On the number of one-factorizations of the complete graph on 12 points

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Abstract: We describe the algorithm being used to enumerate all of the nonisomorphic one-factorizations of K_{12}. We also discuss the current progress of this project.

1 Introduction

We begin with some definitions. A *one-factor* in a graph G is a set of edges in which every vertex appears precisely once. A *one-factorization* of G is a way of partitioning the edge-set of G into one-factors. (We will sometimes refer to a one-factorization as an *OF*). Two one-factorizations F and H of G, say $F = \{f_1, f_2, \ldots, f_k\}$, $H = \{h_1, h_2, \ldots, h_k\}$, are called *isomorphic* if there exists a map ϕ from the vertex-set of G onto itself such that $\{f_1\phi, f_2\phi, \ldots, f_k\phi\} = \{h_1, h_2, \ldots, h_k\}$. Here $f_i\phi$ is the set of all the edges $\{x\phi, y\phi\}$ where $\{x, y\}$ is an edge in F. Obviously, if the complete graph on n vertices K_n has a one-factorization, then necessarily n is even and any such one-factorization contains $n-1$ one-factors each of which contains $n/2$ edges. Figure 1 shows a OF of K_{12}. Each of the rows is a one-factor.

{0, 1}, {2, 3}, {4, 5}, {6, 7}, {8, 9}, {10, 11}
{0, 2}, {1, 4}, {3, 5}, {6, 8}, {7, 10}, {9, 11}
{0, 3}, {1, 6}, {2, 7}, {4, 9}, {5, 10}, {8, 11}
{0, 4}, {1, 8}, {2, 6}, {3, 10}, {5, 9}, {7, 11}
{0, 5}, {1, 7}, {2, 9}, {3, 11}, {4, 6}, {8, 10}
{0, 6}, {1, 9}, {2, 10}, {3, 4}, {5, 11}, {7, 8}
{0, 7}, {1, 10}, {2, 4}, {3, 9}, {5, 8}, {6, 11}
{0, 8}, {1, 11}, {2, 5}, {3, 6}, {4, 7}, {9, 10}
{0, 9}, {1, 2}, {3, 8}, {4, 11}, {5, 7}, {6, 10}
{0, 10}, {1, 3}, {2, 11}, {4, 8}, {5, 6}, {7, 9}
{0, 11}, {1, 5}, {2, 8}, {3, 7}, {4, 10}, {6, 9}

Figure 1: A one-factorization of K_{12}

The exact number of nonisomorphic one-factorizations of K_n is known only for even $n \leq 10$. It is easy to see that there is a unique one-factorization of K_2, K_4, and K_6. There are exactly six for K_8; these were found by Dickson and Safford [4] and a full exposition is given in [16]. In 1973, Gelling [6, 7] proved that there are exactly 396 isomorphism classes of OFs of K_{10}. In both these searches, the orders of the automorphism groups of the factorizations were also found. This information can be used to calculate the exact number of distinct factorizations.

It is also known that the number of nonisomorphic one-factorizations of K_n goes to infinity as n goes to infinity [1, 9]. In fact, if we let $N(n)$ denote the number of nonisomorphic one-factorizations of K_n, then Cameron [3] proved that $N(2n) \sim 2n^2 \ln 2n$. Feeling that the complete enumeration of the nonisomorphic OFs of K_{12} could "not be determined in a reasonable amount of time", Seah and Stinson [12, 15] restricted their search to finding one-factorizations of K_{12} with nontrivial automorphism group. They found that there are exactly 56391 nonisomorphic one-factorizations of K_{12} with nontrivial automorphism groups, excluding those whose automorphism group is of order 2 and consists of six 2-cycles. It is our goal to find the total number of nonisomorphic one-factorizations of K_{12}. We hope to verify the Seah-Stinson number, as well as determine the remaining number of nonisomorphic one-factorizations of K_{12} which they did not count.

This problem was appealing to us as it represents a good example of the so called *combinatorial explosion*. At this time we are estimating that there will be about 2 billion nonisomorphic one-factorizations of K_{12}. We also believe that it will take more than 2 years of CPU time at an execution rate of 100 mips to perform the complete enumeration. This would certainly make this computation impractical if it were not for the fact that our algorithm can be run in parallel on many different processors. We believe that the entire enumeration will be completed in about six months by distributing parts of the problem to workstations that run at rates of 14 to 50 mips. We will have more about this later in this paper.

