for all $2n \geqslant 4(\lambda + p)$. In Section 2 we give constructions for indecomposable 1-factorizations of λK_n for some small values of λ.

1. Existence theorem

We first find an indecomposable 1-factorization of $\lambda K_{2(\lambda+p)}$. We will then use an embedding theorem to construct simple indecomposable 1-factorizations of λK_{2n} for all $2n \geqslant 4(\lambda + p)$.

Theorem 1.1. *For all $\lambda > 2$ there exists an indecomposable 1-factorization of $\lambda K_{2(\lambda+p)}$, where $p < \lambda$ is the smallest prime which does not divide λ.*

Proof. Partition the vertices of $\lambda K_{2(\lambda+p)}$ into two parts, L and R (left and right), each containing $(\lambda + p)$ vertices. We label the points in L with the elements in

$Z_{\lambda+p}$. Similarly label the set R. To distinguish vertices in L from vertices in R we affix a subscript L or R. Thus a typical vertex in L can be described as i_L where $0 \leq i \leq \lambda + p - 1$ and a typical edge from L to R will be denoted as (i_L, j_R). All arithmetic will be performed in $Z_{\lambda+p}$.

We plan to construct an indecomposable 1-factorization of $\lambda K_{2(\lambda+p)}$ which will be called F. We first describe some 1-factors of F in which each edge has one end in L and one end in R. For all $i \in Z_{\lambda+p}$, $i \neq 0$, let

$$g_i = \{(a_L, (a+i)_R) \mid a \in Z_{\lambda+p}\}.$$

Notice that each g_i is a 1-factor of $\lambda K_{2(\lambda+p)}$. The first group of factors in our desired 1-factorization F is the multiset consisting of the 1-factor g_1 exactly $\lambda - p + 1$ times, the 1-factor g_{1-p} exactly $\lambda - 1$ times, and each of the 1-factors g_i, $3 \leq i \leq \lambda + p - 1$, $i \neq 1 - p$ exactly λ times (recall that we do not require simple factorizations). Note that no edge of the form $(i_L, (i+2)_R)$ has been included in any of these factors. Each edge incident with both the sets L and R is in exactly λ factors, except the edges of the form $(i_L, (i+1)_R)$, $(i_L, (i-p+1)_R)$, $(i_L, (i+2)_R)$ and (i_L, i_R). Call the edges of the form (i_L, i_R) the *horizontal* edges.

We next describe the factors f_k, $k = 0, \ldots, \lambda + p - 1$ which cover the horizontal edges and the remainder of the uncovered edges between L and R. Let f_0 consist of the edges $(i_L, (i+1)_R)$ for $0 \leq i \leq p - 2$, $((p-1)_L, 0_R)$ and all (j_L, j_R) with $p \leq j \leq \lambda + p - 1$. For $1 \leq i \leq \lambda + p - 1$ define f_i to be the ith cyclic shift of f_0. More formally, define $(a_L, b_R) + i$ to be the edge $((a+i)_L, (b+i)_R)$ (again, addition is mod $\lambda + p$). Then

$$f_i = f_0 + i = \{(a_L, b_R) + i \mid (a_L, b_R) \in f_0\}.$$

Let $f_i \in F$ for $0 \leq i \leq \lambda + p - 1$. Observe now that each edge (k_L, j_R) is in exactly λ 1-factors in F except the edges $(k_L, (k+2)_R)$ which still are not contained in any 1-factor.

Now we construct the 1-factors which complete F to a 1-factorization. Note that if λ is odd then $p = 2$ and so $\lambda + p$ is odd. If λ is even then p is necessarily odd and again $\lambda + p$ is odd. Thus the number of vertices in each of L and R is an odd number. Let $E(L)$ denote the set of all edges with both ends in L, i.e. edges of the form (i_L, j_L) where $0 \leq i, j \leq \lambda + p - 1$. Also let $E(R)$ be the set of all edges with both ends in R. When a graph has an odd number of vertices it is impossible for it to have a 1-factor and hence a 1-factorization. In this case we define a *near 1-factor* to be a collection of edges which are adjacent to each vertex (except one) exactly once. A *near 1-factorization* is a partitioning of the edges into near 1-factors. Let $G_L = \{g_0, g_1, \ldots, g_{\lambda+p-1}\}$ be a near 1-factorization of the edges in $E(L)$ where the vertex i is not in an edge in the near 1-factor g_i. Let $H_R = \{h_0, h_1, \ldots, h_{\lambda+p-1}\}$ be a near 1-factorization of the edges in $E(R)$ where the vertex i is not in an edge in the near 1-factor h_i. Construct the 1-factor

