There are 526,915,620 nonisomorphic one-factorizations of K_{12}

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Abstract: We enumerate the nonisomorphic and the distinct one-factorizations of K_{12}. We also describe the algorithm used to obtain the result, and the methods we used to verify these numbers.

1 Introduction

We begin with some definitions. A *one-factor* in a graph G is a set of edges in which every vertex appears precisely once. A *one-factorization* of G is a partition of the edge-set of G into one-factors. (We will sometimes refer to a one-factorization as an *OF*). Two one-factorizations F and H of G, say $F = \{f_1, f_2, \ldots, f_k\}$, $H = \{h_1, h_2, \ldots, h_k\}$, are called *isomorphic* if there exists a map ϕ from the vertex-set of G onto itself such that $\{f_1\phi, f_2\phi, \ldots, f_k\phi\} = \{h_1, h_2, \ldots, h_k\}$. Here $f_i\phi$ is the set of all the edges $\{x\phi, y\phi\}$ where $\{x, y\}$ is an edge in F. Obviously, if the complete graph on n vertices K_n has a one-factorization, then necessarily n is even and any such one-factorization contains $n-1$ one-factors each of which contains $n/2$ edges. Figure 1 shows an *OF* of K_{12}. Each of the rows is a one-factor. The *OF* in Figure 1 is the first *OF* of K_{12} under the lexicographical ordering described in Section 2; the order of its automorphism group is 240. There have been several excellent survey papers on one-factorizations and the interested reader is referred to [22], [18], and [12].

The exact number of nonisomorphic one-factorizations of K_{2n} has been known only for $2n \leq 10$. It is easy to see that there is a unique one-factorization of K_2, K_4, and K_6. There are exactly six for K_8; these were found by Dickson and Safford [4] and a full exposition is given in [23]. In 1973, Gelling [8, 9] proved that there are exactly 396 isomorphism classes of *OFs* of K_{10}. In both of these searches, the orders of the automorphism groups of the factorizations were also found. This information can be used to calculate the exact number of distinct factorizations.

It is also known that the number of nonisomorphic one-factorizations of K_n goes

$$\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}, \{10, 11\}$$
$$\{0, 2\}, \{1, 3\}, \{4, 6\}, \{5, 7\}, \{8, 10\}, \{9, 11\}$$
$$\{0, 3\}, \{1, 2\}, \{4, 7\}, \{5, 6\}, \{8, 11\}, \{9, 10\}$$
$$\{0, 4\}, \{1, 5\}, \{2, 8\}, \{3, 9\}, \{6, 10\}, \{7, 11\}$$
$$\{0, 5\}, \{1, 4\}, \{2, 9\}, \{3, 8\}, \{6, 11\}, \{7, 10\}$$
$$\{0, 6\}, \{1, 7\}, \{2, 10\}, \{3, 11\}, \{4, 8\}, \{5, 9\}$$
$$\{0, 7\}, \{1, 6\}, \{2, 11\}, \{3, 10\}, \{4, 9\}, \{5, 8\}$$
$$\{0, 8\}, \{1, 9\}, \{2, 6\}, \{3, 7\}, \{4, 10\}, \{5, 11\}$$
$$\{0, 9\}, \{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 11\}, \{5, 10\}$$
$$\{0, 10\}, \{1, 11\}, \{2, 4\}, \{3, 5\}, \{6, 8\}, \{7, 9\}$$
$$\{0, 11\}, \{1, 10\}, \{2, 5\}, \{3, 4\}, \{6, 9\}, \{7, 8\}$$

Figure 1: The first one-factorization of K_{12}

to infinity as n goes to infinity [1, 11]. In fact, if we let $N(n)$ denote the number of nonisomorphic one-factorizations of K_n, then $\ln N(2n) \sim 2n^2 \ln 2n$, as proved by Cameron [3]. Feeling that the complete enumeration of the nonisomorphic *OFs* of K_{12} could "not be determined in a reasonable amount of time", Seah and Stinson [17, 21] restricted their search to finding one-factorizations of K_{12} with nontrivial automorphism group. They found that there are exactly 56,391 nonisomorphic one-factorizations of K_{12} with nontrivial automorphism groups (excluding those whose automorphism group is of order 2 and consists of six 2-cycles). In this paper we present the results of our search for the total number of nonisomorphic one-factorizations of K_{12}. We corroborate the Seah-Stinson number, as well as determine the remaining number of nonisomorphic one-factorizations of K_{12} which they did not count.

This problem was appealing to us as it represents a good example of the so-called *combinatorial explosion*. In [7] we estimated that we would find about 2 billion nonisomorphic one-factorizations of K_{12}. We also believed that it would take more than 200 MIPS-years of CPU time (200 years on a computer running at 1 MIPS) to perform the complete enumeration. The computation would have been impractical if it were not for the fact that our algorithm can be run in parallel on many different processors. The entire computation required a little over 160 MIPS-years, but we were able to complete the computation in less than eight months by distributing parts of the problem to workstations that run at rates of 12 to 50 mips. We will have more to say about this later in this paper.

