Cycle Systems in the Complete Bipartite Graph Minus a One-Factor

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Dedicated to Curt Lindner on the occasion of his 65$^{\text{th}}$ birthday

Abstract

Let $K_{n,n} - I$ denote the complete bipartite graph with n vertices in each part from which a 1-factor I has been removed. An *m-cycle system* of $K_{n,n} - I$ is a collection of m-cycles whose edges partition $K_{n,n} - I$. Necessary conditions for the existence of such an m-cycle system are that $m \geq 4$ is even, $n \geq 3$ is odd, $m \leq 2n$, and $m \mid n(n-1)$. In this paper, we show these necessary conditions are sufficient except possibly in the case that $m \equiv 0 \pmod 4$ with $n < m < 2n$.

1 Introduction

Throughout this paper, $K_{n,n}$ will denote the complete bipartite graph with n vertices in each partite set; $K_{n,n} - I$ will denote the complete bipartite graph with a 1-factor I removed; and C_m will denote the m-cycle $(v_1, v_2, \ldots, v_m)$. An *m-cycle system* of a graph G is set $\mathcal{C}$ of m-cycles whose edges partition the edge set of G. Several obvious necessary conditions for an m-cycle system $\mathcal{C}$ of a graph G to exist are immediate: $m \leq |V(G)|$, the degrees of the vertices of G must be even, and m must divide the number of edges in G.

There have been many results regarding the existence of m-cycle systems of the complete graph K_v (see, for example, [8]). In this case, the necessary conditions imply that $m \leq v$, v is odd, and that m divides $v(v-1)/2$. In [1, 9], it is shown that these necessary conditions are also sufficient. In the case that v is even, m-cycle systems of $K_v - I$, where I denotes a 1-factor, have been studied. Here the necessary conditions are that $m \leq v$ and that m divides $v(v-2)/2$. These conditions are also known to be sufficient [1, 9].

Cycle systems of complete bipartite graphs have also been studied. The necessary conditions for the existence of an m-cycle system of $K_{n,k}$ are that m, n, and k are even, $n, k \geq m/2$, and m must divide nk. In [10], these necessary conditions were shown to be sufficient. To study m-cycle systems of $K_{n,k}$ when n and k are odd, it is necessary to remove a 1-factor and hence $n = k$. Then, the necessary conditions are that m is even, $n \geq m/2$ with n odd, and m must divide $n(n-1)$. As a consequence of the main result of [6], it is known that $(2n)$-cycle systems of $K_{n,n} - I$ exist. Other results involving cycle systems of $K_{n,n} - I$ are given in [4], and other authors have considered cycle systems of complete multipartite graphs [2, 3, 5, 6, 7].

The main result of this paper is the following.

Theorem 1.1 *Let m and n be positive integers with $m \geq 4$ even and $n \geq 3$ odd. If $m \equiv 0 \pmod 4$ and $m \leq n$, or if $m \equiv 2 \pmod 4$ and $m \leq 2n$, then the graph $K_{n,n} - I$ has an m-cycle system if and only if the number of edges in $K_{n,n} - I$ is a multiple of m.*

Our methods involve Cayley graphs and difference constructions. In Section 2, we give some basic definitions while the proof of Theorem 1.1 is given in Section 3. We shall see that the case $m \equiv 2 \pmod 4$ is fairly easy to handle using known results, but the case $m \equiv 0 \pmod 4$ is more involved.

2 Notation and preliminaries

Let us begin with a few basic definitions. We write $G = H_1 \oplus H_2$ if G is the edge-disjoint union of the subgraphs H_1 and H_2. If $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, where $H_1 \cong H_2 \cong \ldots \cong H_k \cong H$, then the graph G can be *decomposed* into subgraphs isomorphic to H and we say that G is *H-decomposable.* We also shall write $H \mid G$.

The proof of Theorem 1.1 uses Cayley graphs, which we now define. Let S be a subset of a finite group Γ satisfying

(1) $1 \notin S$, where 1 denotes the identity of Γ, and

(2) $S = S^{-1}$; that is, $s \in S$ implies that $s^{-1} \in S$.

A subset S satisfying the above conditions is called a *Cayley subset.* The *Cayley graph* $X(\Gamma; S)$ is defined to be that graph whose vertices are the elements of Γ, with an edge between vertices g and h if and only if $h = gs$ for some $s \in S$. We call S the *connection set* and say that $X(\Gamma; S)$ is a *Cayley graph on the group* Γ.

The graph $K_{n,n}$ is a Cayley graph by selecting the appropriate group; that is, $K_{n,n} = X(\mathbb{Z}_n \times \mathbb{Z}_2; \{(0,1),(1,1),(2,1),\ldots,(n-1,1)\})$. Equivalently, for a positive integer n, let $S \subseteq \{0,1,2,\ldots,n-1\}$ and let $X(n;S)$ denote the graph whose vertices are $u_0, u_1, \ldots, u_{n-1}$ and $v_0, v_1, \ldots, v_{n-1}$ with an edge between u_i and v_j if and only if $j - i \in S$. Clearly, $K_{n,n} = X(n; \{0,1,\ldots,n-1\})$, and we will often write $-s$ for $n-s$ when n is understood.