$$t_i = g_i \cup h_{i+2} \cup \{(i_L, (i+2)_R)\}.$$

Let F contain λ copies of each of the 1-factors t_i for $0 \leq i \leq \lambda + p - 1$. The description of F is now complete. It is not difficult to check that F is a 1-factorization of $\lambda K_{2(\lambda+p)}$.

We have left to prove that this factorization F is indecomposable. For this we will only need to use the factors f_i, $i = 0, \ldots, \lambda+p-1$. Suppose that we can partition the factors of F into two sets F_1 and F_2 such that each edge of the underlying complete graph appears λ_i times in F_i, $i = 1, 2$. Without loss of generality, suppose that f_0 lies in F_1. This factor contains the edge (p_L, p_R) but does not contain $((p-1)_L, (p-1)_R)$. Because these edges are to occur in F_1 with the same frequency, F_1 must have a factor which contains $((p-1)_L, (p-1)_R)$ but not (p_L, p_R). But the only factor such is f_p. Thus $f_0 \in F_1$ implies that $f_p \in F_1$. Now the same argument applies to f_p and implies that $f_{2p} \in F_1$. Following this process $\lambda + p$ times (since $\mathrm{GCD}(p, \lambda) = 1$) we have that $f_i \in F_1$ for all $i = 0, \ldots, \lambda + p - 1$. These factors cover the edge $(0_L, 0_R)$ λ times, so it follows that F is indecomposable. □

Corollary 1.2. *There exists a simple indecomposable* 1*-factorization of* λK_{2n} *for all* $2n \geq 4(\lambda + p)$, *where* $p < \lambda$ *is the smallest prime which does not divide* λ.

Proof. Colbourn, Colbourn and Rosa [1, Cor. 4] have shown that the existence of an IOF$(2n, \lambda)$ with $\lambda \leq 2n-1$ implies the existence of a *simple* indecomposable 1-factorization of λK_{2m} for all $m \geq 2n$. Corollary 1.2 is now immediate from Theorem 1.1. □

2. Starter constructions for small λ

From Corollary 1.2 we have that for a given λ, there exists a simple indecomposable 1-factorization of λK_{2n} for all but a finite number of values of n. In this section we will consider the specific cases when $\lambda \leq 12$. The motivation for this is the desire to fill in the gaps from the following theorem of Colbourn, Colbourn and Rosa.

Theorem 2.1 [1]. *A simple indecomposable* 1*-factorization of* λK_{2n} *exists as follows*:

$\lambda = 2$: *if and only if* $2n \geq 6$;
$\lambda = 3$: *if and only if* $2n \geq 8$;
$\lambda = 4$: *if and only if* $2n \geq 8$;
$\lambda = 5$: *if* $2n = 10, 12, 14$ *or* $2n \geq 20$;
$\lambda = 6$: *if* $2n \geq 12$;
$\lambda = 8$: *if* $2n = 12$ *or* 14 *or* $2n \geq 24$;
$\lambda = 9$: *if* $2n = 12$ *or* 14 *or* $2n \geq 24$;
$\lambda = 10$: *if* $2n = 14$ *or* $2n \geq 28$;
$\lambda = 12$: *if* $2n \geq 32$.

In order to directly construct the desired 1-factorizations, we introduce the concept of starters for 1-factorizations of λK_n. These will be termed λ-starters and are merely the analog of starters for K_n which are well known.

Let G be an additive abelian group of odd order n. A *λ-starter* in G (or order n) is a set $S = \{S_1, S_2, \ldots, S_\lambda\}$ where each S_i is a set of unordered pairs $S_i = \{\{s_{ij}, t_{ij}\} \mid 1 \leq j \leq \frac{1}{2}(n-1)\}$ and which satisfies the following properties:

(1) For each i, $\{s_{ij}\} \cup \{t_{ij}\} = G - \{0\}$,

(2) Each $g \in G - \{0\}$ occurs as a difference $\pm(s_{ij} - t_{ij})$, $1 \leq i \leq \lambda$, $1 \leq j \leq \frac{1}{2}(n-1)$, exactly λ times.