This paper is organized a follows: Section 2 describes the orderly algorithm that is used in this search, Section 3 contains a discussion of our correctness checks for this algorithm, and Section 4 contains our results to this point (January 17, 2017).

2 The Algorithm

The algorithm that we are using is an example of what is called an *orderly* algorithm; it generates the nonisomorphic OFs of K_{12} in lexicographic order. The algorithm builds up each one-factorization by adding one one-factor at a time and rejects a partial one-factorization if it is not the lowest representative (lexicographically) of all the partial one-factorizations in its isomorphism class. In this way, the algorithm is generating only the lowest representative of any isomorphism class of one-factorizations and as such never generates any OFs which are isomorphic to each other. This approach saves both time and space over algorithms which first generate distinct (but possibly isomorphic) one-factorizations and then use methods to winnow isomorphs.

This type of algorithm has been used in other combinatorial searches including enumerating Latin squares [2, 11], strong starters [8], one-factorizations of small graphs [13], Perfect one-factorizations of K_{14} [14], frame factorizations [5] and Howell designs of small order [13]. Our algorithm below is essentially the one that was used by Seah and Stinson to find the nonisomorphic OFs of K_{10} and to find the nonisomorphic one-factorizations of K_{12} with nontrivial automorphism group [15]. Ours has been modified to deal with the explicit case of one-factorizations of K_{12}.

We first give the lexicographic ordering. Suppose that the vertices of K_{12} are numbered $0, 1, \ldots, 11$. An edge e will be written as an ordered pair (x, x') with $1 \leq x < x' \leq 11$. For any two edges $e_1 = (x_1, x'_1)$ and $e_2 = (x_2, x'_2)$, say $e_1 < e_2$ if either $x_1 < x_2$ or $x_1 = x_2$ and $x'_1 < x'_2$. A one-factor f is written as a set of ordered edges, i.e. $f = (e_1, e_2, e_3, e_4, e_5, e_6)$ where $e_i < e_j$ whenever $i < j$. For two one-factors $f_i = (e_{i1}, e_{i2}, \ldots, e_{i6})$ and $f_j = (e_{j1}, e_{j2}, \ldots, e_{j6})$, we say $f_i < f_j$ if there exists a k $(1 \leq k \leq 6)$ such that $e_{il} = e_{jl}$ for all $l < k$, and $e_{ik} < e_{jk}$.

A one-factorization F of K_{12} is written as an ordered set of 11 one-factors, i.e. $F = (f_1, f_2, \ldots, f_{11})$, where $f_i < f_j$ whenever $i < j$. The example in Figure 1 is written in this lexicographic order.

We use F and G to denote one-factorizations and f_i and g_i to denote one-factors contained in F and G, respectively. An ordering for one-factorizations is defined as follows. For two OFs F and G, we say that $F < G$ if there exists some i, $1 \leq i \leq 11$, such that $f_i < g_i$, and $f_j = g_j$ for all $j < i$.

For $1 \leq i \leq 11$, $F_i = (f_1, f_2, \ldots, f_i)$ will denote a *partial* OF consisting of an ordered set of i one-factors. We say that i is the *rank* of the partial one-factorization. Note that $F_{11} = F$, a (complete) one-factorization. We can also extend our ordering to partial OFs of rank i, in an analogous manner.

We say a partial OF $F_i = (f_1, f_2, \ldots, f_i)$ of rank i is *proper* if f_j contains edge $(0, j)$ for $1 \leq j \leq i$. If F_i is not proper, then it is *improper*. A complete one-factorization is necessarily proper.

The automorphism group of the complete graph K_{12} is S_{12}, the symmetric group on 12 elements. Thus given a proper partial OF F_i (of rank i), we can rename the 12 points using a permutation $\alpha \in S_{12}$, and obtain another partial OF (not necessarily proper) of the same graph, denoted F_i^α. We say F_i is *canonical* if $F_i \leq F_i^\alpha$ for all permutations $\alpha \in S_{12}$. Thus, each canonical partial OF F_i is the lexicographically lowest representative of its isomorphism class. The following theorems on canonicity are from Seah [12].