This paper is organized as follows: Section 2 describes the orderly algorithm that we used in the search, Section 3 contains a discussion of our correctness checks for this algorithm, and Section 4 contains our results.

2 The Algorithm

The algorithm that was used is an example of what is called an *orderly* algorithm; it generates the nonisomorphic *OFs* of K_{12} in lexicographic order. The algorithm builds up each one-factorization by adding one one-factor at a time and rejects a partial one-factorization if it is not the lowest representative (lexicographically) of all the partial one-factorizations in its isomorphism class. In this way, the algorithm generates only the lowest representative of any isomorphism class of one-factorizations and as such never generates any *OFs* which are isomorphic to each other. This approach saves both time and space over algorithms which first generate distinct (but possibly isomorphic) one-factorizations and then use methods to winnow isomorphs.

This type of algorithm has been used in other combinatorial searches including enumerating Latin squares [2, 16], strong starters [10], one-factorizations of small graphs [19], perfect one-factorizations of K_{14} [20], holey factorizations [5] and Howell designs of small order [19]. A systematic treatment of this method appears in [6]. Our algorithm below is essentially the one that was used by Seah and Stinson to find the nonisomorphic *OFs* of K_{10} and to find the nonisomorphic one-factorizations of K_{12} with nontrivial automorphism group [21].

We first give the lexicographic ordering. Suppose that the vertices of K_{12} are numbered $0, 1, \ldots, 11$. An edge e will be written as an ordered pair (x, x') with $0 \leq x < x' \leq 11$. For any two edges $e_1 = (x_1, x_1')$ and $e_2 = (x_2, x_2')$, say $e_1 < e_2$ if either $x_1 < x_2$ or $x_1 = x_2$ and $x_1' < x_2'$. A one-factor f is written as a set of ordered edges, i.e. $f = (e_1, e_2, e_3, e_4, e_5, e_6)$ where $e_i < e_j$ whenever $i < j$. For two one-factors $f_i = (e_{i1}, e_{i2}, \ldots, e_{i6})$ and $f_j = (e_{j1}, e_{j2}, \ldots, e_{j6})$, we say $f_i < f_j$ if there exists a k $(1 \leq k \leq 6)$ such that $e_{il} = e_{jl}$ for all $l < k$, and $e_{ik} < e_{jk}$.

A one-factorization F of K_{12} is written as an ordered set of 11 one-factors, i.e. $F = (f_1, f_2, \ldots, f_{11})$, where $f_i < f_j$ whenever $i < j$. The example in Figure 1 is written in this lexicographic order.

We use F and G to denote one-factorizations and f_i and g_i to denote one-factors contained in F and G, respectively. An ordering for one-factorizations is defined as follows. For two *OFs* F and G, we say that $F < G$ if there exists some i, $1 \leq i \leq 11$, such that $f_i < g_i$, and $f_j = g_j$ for all $j < i$.

For $1 \leq i \leq 11$, $F_i = (f_1, f_2, \ldots, f_i)$ will denote a *partial OF* consisting of an ordered set of i edge-disjoint one-factors. We say that i is the *rank* of the partial one-factorization. Note that $F_{11} = F$, a (complete) one-factorization. We can also extend our ordering to partial *OFs* of rank i, in an analogous manner.

We say a partial *OF* $F_i = (f_1, f_2, \ldots, f_i)$ of rank i is *proper* if f_j contains edge $(0, j)$ for $1 \leq j \leq i$. If F_i is not proper, then it is *improper*. A complete one-factorization is necessarily proper.

The automorphism group of the complete graph K_{12} is S_{12}, the symmetric group on 12 elements. Thus given a proper partial *OF* F_i (of rank i), we can rename the 12 points using a permutation $\alpha \in S_{12}$, and obtain another partial *OF* (not necessarily proper) of the same graph, denoted F_i^α. We say F_i is *canonical* if $F_i \leq F_i^\alpha$ for all permutations $\alpha \in S_{12}$. Thus, each canonical partial *OF* F_i is the lexicographically lowest representative of its isomorphism class. The following theorems on canonicity are from Seah [17].