We give an example from [1].

Example 2.2. A 3-starter in the group Z_{11}.

$$S_1 = \{2, 3\}, \{4, 5\}, \{8, 9\}, \{6, 10\}, \{1, 7\},$$
$$S_2 = \{1, 7\}, \{2, 9\}, \{4, 6\}, \{3, 5\}, \{8, 10\},$$
$$S_3 = \{1, 4\}, \{2, 7\}, \{3, 10\}, \{5, 8\}, \{6, 9\}.$$

Note for the vertices labelled by $Z_{11} \cup \{\infty\}$ that $S_i \cup \{0, \infty\}$ is a 1-factor for each i. And for that matter, so is each translate $S_i + g = \{s_{ij} + g, t_{ij} + g\} \cup \{g, \infty\}$ for every $g \in G$. In general we get the following relationship between λ-starters and 1-factorizations of λK_n.

Theorem 2.3. *If there exists a λ-starter of order n then there exists a 1-factorization of λK_{n+1}.*

Proof. Let $S = \{S_1, S_2, \ldots, S_\lambda\}$ be a λ-starter of order n. As noted above each S_i generates n 1-factors $S_i + g$ of λK_{n+1}. Each pair $\{a, b\} \subset G$, occurs exactly λ times in these 1-factors by property (2) of the definition. The pair $\{\infty, g\}$ also occurs exactly λ times, as $\{\infty, g\} \in S_i + g$ for $1 \leq i \leq \lambda$. $\square$

Notice that if the S_i's are distinct then the resulting 1-factorization will be simple. Under certain conditions the 1-factorization generated by the λ-starter will also be indecomposable. One of those conditions is given in the following theorem. We will say that a difference $d \in G$ is *entirely contained* in S_i if all λ occurrences of pairs with difference d are in S_i. The following theorem was first given in [2] but we will also give it here for the sake of completeness.

Theorem 2.4. *Let $S = \{S_1, S_2, \ldots, S_\lambda\}$ be a λ-starter in the group G of order n. If there exists a difference $d \in G$ which is entirely contained in S_i for some i, and if* GCD$(\lambda, n) = 1$, *then S generates an indecomposable 1-factorization of λK_{n+1}.*

Proof. Assume difference d is entirely contained in S_i. Also assume that the resulting 1-factorization F of λK_{n+1} contains a 1-factorization F' of $\lambda_1 K_{n+1}$ where $\lambda_1 \leq \lambda$. The total number of pairs occurring in F' with difference d is $\lambda_1 \times n$. If t

translates of S_i are contained in F', then we get that the total number of pairs occurring in F' with difference d is $t \times \lambda$. Equating, we get $\lambda_1 \times n = t \times \lambda$. Since $\mathrm{GCD}(\lambda, n) = 1$ and since $\lambda_1 \leqslant \lambda$, we conclude that $\lambda_1 = \lambda$. Thus F is indecomposable. □

One can see that in Example 2.2 the difference 1 is entirely contained in S_1. Since $\mathrm{GCD}(3, 11) = 1$, we have by Theorem 2.4 that this starter generates a simple IOF(12, 3). If a λ-starter S generates an indecomposable 1-factorization we will say S is an *indecomposable* starter. The following is our main theorem on simple indecomposable 1-factorizations of λK_n with small λ.

Theorem 2.5. *A simple indecomposable* 1*-factorization of* λK_{2n} *exists as follows*:
$\lambda = 2$: *if and only if* $2n \geqslant 6$;
$\lambda = 3$: *if and only if* $2n \geqslant 8$;
$\lambda = 4$: *if and only if* $2n \geqslant 8$;
$\lambda = 5$: *if* $2n \geqslant 10$;
$\lambda = 6$: *if* $2n \geqslant 12$;
$\lambda = 7$: *if* $2n \geqslant 16$;
$\lambda = 8$: *if* $2n \geqslant 12$;
$\lambda = 9$: *if* $2n \geqslant 12$;
$\lambda = 10$: *if* $2n \geqslant 14$;
$\lambda = 11$: *if* $2n \geqslant 52$;
$\lambda = 12$: *if* $2n \geqslant 32$.