Theorem 2.1 *If two proper partial OFs of rank i, F_i and G_i, are distinct and are both canonical, then F_i and G_i are nonisomorphic.*

Theorem 2.2 *If a partial proper one-factorization $F_i = (f_1, f_2, \ldots, f_i)$ is canonical, and $1 \leq j \leq i$ then $F_j = (f_1, f_2, \ldots, f_j)$ is also canonical.*

Theorem 2.3 *If a partial proper OF $F_i = (f_1, f_2, \ldots, f_i)$ is not canonical, then any complete OF extended from F_i is also not canonical.*

Note that one can form a rooted tree in which each node represents one of the partial proper canonical OFs of K_{12}. The root represents the unique canonical F_1 which consists of the following one-factor, f_a:

$$f_a = \{(0,1), (2,3), (4,5), (6,7), (8,9), (10,11)\}$$

If a node v represents F_i, then the children of v represent each of the F_{i+1} which are proper canonical extensions of F_i. The nodes at level 11 of the tree represent the canonical OFs of K_{12}.

We can now describe the orderly algorithm that we use to construct canonical (non-isomorphic) OFs of the complete graph K_{12}; it is based on a depth-first traversal of the tree. The following recursive pseudo-coded procedure describes how to generate, from a given canonical F_i, all of the canonical F_{i+1} extending F_i, for $0 \leq i \leq 10$. Let F_0 be the partial OF of rank 0 (an empty set), and note that $F_0^\alpha = F_0$ for all $\alpha \in S_{12}$. We invoke the procedure using **Generate**$(F_0, 0)$.

procedure **Generate**(F_i, i)**:**
    if $i = 11$ then
        F_i is a canonical OF
    else
(1)   for each f, containing $(0, i+1)$, disjoint from each 1-factor in F_i do
(2)   for each permutation α do
(3)   if $F_i^\alpha \bigcup \{f^\alpha\} < F_i \bigcup \{f\}$ then
                    $F_i \bigcup \{f\}$ is not canonical, discard it and go on to next f
            {Here $F_i^\alpha \bigcup \{f^\alpha\} \geq F_i \bigcup \{f\}$ for all α. Hence $F_i \bigcup \{f\}$ is}
            {canonical and proper.}
            Generate$(F_i \bigcup \{f\}, i+1)$

There are several opportunities for improving the efficiency of the algorithm. We first note that the loop controlled by statement (1) potentially has $3 \cdot 5 \cdot 7 \cdot 9 = 945$ one-factors f to test as candidates for extensions of F_i. However, backtracking for each set of edges that comprise a one-factor disjoint from F_i reduces the number of one-factors that need to be considered.

As noted above, for all canonical $F_i = \{f_1, f_2, \ldots, f_i\}$, $i \geq 1$, $f_1 = f_a$. Since the union of two disjoint one-factors is a union of disjoint cycles of even length, then for any one-factor f which is edge disjoint from f_a, $\{f\} \bigcup \{f_a\}$ will form a graph isomorphic to either three disjoint 4-cycles; a 4-cycle and an 8-cycle; two 6-cycles; or a single 12-cycle. Thus, each F_2 is in one of four isomorphism classes. Following are the four one-factors which, when unioned with f_a, yield in turn each of the four canonical rank 2 one-factorizations of K_{12}.

1. $\{(0,2), (1,3), (4,6), (5,7), (8,10), (9,11)\} \bigcup \{f_a\}$ forms three disjoint 4-cycles

2. $\{(0,2), (1,3), (4,6), (5,8), (7,10), (9,11)\} \bigcup \{f_a\}$ forms a 4-cycle and an 8-cycle

3. $\{(0,2), (1,4), (3,5), (6,8), (7,10), (9,11)\} \bigcup \{f_a\}$ forms two disjoint 6-cycles

4. $\{(0,2), (1,4), (3,6), (5,8), (7,10), (9,11)\} \bigcup \{f_a\}$ forms a 12-cycle

We refer to the isomorphism class of a pair of one-factors as their *cycle structure*, and label these classes *type 1*, *type 2*, *type 3*, and *type 4* respectively. We further note that this ordering of the types is the same as the lexicographic ordering of the canonical representatives of the types. We extend the definition of *type* to apply to all canonical partial OFs. For all canonical $F_i = \{f_1, f_2, \ldots, f_i\}$, $i \geq 2$, $\{f_1\} \bigcup \{f_2\}$ is one of the four canonical rank 2 one-factorizations of K_{12}; we define the *type* of F_i to be the type of $\{f_1\} \bigcup \{f_2\}$. We note that all canonical rank i one-factorizations of K_{12} which have type s lexicographically precede all canonical rank i one-factorizations of K_{12} which have type t, for $s < t$.