Theorem 2.1 *If two proper partial OFs of rank i, F_i and G_i, are distinct and are both canonical, then F_i and G_i are nonisomorphic.*

Theorem 2.2 *If a partial proper one-factorization $F_i = (f_1, f_2, \ldots, f_i)$ is canonical, and $1 \leq j \leq i$ then $F_j = (f_1, f_2, \ldots, f_j)$ is also canonical.*

Theorem 2.3 *If a partial proper one-factorization $F_i = (f_1, f_2, \ldots, f_i)$ is not canonical, then any complete OF extended from F_i is also not canonical.*

Note that one can form a rooted tree in which each node represents one of the partial proper canonical *OFs* of K_{12}. The root represents the unique canonical F_1 which consists of the following one-factor, f_a:

$$f_a = \{(0,1), (2,3), (4,5), (6,7), (8,9), (10,11)\}$$

If a node v represents F_i, then the children of v represent each of the F_{i+1} which are proper canonical extensions of F_i. The nodes at level 11 of the tree represent the canonical *OFs* of K_{12}.

We can now describe the orderly algorithm that we use to construct canonical (non-isomorphic) *OFs* of the complete graph K_{12}; it is based on a depth-first traversal of the tree. The following recursive pseudo-coded procedure describes how to generate, from a given canonical F_i, all of the canonical F_{i+1} extending F_i, for $0 \leq i \leq 10$. Let F_0 be the partial *OF* of rank 0 (an empty set), and note that $F_0^\alpha = F_0$ for all $\alpha \in S_{12}$. We invoke the procedure using **Generate**$(F_0, 0)$.

procedure **Generate**(F_i, i):
    if $i = 11$ then
        F_i is a canonical OF
    else
(1)   for each f, containing $(0, i+1)$, disjoint from each 1-factor in F_i do
(2)   for each permutation α do
(3)   if $F_i^\alpha \bigcup \{f^\alpha\} < F_i \bigcup \{f\}$ then
                    $F_i \bigcup \{f\}$ is not canonical, discard it and go on to next f
                endif
            endfor
(4)   Generate$(F_i \bigcup \{f\}, i+1)$
        endfor
    endif

Statement (4) is reached if, and only if, $F_i^\alpha \bigcup \{f^\alpha\} \geq F_i \bigcup \{f\}$ for all α. Thus, the recursive call to Generate is made precisely when $F_i \bigcup \{f\}$ is canonical and proper.

There are several opportunities for improving the efficiency of the algorithm. We first note that the loop controlled by statement (1) potentially has $3 \cdot 5 \cdot 7 \cdot 9 = 945$ one-factors f to test as candidates for extensions of F_i. However, backtracking for each set of edges that comprise a one-factor disjoint from F_i reduces the number of one-factors that need to be considered.

As noted above, for all canonical $F_i = \{f_1, f_2, \ldots, f_i\}$, $i \geq 1$, $f_1 = f_a$. Since the union of two disjoint one-factors is a union of disjoint cycles of even length, then for any one-factor f which is edge disjoint from f_a, $\{f\} \bigcup \{f_a\}$ will form a graph isomorphic to either three disjoint 4-cycles; a 4-cycle and an 8-cycle; two 6-cycles; or a single 12-cycle. Thus, each F_2 is in one of four isomorphism classes. The following are the four one-factors which, when unioned with f_a, yield in turn each of the four canonical rank 2 one-factorizations of K_{12}.

1. $\{(0,2),(1,3),(4,6),(5,7),(8,10),(9,11)\} \bigcup \{f_a\}$ forms three disjoint 4-cycles
2. $\{(0,2),(1,3),(4,6),(5,8),(7,10),(9,11)\} \bigcup \{f_a\}$ forms a 4-cycle and an 8-cycle
3. $\{(0,2),(1,4),(3,5),(6,8),(7,10),(9,11)\} \bigcup \{f_a\}$ forms two disjoint 6-cycles
4. $\{(0,2),(1,4),(3,6),(5,8),(7,10),(9,11)\} \bigcup \{f_a\}$ forms a 12-cycle

We refer to the isomorphism class of a pair of one-factors as their *cycle structure*, and label these classes *type 1*, *type 2*, *type 3*, and *type 4* respectively. We further note that this ordering of the types is the same as the lexicographic ordering of the canonical representatives of the types. We extend the definition of *type* to apply to all canonical partial *OFs*. For all canonical $F_i = \{f_1, f_2, \ldots, f_i\}$, $i \geq 2$, $\{f_1\} \bigcup \{f_2\}$ is one of the four canonical rank 2 one-factorizations of K_{12}; we define the *type* of F_i to be the type of $\{f_1\} \bigcup \{f_2\}$. We note that all canonical rank i one-factorizations of K_{12} which have type s lexicographically precede all canonical rank i one-factorizations of K_{12} which have type t, for $s < t$.