Proof. The result stated for $\lambda = 11$ is from Corollary 1.2. In view of Theorem 2.1 to complete the proof we need only construct simple indecomposable 1-factorizations of the following graphs: $5K_{16}$, $5K_{18}$, $8K_{16}$, $8K_{18}$, $8K_{20}$, $8K_{22}$, $9K_{16}$, $9K_{18}$, $9K_{20}$, $9K_{22}$, $10K_{16}$, $10K_{18}$, $10K_{20}$, $10K_{22}$, $10K_{24}$, $10K_{26}$ and $7K_{2n}$ for all $n \geqslant 8$. By the theorem of Colbourn, Colbourn and Rosa [1] mentioned in the proof of Corollary 1.2, if simple indecomposable 1-factorizations of the graphs $7K_{16}$, $7K_{18}$, $7K_{20}$, $7K_{22}$, $7K_{24}$, $7K_{26}$, $7K_{28}$, $7K_{30}$ can be constructed, then there exist simple IOF($2n$, 7) for all $n \geqslant 8$. In the Appendix we give indecomposable λ-starters which generate these desired 1-factorizations. All of these starters obviously generate simple 1-factorizations but to prove that they give indecomposable 1-factorizations requires some discussion.

By Theorem 2.4, the starters given in the appendix for the following graphs are indecomposable: $5K_{18}$, $7K_{16}$, $7K_{18}$, $7K_{20}$, $7K_{22}$, $7K_{24}$, $7K_{26}$, $7K_{28}$, $7K_{30}$, $8K_{16}$, $8K_{18}$, $8K_{20}$, $8K_{22}$, $9K_{20}$, $9K_{22}$, $10K_{22}$ and $10K_{24}$. Easy ad hoc arguments were used to show that the starters given in the appendix for the following graphs are indecomposable: $5K_{16}$, $7K_{22}$, $9K_{18}$, $10K_{18}$ and $10K_{20}$. Finally, a computer was required to decide that the starters given for $9K_{16}$, $10K_{16}$ and $10K_{26}$ are also indecomposable. □

Appendix

Indecomposable λ-starters of $5K_{16}$.
9,10 1,2 7,8 4,5 13,14 6,12 11,3
3,5 8,10 12,14 11,13 4,6 1,7 2,9
7,10 13,1 5,8 6,9 14,2 12,3 4,11
4,8 9,13 12,1 7,11 2,6 14,5 3,10
3,8 6,11 14,4 12,2 5,10 7,13 9,1

Indecomposable λ-starters of $5K_{18}$
6,7 9,10 15,16 2,3 13,14 5,11 1,8 4,12
10,12 4,6 14,16 13,15 5,7 3,9 1,8 11,2
5,8 13,16 7,10 11,14 3,6 15,4 12,2 1,9
7,11 10,14 1,5 12,16 2,6 3,9 8,15 13,4
15,3 1,6 7,12 5,10 9,14 13,2 4,11 8,16

Indecomposable λ-starters of $7K_{16}$.
7,8 3,4 11,12 9,10 13,14 1,2 5,6
14,1 7,9 3,5 6,8 10,12 11,13 2,4
1,5 3,6 14,2 10,13 9,12 4,7 8,11
1,4 7,11 6,10 5,9 13,2 14,3 8,12
9,14 1,6 5,10 7,12 3,8 13,4 11,2
7,13 14,5 10,1 2,8 12,3 6,11 4,9
3,10 5,12 6,13 4,11 2,9 7,14 1,8

Indecomposable λ-starters of $7K_{18}$.
1,2 13,14 10,11 6,7 4,5 8,9 15,16 12,3
16,1 13,15 3,5 8,10 12,14 4,6 7,9 11,2
3,7 13,16 5,8 12,15 6,9 11,14 1,4 2,10
15,1 5,9 16,3 8,12 2,6 10,14 7,11 13,4
4,9 14,2 3,8 10,15 13,1 6,12 5,11 16,7
10,16 9,15 5,11 6,12 1,7 3,8 14,2 13,4
8,15 11,1 5,12 16,6 2,9 3,10 7,14 13,4