Suppose we wish to consider extending some proper canonical F_i, $2 \leq i \leq 10$, by adding one-factor f. Further, assume that F_i has type t, $2 \leq t \leq 4$. Let g be a one-factor in F_i such that the type of $\{f\} \bigcup \{g\}$ (call it s) is minimal. If $s < t$, then $F_i \bigcup \{f\}$ is not canonical because there exists a permutation α that maps $F_i \bigcup \{f\}$ to a canonical rank i OF of type s. This observation leads to the following improvement of the algorithm. If F_i has type t, and f, at statement (1), forms a type s cycle structure with some one-factor in F_i such that $s < t$, then f can be discarded as a candidate for extending F_i to a proper canonical F_{i+1}.

The classification scheme permits an additional optimization of the algorithm. At statement (2) of the algorithm, α is chosen from the 12! elements of S_{12}. However, the algorithm only needs to consider those permutations which might map $F_i \bigcup \{f\}$ into a lexicographically lower isomorph. Thus, if F_i is of type t, then the only permutations which need to be considered are those which map some pair of one-factors in $F_i \bigcup \{f\}$ onto the canonical rank 2 factorization of type t.

Improvements based on the types of partial factorizations were used in [12] and [13]. Our implementation of the algorithm also uses *lazy evaluation* techniques; we postpone parts of the computation on the chance that the current F_i can not be extended to the next rank. We also use *dynamic programming* techniques, saving information from the generation of permutations at rank i to speed up the generation of the permutations at rank $i + 1$.

The algorithm outlined above can easily be modified for certain classes of OFs that are of interest. Indeed, it has been modified to find *perfect* one-factorizations of K_{12} and K_{14} [12, 14], and to find so-called *frame* factorizations of K_n for $n \leq 10$ [5]. We are also using the algorithm to conduct a *bushiest-first* search for new perfect one-factorizations of K_{16}; we probe the tree of proper canonical one-factorizations of K_{16} visiting first the nodes with the most children.

3 Correctness

Based on the four types of rank 2 factorizations, it is easy to verify by hand that our program correctly generates the four canonical proper rank 2 one-factorizations. We now describe the techniques we used to verify the correctness of the program at ranks 3 and 4.

We begin by determining, independently of our program, the number of distinct $F_3 = \{f_1, f_2, f_3\}$, the rank 3 one-factorizations of K_{12}. We note that $(0, a)$ is an edge in f_1, $(0, b) \in f_2$, and $(0, c) \in f_3$ where a, b, and c are distinct, and $1 \leq a, b, c \leq 11$. If we fix a, then there are $9 \cdot 7 \cdot 5 \cdot 3 = 945$ ways to choose the remaining 5 edges in f_1. If b and c are also fixed, then 105 of the choices for f_1 contain edge (b, c), and 840 do not. Using a simple backtracking algorithm, we counted 220,156 distinct F_3 for a fixed f_1 containing $\{(0, a), (b, c)\}$ when $(0, b) \in f_2$ and $(0, c) \in f_3$. When f_1 is fixed, but $(a, b) \notin f_1$, there are 223,632 distinct F_3. Since there are $\binom{11}{3} = 165$ ways to select a, b, and c, then the total number of distinct F_3 is

$$165(105 \cdot 220156 + 840 \cdot 223632) = 34{,}809{,}597{,}900$$

We now describe how we used our program to determine the number of distinct F_3. Our program found 76 proper canonical F_3. Additionally, we used the program to determine that there are 81 improper canonical F_3. Let $\mathcal{C}^3 = \{\mathcal{C}^3_1, \mathcal{C}^3_2, \ldots \mathcal{C}^3_{157}\}$ be the ordered set of all canonical rank 3 one-factorizations of K_{12} in lexicographic order, with $\{\mathcal{C}^3_1, \mathcal{C}^3_2, \ldots \mathcal{C}^3_{76}\}$ as the proper rank 3 canonical OFs. We determine the size of the isomorphism class of $\mathcal{C}^3_i$, $1 \leq i \leq 157$, by counting the number of F_3 that can be mapped onto $\mathcal{C}^3_i$.