Suppose we wish to consider extending some proper canonical F_i, $2 \leq i \leq 10$, by adding one-factor f. Further, assume that F_i has type t, $2 \leq t \leq 4$. Let g be a one-factor in F_i such that the type of $\{f\} \bigcup \{g\}$ (call it s) is minimal. If $s < t$, then $F_i \bigcup \{f\}$ is not canonical because there exists a permutation α that maps $F_i \bigcup \{f\}$ to a canonical rank i *OF* of type s. This observation leads to the following improvement of the algorithm. If F_i has type t, and f, at statement (1), forms a type s cycle structure with some one-factor in F_i such that $s < t$, then f can be discarded as a candidate for extending F_i to a proper canonical F_{i+1}.

The classification scheme permits an additional optimization of the algorithm. At statement (2) of the algorithm, α is chosen from the 12! elements of S_{12}. However, the algorithm only needs to consider those permutations which might map $F_i \bigcup \{f\}$ into a lexicographically lower isomorph. Thus, if F_i is of type t, then the only permutations which need to be considered are those which map some pair of one-factors in $F_i \bigcup \{f\}$ onto the canonical rank 2 factorization of type t.

Improvements based on the types of partial factorizations were used in [17] and [19]. Our implementation of the algorithm also uses *dynamic programming* techniques, saving information from the generation of permutations at rank i to speed up the generation of the permutations at rank $i+1$. In particular, we maintain a stack of the $\binom{i}{2}$ pairs of factors in the current F_i, where each pair is stored as the set of cycles formed by the union. When factor f is added to F_i, we push onto the stack the i unions of f and f_j where $1 \leq j \leq i$. The desired permutations are generated by traversing the cycles.

The algorithm outlined above can easily be modified for certain classes of *OFs* that are of interest. Indeed, it has been modified to find *perfect* one-factorizations of K_{12} and K_{14} [17, 20], and to find so-called *holey* factorizations of K_n for $n \leq 10$ [5].

3 Results

The essential feature of the algorithm is that, given any partial one-factorization F, it attempts to generate a lexicographically lower member of the isomorphism class of F.

Thus, a search for complete canonical *OFs* can proceed independently from any proper canonical partial *OF*. We do not need to store the one-factorizations that are constructed (we do count them and store information about some of them) and we do not need to construct the one-factorizations in order. This allows us to work on many processors that do not even need to communicate with each other. Thus, in less than eight months we were able to obtain the 8.15 years of cpu time (at 20 mips) that were required to compute the following result.

Theorem 3.1 *There are* 526, 915, 620 *nonisomorphic one-factorizations of* K_{12}.

For each complete one-factorization that we generated, we recorded the size of its automorphism group. Seah and Stinson [21] counted the number of nonisomorphic one-factorizations of K_{12} with nontrivial automorphism groups, with the exception of those whose automorphism group is of order 2 and consists of six 2-cycles. Our final count (in Table 1) is consistent with their results. By subtracting the number of one-factorizations they found with automorphism groups of order 2 from our count, one can determine that there are $437,436 - 39,706 = 397,730$ nonisomorphic one-factorizations of K_{12} whose automorphism group is of order 2, and consists of six 2-cycles.

Aut	1	2	3	4	5	6	8	10
n	526,461,499	437,436	669	14,801	92	245	610	10

Aut	11	12	16	20	24	32	48	55	110	240	660
n	2	138	76	2	25	4	6	1	1	2	1

Table 1: The number of one-factorizations with each automorphism group order

Label the nonisomorphic one-factorizations as $C_i, 1 \leq i \leq 526,915,620$, then use Burnside's Lemma to compute the number of distinct one-factorizations of K_{12} as

$$\sum_{i=1}^{526,915,620} \frac{12!}{|Aut(C_i)|}$$

This yields the following result.

Theorem 3.2 *There are* 252, 282, 619, 805, 368, 320 *distinct OFs of* K_{12}.

Let $C^i = \{C_1^i, C_2^i, \ldots\}$ be the lexicographically ordered set of all proper canonical partial one-factorizations with i levels. As discussed in Section 2, there are four types of *OFs* based on the cycle structure of the first pair of one-factors in an *OF*. These correspond to the four elements of $\mathcal{C}^2$; by the definition of *type*, C_i^2 is of type i. The edges of $\mathcal{C}_4^2$ form a 12-cycle. If a partial *OF*, F_i, of type 4 contained a pair of factors with a cycle structure of type $t < 4$, then F_i could be mapped into F_i' of type t. So, *every* pair of one-factors in a canonical type 4 one-factorization of K_{12} has a type 4 cycle structure. The complete *OFs* of type 4 are called *perfect* (wherein the union of any pair of one-factors is a hamilton circuit of the complete graph). Petrenyuk and Petrenyuk [15] found that there are five perfect one-factorizations of K_{12}, and our results concur.

We also corroborate that there is a unique OF that is type 3 uniform; that is, every pair of one-factors forms a pair of disjoint 6-cycles [3]. It is the unique OF that derives from C^3_{41}. Cameron showed that there are neither type 1 uniform nor type 2 uniform OFs of K_{12}. Again, our results concur. In fact, Cameron showed that there exists a OF in which each pair of one-factors forms a union of disjoint 4-cycles if, and only if, n is a power of 2 [3]. The unique type 3 uniform OF, and the five perfect one-factorizations, are listed in the Appendix.