Indecomposable λ-starters of $7K_{20}$.
14,15 12,13 17,18 4,5 7,8 10,11 2,3 1,9 16,6
1,3 16,18 5,7 4,6 13,15 12,14 9,11 2,10 8,17
4,8 17,1 7,10 18,2 12,15 13,16 6,9 3,11 5,14
15,18 12,16 10,14 1,5 9,13 4,8 2,6 3,11 17,7
7,12 15,1 13,18 5,10 11,16 2,8 3,9 17,6 14,4
6,12 16,3 17,4 14,1 9,15 8,13 5,10 18,7 2,11
11,18 7,14 1,8 3,10 6,13 17,5 9,16 15,4 12,2

Indecomposable λ-starters of $7K_{22}$.
12,13 5,6 10,11 14,15 19,20 3,4 1,2 9,17 7,16 8,18
18,20 14,16 17,19 9,11 4,6 8,10 3,5 7,15 13,1 2,12
5,9 14,17 1,4 13,16 15,18 3,6 8,11 20,7 10,19 2,12
10,13 11,15 5,9 20,3 4,8 19,2 12,16 14,1 18,6 7,17
11,16 5,10 20,4 1,6 12,17 13,19 3,9 7,15 14,2 8,18
14,20 5,11 7,13 9,15 6,12 17,1 19,3 2,10 16,4 8,18
15,1 7,14 6,13 18,4 5,12 9,16 17,3 2,10 11,20 19,8

Indecomposable λ-starters of $7K_{24}$.
7,8 1,2 18,19 11,12 13,14 21,22 3,4 9,17 6,15 10,20 5,16
17,19 2,4 6,8 11,13 1,3 18,20 10,12 14,22 7,16 5,15 21,9
3,7 9,12 15,18 17,20 10,13 5,8 19,22 21,6 2,11 14,1 16,4
17,20 18,22 7,11 10,14 9,13 1,5 21,2 8,16 3,12 19,6 4,15
1,6 8,13 9,14 20,2 12,17 16,22 4,10 11,19 21,7 18,5 15,3
22,5 3,9 14,20 12,18 21,4 2,7 10,15 11,19 8,17 6,16 13,1
14,21 22,6 8,15 2,9 11,18 3,10 12,19 16,1 4,13 20,7 17,5

Indecomposable λ-starters of $7K_{26}$.
12,13 2,3 5,6 7,8 22,23 20,21 14,15 9,17 10,19 16,1 18,4 24,11
8,10 7,9 2.4 11,13 19,21 18,20 1,3 14,22 15,24 6,16 12,23 5,17
22,1 24,2 4,7 5,8 16,19 15,18 11,14 9,17 12,21 3,13 20,6 23,10
7,10 19,23 2,6 5,9 18,22 11,15 20,24 8,16 12,21 4,14 17,3 1,13
14,19 12,17 2,7 15,20 6,11 23.4 24,5 1,9 13,22 8,18 10,21 16,3
2,8 1,7 9,15 4,10 5,11 16,21 18,23 12,20 13,22 14,24 17,3 19,6
8,15 13.20 21,3 24,6 11,18 23,5 7,14 1,9 10,19 12,22 16,2 17,4

Indecomposable λ-starters of $7K_{28}$.
7,8 3,4 11,12 19,20 1,2 14,15 23,24 17,25 9,18 6,16 21,5 10,22 13,26
20,22 16,18 10,12 6,8 15,17 26,1 11,13 23,4 21,3 24,7 25,9 2,14 19,5
12,16 1,4 11,14 22,25 20,23 15,18 3,6 2,10 26,8 7,17 13,24 9,21 19,5
19,22 1,5 9,13 3,7 20,24 17,21 4,8 10,18 6,15 16,26 12,23 2,14 25,11
3,8 7,12 13,18 1,6 21,26 19,25 11,17 23,4 15,24 10,20 5,16 2,14 9,22
8,14 22,1 6,12 25,4 20,26 11,16 10,15 24,5 9,18 13,23 19,3 17,2 21,7
5,12 4,11 26,6 16,23 17,24 15,22 13,20 2,10 25,7 8,18 19,3 9,21 1,14