1. Each of the 12 vertices in K_{12} can be mapped onto vertex 0 in $\mathcal{C}^3_i$;

2. If vertex v is mapped onto 0, there are 3! ways of mapping the neighbors of v in some F_3 into the neighbors of 0 in $\mathcal{C}^3_i$;

3. There are 8! ways of mapping the remaining 8 vertices of an F_3 into the remaining 8 vertices of $\mathcal{C}^3_i$.

Thus, if $Aut(\mathcal{C}^3_i)$ is the size of the automorphism group of $\mathcal{C}^3_i$, then there are $(12 \cdot 3! \cdot 8!)/A(\mathcal{C}^3_i)$ elements in the isomorphism class of $\mathcal{C}^3_i$.

We determined the sizes of the automorphism groups of the elements of $\mathcal{C}^3$ in two ways. First, when **Generate** was called with parameter i equal to 10 (thus generating OFs with rank $i + 1 = 11$), we counted at statement (3) the number of permutations which were automorphisms on $F_i \bigcup \{f\}$. The second technique was to generate a line graph for each of the 157 elements of $\mathcal{C}^3$, and then use McKay's *nauty* program [10] to determine the sizes of the automorphism groups. The results from these two approaches agreed.

Based on the sizes of the automorphism groups of the elements of $\mathcal{C}^3$, we calculated the number of distinct F_3 to be

$$\sum_{i=1}^{157} \frac{12 \cdot 3! \cdot 8!}{Aut(\mathcal{C}^3_i)} = 34{,}809{,}597{,}900$$

This result agrees with that derived by our independent method described above.

We repeated this verification process for the rank 4 OFs. Based on the 945 choices for a first factor containing $(0, a)$, 315 generated 47,323,968 distinct F_4 each, and 630 generated 48,190,032 distinct F_4 each; 315 of the 945 first factors contain an edge (b, c) where $\{(0, b), (0, c)\} \subset f_2 \bigcup f_3 \bigcup f_4$. Thus, the number of distinct F_4 is

$$\binom{11}{4}(315 \cdot 47323968 + 630 \cdot 48190032) = 45{,}266{,}770{,}080$$

Using our program we found 32,741 elements of $\mathcal{C}^4$, the set of canonical rank 4 OFs; 5,108 are proper and 27,633 are improper. We again determined the sizes of the automorphism groups using both our own program and *nauty*. Indeed, we found that

$$\sum_{i=1}^{32741} \frac{12 \cdot 4! \cdot 7!}{Aut(\mathcal{C}^4_i)} = 45{,}266{,}770{,}080$$

We have not checked results beyond rank 4. All of the computations for the verification at rank 4 required a total of about one hour on a 24 mips DECstation 5000. We have not pursued verification at level 5; there are 5,108 proper canonical F_4, however there are 291,031 proper canonical F_5. We do not know the number of improper canonical F_5.

Finally, we note that we used a modified version of the program to generate the proper canonical one-factorizations of K_{10} (both partial and complete). Our results agreed with those in [13].

4 Current Results

The essential feature of the algorithm is that, given any partial one-factorization F, it attempts to generate a lexicographically lower member of the isomorphism class of F. Thus, a search for complete canonical OFs can proceed independently from any proper canonical partial OF. We do not need to store the one-factorizations that are constructed (we do count them and store information about some of them) and we do not need to construct the one-factorizations in order. This allows us to work on many processors that do not even need to communicate with each other.

As discussed in Section 2, there are four types of OFs based on the cycle structure of the first pair of one-factors in an OF. These correspond to the four elments of $\mathcal{C}^2$, all of which are proper. The edges of $\mathcal{C}_4^2$ form a 12-cycle. If a partial OF, F_i, of type 4 contained a pair of factors with a cycle structure of type $t < 4$, then F_i could be mapped into F_i' of type t. So, *every* pair of one-factors in a canonical type 4 one-factorization of K_{12} has a type 4 cycle structure. The complete OFs of type 4 are called *perfect* (wherein the union of any pair of one-factors is a hamilton circuit of the complete graph). Seah and Stinson [15] found that there are five perfect one-factorizations of K_{12}, and our results concur.