Because of the tight constraint on the cycle structures, the search for all type 4 canonical OFs is fast. We proceeded directly from $\mathcal{C}^2_4$ to find all complete canonical OFs of type 4 in about thirty minutes running at a rate of 20 mips. For similar reasons, it is tractable to conduct the search for all type 3 OFs directly from $\mathcal{C}^2_3$; this required about thirty-five hours running at 20 mips. However, the size of the problem makes it impractical to proceed directly from $\mathcal{C}^2_1$ or $\mathcal{C}^2_2$. In these cases we start the search independently from each of their proper rank three descendants in $\mathcal{C}^3$.

In Table 2 we show the numbers of partial proper canonical one-factorizations derived from each of the four elements in $\mathcal{C}^2$; the rank 11 OFs are the complete one-factorizations. In Tables 3 through 6 we list the number of proper canonical OFs at each rank for each of the rank three descendants of the four elements of $\mathcal{C}^2$ respectively. Note that from the second column of Table 2, there will be 13 rows in Table 3 numbered 1 to 13, 19 rows in Table 4 numbered 14 to 32, 20 rows in Table 5 numbered 33 to 52, and 24 rows in Table 6 numbered 53 to 76. The times in cpu hours are based on a rate of 20 mips.

	Rank									cpu
$\mathcal{C}^2_i$	3	4	5	6	7	8	9	10	11	hrs
1	13	1281	90035	3227652	50861347	334401809	764368898	486360795	343101895	52575
2	19	2661	183345	5198549	68042364	391198201	696072421	344841630	183813540	18750
3	20	771	14972	123762	371692	353455	76933	3796	180	35
4	24	395	2679	10987	13791	3491	209	6	5	.5
tot	76	5108	291031	8560950	119289194	725956956	1460518461	831206227	526915620	71360

Table 2: Numbers of proper partial canonical OFs derived from $\mathcal{C}^2$

$\mathcal{C}_i^3$	Rank								cpu hrs
	4	5	6	7	8	9	10	11	
1	4	152	4355	55414	437492	1114428	794716	565834	225
2	71	4098	141279	2387510	20203917	56986186	44333912	37629746	8335
3	21	994	29144	503796	3954919	9976612	7387669	5821099	2345
4	43	2100	75117	1372820	10464057	26645253	19673322	15676934	3600
5	44	2261	75018	1286869	10186389	26384426	19536258	15260750	2065
6	68	3672	133437	2431214	19110006	49479259	35697642	27570997	4950
7	179	15484	748238	15500063	117568281	298820818	205099284	150517505	18700
8	150	11757	469128	5765152	43359194	98826193	57992703	40743607	4470
9	119	7250	231260	4109878	26190683	63400098	38920197	27578167	3560
10	167	13495	490676	7641861	34880702	64484208	31148861	14194448	1905
11	156	12224	410426	5676581	35369247	55084008	22589908	7117166	1990
12	134	8562	219920	2322418	7939530	8778320	2433531	384149	260
13	125	7986	199654	1807771	4737392	4389089	752792	41493	170
tot	1281	90035	3227652	50861347	334401809	764368898	486360795	343101895	52575

Table 3: Numbers of type 1 proper partial canonical *OFs*

$\mathcal{C}_i^3$	Rank								cpu hrs
	4	5	6	7	8	9	10	11	
14	86	4267	160048	2776011	20914216	50257223	35083672	25704526	1815
15	70	3834	107585	1666752	13517003	31122923	20266304	14572837	880
16	171	13943	591866	10930471	75704279	175092822	104977202	62858850	5860
17	195	19943	943364	18867458	141289174	287529951	146106090	72109582	6820
18	175	16429	622742	9484427	54216317	73897948	22003950	5821093	1225
19	153	13030	450220	6011875	28669360	31937747	7374945	1433411	710
20	172	14594	412017	4799778	19804695	21561259	5002775	845086	605
21	187	15577	533689	6778202	24424413	19113862	3523357	435112	615
22	177	13362	361466	1800072	4903376	2572238	284957	21109	80
23	168	12261	310962	1385055	3168130	1375432	129127	7572	50
24	171	12457	243865	1785258	2992324	1214618	73828	3890	35
25	122	6416	81064	264748	267286	55672	1761	10	10
26	156	9278	160040	806027	838355	249353	10759	400	15
27	98	4395	37341	105276	54886	7719	279	0	2.5
28	144	6864	55503	198465	144133	33190	1184	27	10
29	135	6777	56612	201339	173118	35119	1047	32	10
30	90	3364	19099	35230	22504	1464	19	0	.5
31	134	5592	48224	142467	93734	13850	374	3	5
32	57	962	2842	3453	898	31	0	0	.05
tot	2661	183345	5198549	68042364	391198201	696072421	344841630	183813540	18750