Indecomposable λ-starters of $7K_{30}$.
22,23 14,15 1,2 8,9 24,25 5,6 11,12 19,27 17,26 10,20 7,18 21,4 3,16 28,13
4,6 26,28 3,5 16,18 21,23 10,12 17,19 14,22 11,20 27,8 25,7 1,13 2,15 24,9
16,20 9,12 1,4 23,26 7,10 11,14 18,21 27,6 22,2 24,5 8,19 13,25 15,28 3,17
16,19 13,17 6,10 27,2 21,25 22,26 4,8 15,23 5,14 1,11 7,18 20,3 28,12 24,9
9,14 13,18 2,7 12,17 3,8 20,26 22,28 15,23 1,10 24,5 16,27 21,4 6,19 11,25
11,17 22,28 15,21 8,14 20,26 2,7 1,6 24,3 9,18 13,23 5,16 27,10 12,25 19,4
11,18 26,4 28,6 9,16 13,20 17,24 1,8 15,23 27,7 21,2 3,14 10,22 12,25 5,19

Indecomposable λ-starters of $8K_{16}$.

7,8 3,4 11,12 9,10 13,14 1,2 5,6
13,14 9,11 1,3 5,7 2,4 10,12 6,8
10,12 3,5 14,2 8,11 4,7 13,1 6,9
10,13 11,14 1,4 8,12 2,6 5,9 3,7
7,11 4,8 6,10 1,5 9,14 13,3 12,2
4,9 2,7 5,10 11,1 13,3 6,12 8,14
5,11 13,4 2,8 6,12 3,9 10,1 7,14
6,13 3,10 7,14 5,12 2,9 4,11 1,8

Indecomposable λ-starters of $8K_{18}$.

7,8 11,12 15,16 1,2 9,10 13,14 3,4 5,6
6,8 5,7 2,4 1,3 14,16 10,12 13,15 9,11
16,2 11,14 5,8 9,12 4,7 3,6 10,13 15,1
10,14 2,6 1,5 12,16 4,8 11,15 9,13 3,7
7,12 11,16 14,2 4,9 5,10 1,6 8,13 15,3
3,9 10,16 1,7 6,12 13,2 15,4 8,14 5,11
7,14 15,5 9,16 6,13 12,2 4,11 3,10 1,8
6,14 3,11 8,16 4,12 2,10 7,15 1,9 5,13

Indecomposable λ-starters of $8K_{20}$.

17,18 2,3 8,9 15,16 11,12 13,14 4,5 6,7 1,10
7,9 16,18 6,8 1,3 10,12 2,4 11,13 15,17 5,14
8,11 3,6 13,16 1,4 7,10 14,17 9,12 18,2 15,5
11,15 16,1 6,10 5,9 17,2 14,18 3,7 8,12 4,13
11,16 13,18 4,9 12,17 2,7 1,6 3,8 10,15 5,14
10,16 3,9 17,4 5,11 14,1 7,13 2,8 12,18 6,15
8,15 13,1 4,11 14,2 5,12 18,6 9,16 3,10 17,7
3,11 7,15 16,5 4,12 1,9 13,2 6,14 10,18 8,17

Indecomposable λ-starters of $8K_{22}$.

6,7 4,5 2,3 14,15 12,13 18,19 16,17 9,10 20,8 1,11
8,10 12,14 17,19 2,4 11,13 1,3 7,9 18,20 6,15 16,5
17,20 15,18 1,4 16,19 3,6 9,12 8,11 7,10 5,14 13,2
11,15 1,5 8,12 13,17 3,7 19,2 6,10 16,20 9,18 4,14
3,8 5,10 15,20 17,1 4,9 7,12 11,16 14,19 18,6 13,2
14,20 19,4 10,16 6,12 18,3 9,15 7,13 2,8 17,5 1,11
5,12 16,2 18,4 8,15 7,14 10,17 19,6 1,9 11,20 3,13
19,6 12,20 18,5 7,15 8,16 3,11 2,9 10,17 13,1 4,14

Indecomposable λ-starters of $9K_{16}$.

7,8 3,4 11,12 9,10 13,14 1,2 5,6
10,11 12,13 2,4 6,8 3,5 14,1 7,9
12,14 11,13 1,3 8,10 2,5 4,7 6,9

5,8 9,12 3,6 1,4 14,2 10,13 7,11
12,1 13,2 7,11 5,9 14,3 4,8 6,10
9,13 14,4 1,6 5,10 3,8 7,12 11,2
9,14 8,13 1,6 2,7 12,3 4,10 5,11
1,7 11,2 8,14 4,10 3,9 6,13 5,12
5,12 1,8 2,9 3,10 6,13 4,11 7,14