Because of this tight constraint on the cycle structures, the search for all type 4 canonical OFs is quite fast. We could proceed directly from $\mathcal{C}_4^2$ to find all complete canonical OFs of type 4 in about six minutes running at a rate of 100 mips. For similar reasons, it is tractable to conduct the search for all type 3 OFs directly from $\mathcal{C}_3^2$; this required about seven hours running at 100 mips. However, the size of the problem makes it impractical to proceed directly from $\mathcal{C}_1^2$ or $\mathcal{C}_2^2$. In these cases we start the search independently from each of their proper rank three descendants in $\mathcal{C}^3$.

In Table 1 we show the numbers of partial proper canonical one-factorizations derived directly from $\mathcal{C}_3^3$ and $\mathcal{C}_4^3$; the rank 11 OFs are the complete one-factorizations. For the sake of completeness, we also summarize the partial results derived from $\mathcal{C}_1^2$ and $\mathcal{C}_2^2$, and totals which include the partial results. All lower bounds derived from partial results are underlined in the table.

In Table 2 we list for each of the first 32 elements of $\mathcal{C}^3$ the number of proper canonical OFs generated at each rank. The first 13 elements of $\mathcal{C}^3$ are of type 1, and the next 19 are of type 2. Again, partial results are underlined. About 3,100 hours of cpu time at 100 mips were required to obtain the partial results in Table 2.

The OFs with automorphism group size greater than two were completely enumerated in [12]; our results so far are consistent with that enumeration. As well, [12] found that there are at least 39,706 canonical one-factorizations of K_{12} with automorphism group size of two. So far, we have found 77,016 with automorphism group size of two. We have found 157,000,971 canonical OFs of K_{12} to date, and anticipate finding about two billion in all. We estimate that we need about eighteen more months of cpu time at 100 mips to complete the enumeration. However, we hope to finish in about six months by distributing the computation.

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$\mathcal{C}_i^2$	Rank								
	3	4	5	6	7	8	9	10	11
1	13	1281	90035	834355	9862184	48483661	69365845	95936792	39225013
2	19	2661	183345	4446635	54335882	294579706	470722376	204780756	117775773
3	20	771	14972	123762	371692	353455	76933	3796	180
4	24	395	2679	10987	13791	3491	209	6	5
tot	76	5108	291031	5415739	64583549	343420313	540165363	300721350	157000971

Table 1: Numbers of proper partial canonical OFs derived from $\mathcal{C}^2$

$\mathcal{C}_i^3$	Rank							
	4	5	6	7	8	9	10	11
1	4	152	4355	55414	437492	1114428	794716	565834
2	71	4098						7512029
3	21	994						1598245
4	43	2100						2563696
5	44	2261						3568101
6	68	3672						
7	179	15484						1504031
8	150	11757						645457
9	119	7250						992390
10	167	13495						12732422
11	156	12224	410426	5676581	35369247	55084008	22589908	7117166
12	134	8562	219920	2322418	7939530	8778320	2433531	384149
13	125	7986	199654	1807771	4737392	4389089	752792	41493
tot	1281	90035	834355	9862184	48483661	69365845	95936792	39225013
14	86	4267						12182653
15	70	3834	107585	1666752	13517003	31122923	20266304	14572837
16	171	13943						10342956
17	195	19943	943364	18867458	141289174	287529951	146106090	72109582
18	175	16429	622742	9484427	54216317	73897948	22003950	5821093
19	153	13030	450220	6011875	28669360	31937747	7374945	1433411
20	172	14594	412017	4799778	19804695	21561259	5002775	845086
21	187	15577	533689	6778202	24424413	19113862	3523357	435112
22	177	13362	361466	1800072	4903376	2572238	284957	21109
23	168	12261	310962	1385055	3168130	1375432	129127	7572
24	171	12457	243865	1785258	2992324	1214618	73828	3890
25	122	6416	81064	264748	267286	55672	1761	10
26	156	9278	160040	806027	838355	249353	10759	400
27	98	4395	37341	105276	54886	7719	279	0
28	144	6864	55503	198465	144133	33190	1184	27
29	135	6777	56612	201339	173118	35119	1047	32
30	90	3364	19099	35230	22504	1464	19	0
31	134	5592	48224	142467	93734	13850	374	3
32	57	962	2842	3453	898	31	0	0
tot	2661	183345	4446635	54335882	294579706	470722376	204780756	117775773

Table 2: Numbers of proper partial canonical OFs derived from $\mathcal{C}_1^3$ through $\mathcal{C}_{32}^3$