Table 4: Numbers of type 2 proper partial canonical *OFs*

$\mathcal{C}_i^3$	Rank								cpu hrs
	4	5	6	7	8	9	10	11	
33	21	392	3224	12477	15444	4345	292	16	2
34	7	124	1285	5955	5771	1653	99	10	2
35	84	3305	44570	169808	203416	49778	2743	119	25
36	71	1484	15484	45006	36575	8021	274	16	2
37	72	1615	16408	45949	37796	5774	183	7	1.2
38	71	1584	11417	32342	23881	4239	121	4	1.2
39	70	1473	8833	22668	14659	1921	69	4	.6
40	71	1399	9732	21483	10069	849	12	3	.52
41	33	258	626	309	107	6	1	1	.04
42	50	686	1545	1325	404	22	0	0	.07
43	44	546	1954	2279	724	41	1	0	.07
44	32	365	645	658	224	12	0	0	.02
45	39	554	3650	6229	2626	171	0	0	.06
46	28	344	1947	2733	1157	76	1	0	.03
47	19	219	771	833	285	14	0	0	.01
48	20	231	683	714	153	6	0	0	.01
49	14	161	383	409	95	2	0	0	$< .01$
50	10	108	304	233	37	3	0	0	$< .01$
51	12	104	268	276	31	0	0	0	$< .01$
52	3	20	33	6	1	0	0	0	$< .01$
tot	771	14972	123762	371692	353455	76933	3796	180	35

Table 5: Numbers of type 3 proper partial canonical *OFs*

$\mathcal{C}_i^3$	Rank								cpu hrs
	4	5	6	7	8	9	10	11	
53	52	478	2855	5008	1300	78	3	3	.14
54	60	1058	5942	7993	2145	130	2	1	.23
55	48	290	994	561	38	0	0	0	.03
56	34	318	707	155	6	0	0	0	.02
57	30	184	317	62	1	0	0	0	.01
58	13	25	9	0	0	0	0	0	$< .01$
59	31	156	110	8	0	0	0	0	$< .01$
60	25	83	43	3	0	0	0	0	$< .01$
61	24	40	6	0	0	0	0	0	$< .01$
62	20	21	3	1	1	1	1	1	$< .01$
63	17	12	1	0	0	0	0	0	$< .01$
64	13	11	0	0	0	0	0	0	$< .01$
65	10	3	0	0	0	0	0	0	$< .01$
66	8	0	0	0	0	0	0	0	$< .01$
67	7	0	0	0	0	0	0	0	$< .01$
68	0	0	0	0	0	0	0	0	$< .01$
69	2	0	0	0	0	0	0	0	$< .01$
70	1	0	0	0	0	0	0	0	$< .01$
71	0	0	0	0	0	0	0	0	$< .01$
72	0	0	0	0	0	0	0	0	$< .01$
73	0	0	0	0	0	0	0	0	$< .01$
74	0	0	0	0	0	0	0	0	$< .01$
75	0	0	0	0	0	0	0	0	$< .01$
76	0	0	0	0	0	0	0	0	$< .01$
tot	395	2679	10987	13791	3491	209	6	5	.5

Table 6: Numbers of type 4 proper partial canonical *OFs*

4 Verification

Based on the four types of rank 2 factorizations, it is easy to verify by hand that our program correctly generates the four canonical proper rank 2 one-factorizations. In [7] we describe how we verified the results for ranks 3 and 4 using a complete enumeration of the distinct partial one-factorizations.

We used the following method to verify the correctness of Theorem 3.2, which in turn leads us to believe that Theorem 3.1 is also the correct value. Define $F(n,k)$ to be the number of proper partial *OFs* of K_n with exactly k levels; $F(n,0) = 1$. We can compute $F(n,k)$ in the following manner. Let $Reg(n,k)$ denote the set of (isomorphism classes of) regular graphs of order n and degree k. The numbers of them, for $n = 12$ and $0 \leq k < 12$, are $1, 1, 9, 94, 1547, 7849, 7849, 1547, 94, 9, 1$, and 1. The graphs were generated using the algorithm described in [14], and the orders of their automorphism groups were found using the program `nauty` [13]. The numbers of graphs agree with the numbers found by Faradzhev [6].