Indecomposable λ-starters of $9K_{18}$.
7,8 11,12 15,16 1,2 9,10 13,14 3,4 5,6
14,15 11,13 6,8 2,4 7,9 10,12 16,1 3,5
9,11 12,14 16,2 3,6 15,1 5,8 10,13 4,7
9,12 11,14 13,16 15,2 4,8 1,5 6,10 3,7
9,13 4,8 14,1 2,6 5,10 11,16 7,12 15,3
5,10 2,7 8,13 6,11 16,4 12,1 14,3 9,15
5,11 14,3 4,10 12,1 9,15 7,13 2,8 16,6
8,15 3,10 14,4 11,1 6,13 2,9 5,12 16,7
7,15 4,12 2,10 6,14 8,16 3,11 1,9 5,13

Indecomposable λ-starters of $9K_{20}$.
5,6 1,2 15,16 17,18 13,14 3,4 7,8 11,12 9,10
18,1 2,4 7,9 6,8 11,13 10,12 3,5 15,17 14,16
2,5 1,4 3,6 7,10 15,18 14,17 9,12 13,16 8,11
9,13 14,18 17,2 1,5 11,15 3,7 12,16 4,8 6,10
5,10 15,1 9,14 8,13 17,3 18,4 6,11 16,2 7,12
16,3 2,8 9,15 7,13 14,1 4,10 11,17 6,12 18,5
6,13 7,14 2,9 15,3 10,17 11,18 5,12 1,8 16,4
3,11 12,1 6,14 15,4 2,10 8,16 9,17 5,13 18,7
8,17 3,12 7,16 1,10 4,13 5,14 9,18 6,15 2,11

Indecomposable λ-starters of $9K_{22}$.
5,6 3,4 1,2 8,9 19,20 16,17 10,11 14,15 12,13 18,7
20,1 17,19 9,11 13,15 10,12 6,8 16,18 2,4 5,7 14,3
7,10 15,18 17,20 11,14 5,8 1,4 3,6 16,19 9,12 13,2
3,7 11,15 14,18 4,8 13,17 5,9 16,20 6,10 19,2 12,1
7,12 18,2 10,15 9,14 19,3 17,1 6,11 8,13 20,4 16,5
9,15 6,12 7,13 19,4 20,5 18,3 17,2 10,16 8,14 1,11
20,6 7,14 11,18 19,5 8,15 10,17 16,2 1,9 4,12 3,13
7,15 8,16 18,5 6,14 1,9 17,4 11,19 3,10 13,20 2,12
5,14 8,17 6,15 13,1 2,11 9,18 3,12 19,7 16,4 10,20

Indecomposable λ-starters of $10K_{16}$.
7,8 3,4 11,12 9,10 13,14 1,2 5,6
5,6 7,8 13,14 2,4 9,11 10,12 1,3

11,13 12,14 1,3 8,10 2,4 5,7 6,9
9,12 13,1 8,11 3,6 4,7 2,5 10,14
3,6 2,5 1,4 9,13 7,11 8,12 10,14
1,5 10,14 9,13 3,7 4,8 12,2 6,11
9,14 11,1 2,7 5,10 3,8 13,4 6,12
8,13 14,4 5,10 6,12 11,2 1,7 3,9
4,10 6,12 7,13 14,5 11,3 1,8 2,9
2,9 3,10 7,14 1,8 4,11 5,12 6,13

Indecomposable λ-starters of $10K_{18}$.
7,8 11,12 15,16 1,2 9,10 13,14 3,4 5,6
3,4 1,2 14,16 5,7 6,8 13,15 10,12 9,11
12,15 14,16 1,3 4,6 8,11 9,12 7,10 2,5
16,2 3,6 4,7 11,15 9,13 8,12 1,5 10,14
16,3 7,11 9,13 2,6 10,14 5,8 1,4 12,15
6,11 16,4 7,12 8,13 5,10 15,3 14,2 1,9
15,3 9,14 6,11 16,5 2,8 4,10 7,13 12,1
8,14 12,1 15,4 5,11 7,13 3,10 2,9 16,6
5,12 8,15 11,1 6,13 3,10 2,9 14,4 16,7
5,13 4,12 7,15 8,16 3,11 2,10 1,9 6,14

Indecomposable λ-starters of $10K_{20}$.
5,6 1,2 15,16 17,18 13,14 3,4 7,8 11,12 9,10
9,10 15,17 2,4 5,7 6,8 16,18 12,14 1,3 11,13
9,11 14,16 17,1 10,13 12,15 3,6 5,8 18,2 4,7
4,7 13,16 6,9 10,14 18,3 8,12 1,5 17,2 11,15
17,2 1,5 3,7 12,16 18,4 6,11 10,15 9,14 8,13
4,9 17,3 1,6 10,15 16,2 7,13 8,14 5,11 12,18
10,16 3,9 1,7 17,4 12,18 5,11 6,13 14,2 8,15
5,12 2,9 16,4 6,13 11,18 10,17 1,8 7,15 14,3
17,6 7,15 4,12 13,2 3,11 1,9 8,16 10,18 5,14
6,15 2,11 3,12 5,14 7,16 8,17 1,10 4,13 9,18

Indecomposable λ-starters of $10K_{22}$.
3,4 11,12 17,18 19,20 15,16 7,8 9,10 5,6 1,2 13,14
17,19 2,4 5,7 18,20 9,11 1,3 6,8 10,12 13,15 14,16
6,9 10,13 17,20 1,4 16,19 15,18 2,5 11,14 3,7 8,12
20,3 12,16 19,2 1,5 7,11 6,10 4,8 9,13 14,17 15,18
10,15 14,19 17,1 13,18 3,8 4,9 11,16 2,7 20,5 6,12
4,10 11,17 7,13 20,5 6,12 16,1 9,15 8,14 18,2 19,3
8,15 6,13 17,3 4,11 16,2 19,5 7,14 10,18 12,20 1,9
19,6 15,2 1,9 17,4 8,16 10,18 3,11 7,14 13,20 5,12
16,4 19,7 6,15 9,18 11,20 1,10 8,17 5,14 3,12 13,2
10,20 1,11 3,13 5,15 9,19 4,14 17,6 8,18 2,12 7,16

Indecomposable λ-starters of $10K_{24}$.

8,9 10,11 5,6 16,17 3,4 14,15 12,13 19,20 21,22 1,2 7,18
16,18 12,14 2,4 13,15 9,11 17,19 6,8 20,22 1,3 5,7 10,21
8,11 17,20 13,16 6,9 7,10 19,22 1,4 12,15 2,5 18,21 3,14
6,10 4,8 7,11 1,5 15,19 13,17 22,3 12,16 14,18 21,2 9,20
16,21 17,22 6,11 7,12 4,9 13,18 5,10 19,1 3,8 15,20 14,2
3,9 1,7 16,22 13,19 5,11 4,10 2,8 14,20 12,18 15,21 6,17
7,14 3,10 4,11 12,19 15,22 9,16 1,8 13,21 20,5 18,2 6,17
20,5 8,16 2,10 18,3 7,15 13,21 14,22 4,11 12,19 1,9 6,17
21,7 9,18 8,17 10,19 5,14 2,11 3,12 15,1 20,6 13,22 16,4
6,16 10,20 5,15 14,1 17,4 8,18 2,12 22,9 11,21 3,13 19,7

Indecomposable λ-starters of $10K_{26}$.

11,12 4,5 2,3 6,7 20,21 22,23 16,17 14,15 8,9 18,19 24,10 1,13
24,1 7,9 20,22 17,19 6,8 11,13 21,23 16,18 3,5 10,12 4,15 2,14
7,10 17,20 11,14 15,18 3,6 19,22 9,12 23,1 21,24 5,8 2,13 4,16
16,20 2,6 18,22 24,3 19,23 13,17 1,5 8,12 10,14 7,11 4,15 9,21
5,10 19,24 21,1 18,23 9,14 7,12 6,11 22,2 15,20 8,13 17,3 4,16
3,9 11,17 24,5 13,19 16,22 6,12 1,7 15,21 2,8 14,20 18,4 23,10
11,18 3,10 7,14 17,24 19,1 16,23 2,9 5,13 4,12 15,22 20,6 21,8
2,10 6,14 20,3 13,21 16,24 9,17 22,5 8,15 4,11 18,1 12,23 7,19
11,20 15,24 7,16 4,13 21,5 8,17 1,10 19,3 22,6 9,18 12,23 2,14
17,2 16,1 23,8 10,20 12,22 5,15 21,6 4,14 9,19 3,13 7,18 24,11

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