For G in $Reg(n,k)$, let $f(G)$ denote the number of level k partial *OFs* of G (with one-factors $\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_k$) such that the neighbors of 0 appear in increasing order. $f(G)$ can be computed recursively:

$$\begin{aligned} f(\text{empty graph}) &= 1 \\ f(G) &= \textstyle\sum_{\hat{f}} f(G - \hat{f}) \end{aligned}$$

where the sum is over all one-factors $\hat{f}$ of G such that the neighbor of 0 in $\hat{f}$ is the greatest-numbered neighbor of 0 in G. This recursion gave all $f(G)$ for $Reg(12,*)$ and $Reg(10,*)$ in 9 minutes cpu at 20 mips.

For G in $Reg(n,k)$ define $a(G) = nk!(n-k-1)!/|Aut(G)|$. Interpret $a(G)$ as the number of labellings of G such that vertex 0 has neighbors $\{1, 2, ..., k\}$. Now we have

$$F(n,k) = \sum_{G \in Reg(n,k)} a(G) f(G)$$

The values of $F(n,k)$ for $n \in \{10, 12\}$ and $0 \leq k < n$ appear in Table 7.

$F(10,0)$	=	1	$F(10,5)$	=	91847664
$F(10,1)$	=	105	$F(10,6)$	=	556513920
$F(10,2)$	=	7140	$F(10,7)$	=	1385809920
$F(10,3)$	=	298560	$F(10,8)$	=	1225566720
$F(10,4)$	=	7193520	$F(10,9)$	=	1225566720[1]
$F(12,0)$	=	1	$F(12,6)$	=	308965522498560
$F(12,1)$	=	945	$F(12,7)$	=	8140425332843520
$F(12,2)$	=	570780	$F(12,8)$	=	83934109620264960
$F(12,3)$	=	210967260	$F(12,9)$	=	281058390107873280
$F(12,4)$	=	45266770080	$F(12,10)$	=	252282619805368320
$F(12,5)$	=	5283249732000	$F(12,11)$	=	252282619805368320

Table 7: Values of $F(10,k)$ and $F(12,k)$

As a check of the computations, consider that a complete one-factorization can be written as the one-factorization of some k-regular graph together with a one-factorization of its complement, for any k. Thus,

$$F(n,n-1) = \sum_{G \in Reg(n,k)} a(G) f(G) f(\overline{G})$$

where $\overline{G}$ is the complement of G. The interesting thing is that this expression must be independent of k. This test was passed successfully.

The value of $F(12,11)$ in Table 7 agrees with the number in Theorem 3.2 which expresses the total number of distinct *OFs*. Since we have obtained this 18 digit number in two different ways, we are confident it is the correct value. Also, the values of $F(12,3)$ and $F(12,4)$ agree with the results in [7].

Finally, we note that we used a modified version of the program to generate the proper canonical one-factorizations of K_{10} (both partial and complete). Our results, as well as our computed value of $F(10,9)$, agree with the results in [8], [9], and [19].

5 Conclusion

There are precisely 526,915,620 nonisomorphic and 252,282,619,805,368,320 distinct one-factorizations of K_{12}. We have derived this in two independent ways. Furthermore, our numbers agree with all previous computations of *OFs* of K_{12}. In particular, we found that there are five perfect *OFs* of K_{12}, and that for every automorphism group order greater than two, we found the same number of *OFs* as Seah and Stinson [21].

The computation required 8.15 years of cpu time at a rate of 20 mips. However, since sub-trees of the tree of partial one-factorizations could be searched independently, we were able to distribute the computation to many processors and perform the complete computation in less than eight months.

We performed some preliminary investigations into the number of one-factorizations of K_{14}, K_{16} and K_{18}. Partial searches of the trees of labeled one-factorizations of K_n have yielded the following estimates. The number of distinct OFs of K_{14} is approximately 9.876×10^{28}, for K_{16} the number is 1.48×10^{44}, and for K_{18} it is 1.52×10^{63}. If we assume that most distinct OFs have only trivial automorphisms, then we can derive estimates of the number of nonisomorphic OFs by dividing the number of distinct OFs of K_n by $n!$.

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[1] In the recent survey [22] this number is given incorrectly as 1255266720.

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Appendix: The uniform one-factorizations of K_{12}

These are the six uniform one-factorizations of K_{12}. For each *OF*, the union of any pair of one-factors is isomorphic to the union of any other pair of one-factors in the *OF*. The first *OF* below is the unique type 3 uniform one-factorization; the union of any pair of one-factors forms two disjoint 6-cycles. The other one-factorizations below are the five perfect one-factorizations of K_{12}; for each such *OF*, the union of any pair of one-factors forms a 12-cycle.

These six one-factorizations are listed in lexicographical order, and are the lexicographically last six canonical one-factorizations of K_{12}. Each *OF* is written with one one-factor per line, and each successive pair of vertices indicates an edge. Thus, the first line of the first *OF* specifies the one-factor $\{(0,1),(2,3),(4,5),(6,7),(8,9),(10,11)\}$. For each of the *OFs* we identify the element of $\mathcal{C}^3$ from which it is descended, and the order of its automorphism group.

The six uniform one-factorizations of K_{12}

OF # 526,915,615 (type 3 uniform)
derived from $\mathcal{C}^3_{41}$
$|Aut| = 660$

0 1 2 3 4 5 6 7 8 9 10 11
0 2 1 4 3 5 6 8 7 10 9 11
0 3 1 6 2 7 4 9 5 10 8 11
0 4 1 8 2 6 3 10 5 9 7 11
0 5 1 7 2 9 3 11 4 6 8 10
0 6 1 9 2 10 3 4 5 11 7 8
0 7 1 10 2 4 3 9 5 8 6 11
0 8 1 11 2 5 3 6 4 7 9 10
0 9 1 2 3 8 4 11 5 7 6 10
0 10 1 3 2 11 4 8 5 6 7 9
0 11 1 5 2 8 3 7 4 10 6 9

OF # 526,915,616 (perfect)
derived from $\mathcal{C}^3_{53}$
$|Aut| = 1$

0 1 2 3 4 5 6 7 8 9 10 11
0 2 1 4 3 6 5 8 7 10 9 11
0 3 1 5 2 7 4 9 6 11 8 10
0 4 1 6 2 8 3 11 5 10 7 9
0 5 1 10 2 9 3 4 6 8 7 11
0 6 1 11 2 10 3 9 4 8 5 7
0 7 1 2 3 8 4 11 5 9 6 10
0 8 1 3 2 6 4 7 5 11 9 10
0 9 1 7 2 4 3 10 5 6 8 11
0 10 1 9 2 11 3 5 4 6 7 8
0 11 1 8 2 5 3 7 4 10 6 9

OF # 526,915,617 (perfect)
derived from $\mathcal{C}^3_{53}$
$|Aut| = 110$

0 1 2 3 4 5 6 7 8 9 10 11
0 2 1 4 3 6 5 8 7 10 9 11
0 3 1 5 2 7 4 9 6 11 8 10
0 4 1 8 2 6 3 10 5 11 7 9
0 5 1 9 2 11 3 7 4 10 6 8
0 6 1 11 2 10 3 9 4 8 5 7
0 7 1 10 2 8 3 11 4 6 5 9
0 8 1 7 2 9 3 5 4 11 6 10
0 9 1 6 2 4 3 8 5 10 7 11
0 10 1 3 2 5 4 7 6 9 8 11
0 11 1 2 3 4 5 6 7 8 9 10

OF # 526,915,618 (perfect)
derived from $\mathcal{C}^3_{53}$
$|Aut| = 10$

0 1 2 3 4 5 6 7 8 9 10 11
0 2 1 4 3 6 5 8 7 10 9 11
0 3 1 5 2 7 4 9 6 11 8 10
0 4 1 8 2 10 3 7 5 11 6 9
0 5 1 9 2 6 3 11 4 10 7 8
0 6 1 11 2 9 3 10 4 8 5 7
0 7 1 10 2 11 3 8 4 6 5 9
0 8 1 6 2 4 3 9 5 10 7 11
0 9 1 7 2 8 3 5 4 11 6 10
0 10 1 2 3 4 5 6 7 9 8 11
0 11 1 3 2 5 4 7 6 8 9 10

OF # 526,915,619 (perfect)
derived from $\mathcal{C}^3_{54}$
$|Aut| = 5$

0 1 2 3 4 5 6 7 8 9 10 11
0 2 1 4 3 6 5 8 7 10 9 11
0 3 1 5 2 7 4 10 6 9 8 11
0 4 1 8 2 10 3 9 5 6 7 11
0 5 1 7 2 9 3 11 4 8 6 10
0 6 1 11 2 4 3 8 5 7 9 10
0 7 1 10 2 11 3 4 5 9 6 8
0 8 1 2 3 7 4 9 5 10 6 11
0 9 1 3 2 6 4 7 5 11 8 10
0 10 1 6 2 8 3 5 4 11 7 9
0 11 1 9 2 5 3 10 4 6 7 8

OF # 526,915,620 (perfect)
derived from $\mathcal{C}^3_{62}$
$|Aut| = 55$

0 1 2 3 4 5 6 7 8 9 10 11
0 2 1 4 3 6 5 8 7 10 9 11
0 3 1 6 2 5 4 11 7 8 9 10
0 4 1 8 2 9 3 10 5 6 7 11
0 5 1 11 2 10 3 8 4 7 6 9
0 6 1 2 3 9 4 10 5 7 8 11
0 7 1 9 2 11 3 5 4 6 8 10
0 8 1 7 2 4 3 11 5 9 6 10
0 9 1 10 2 7 3 4 5 11 6 8
0 10 1 5 2 8 3 7 4 9 6 11
0 11 1 3 2 6 4 8 5 10 7 9