Many of our decompositions arise from the action of a permutation on a fixed subgraph. Let ρ be a permutation of the vertex set V of a graph G. For any subset U of V, ρ acts as a function from U to V by considering the restriction of ρ to U. If H is a subgraph of G with vertex set U, then $\rho(H)$ is a subgraph of G provided that for each edge $xy \in E(H)$, $\rho(x)\rho(y) \in E(G)$. In this case, $\rho(H)$ has vertex set $\rho(U)$ and edge set $\{\rho(x)\rho(y) : xy \in E(H)\}$. Note that $\rho(H)$ may not be defined for all subgraphs H of G since ρ is not necessarily an automorphism. In this paper, however, ρ will be an automorphism, so $\rho(H)$ will be defined for all subgraphs H.

For a set D of integers and an integer x, we define the sets $\pm D = \{\pm d \mid d \in D\}$, $D + x = \{d + x \mid d \in D\}$, and $x - D = \{x - d \mid d \in D\}$.

3 The proof of the main theorem

In this section, we shall prove Theorem 1.1. It turns out that when $m \equiv 2 \pmod 4$, an m-cycle system of $K_{n,n} - I$ can be found from an $(m/2)$-cycle system of K_n as we now show.

Lemma 3.1 *For positive integers m and n with $m \equiv 2 \pmod 4$, n odd, and $6 \le m \le 2n$, the graph $K_{n,n}$ has a decomposition into m-cycles and a 1-factor if and only if $m \mid n(n-1)$.*

Proof. Let m and n be integers with $m \equiv 2 \pmod 4$, n odd, and $6 \le m \le 2n$. Let the partite sets of $K_{n,n}$ be denoted by $\{u_0, u_1, \ldots, u_{n-1}\}$ and $\{v_0, v_1, \ldots, v_{n-1}\}$. Since $m \equiv 2 \pmod 4$, we have $m = 2k$ for some odd integer k. Then $k \le n$ and $k \mid n(n-1)/2$. Hence, by [1, 9], K_n has a decomposition into k-cycles. Let the vertices of K_n be labelled with $w_0, w_1, \ldots, w_{n-1}$ and let $\mathcal{C}$ be a decomposition of K_n into k-cycles. Suppose that $C = (w_{i_0}, w_{i_1}, w_{i_2}, w_{i_3}, \ldots, w_{i_{k-1}})$ is a k-cycle in $\mathcal{C}$. Then the cycle

$$C' = (u_{i_0}, v_{i_1}, u_{i_2}, v_{i_3} \ldots, u_{i_{k-1}}, v_{i_0}, u_{i_1}, v_{i_2}, u_{i_3}, \ldots, v_{i_{k-1}})$$

is of length $2k$ in $K_{n,n}$. Furthermore, for each edge $w_i w_j$ of C, the edges $u_i v_j$ and $v_i u_j$ appear on C'. Thus, the collection

$$\begin{aligned}\mathcal{C}' \;=\; & \{(u_{i_0}, v_{i_1}, u_{i_2}, v_{i_3} \ldots, u_{i_{k-1}}, v_{i_0}, u_{i_1}, v_{i_2}, u_{i_3}, \ldots, v_{i_{k-1}}) \\ & \mid (w_{i_0}, w_{i_1}, w_{i_2}, w_{i_3}, \ldots, w_{i_{k-1}}) \in \mathcal{C}\}\end{aligned}$$

together with $\{u_iv_i \mid 0 \le i \le n-1\}$ is a decomposition of $K_{n,n}$ into m-cycles and a 1-factor. ■

The case $m \equiv 0 \pmod 4$ cannot be obtained by using a similar argument as in Lemma 3.1. Suppose that $m \equiv 0 \pmod 4$, say $m = 2k$ with k even and let $n \ge 3$ be odd with $m \le 2n$ and $m \mid n(n-1)$. As before, $k \mid n(n-1)/2$ and $k \le n$ so that a k-cycle system $\mathcal{C}$ of K_n exists. However, for each cycle $C = (w_{i_0}, w_{i_1}, w_{i_2}, w_{i_3}, \dots, w_{i_{k-1}})$ in $\mathcal{C}$, we obtain the two k-cycles

$$C' = (u_{i_0}, v_{i_1}, u_{i_2}, v_{i_3} \dots, v_{i_{k-1}})$$

and

$$C'' = (v_{i_0}, u_{i_1}, v_{i_2}, u_{i_3}, \dots, u_{i_{k-1}})$$

in $K_{n,n}$ rather than one $2k$-cycle. Thus, we need more elaborate constructions for the case $m \equiv 0 \pmod 4$.

To help guide the reader, we will now give a rough outline of these constructions. Suppose that $m < n$ and $n(n-1)$ is a multiple of m. Let $n = qm + r$. The first construction, given in Lemma 3.2, generates n cycles, each of length m. Collectively, these cycles contain all edges u_iv_j where $j - i \in \pm D$ for a given set D of $m/2$ nonzero differences. This construction will be applied q times, leaving r differences. If $r = 1$, then this will give the required 1-factor, while if $r > 2$, we proceed as follows. In Lemma 3.5, we show that $r - 1 = s(m/g)$, where $g = \gcd(m, n)$. Lemma 3.3 generates $2n/g$ cycles where these cycles contain all edges u_iv_j where $j - i \in \pm(D \cup (D + n/g))$ for a given set D of $m/(2g)$ differences. This construction will be applied $\lfloor s/2 \rfloor$ times, leaving either 1 difference (the missing 1-factor) or $m/g + 1$ differences. In the latter case, we apply the construction of Lemma 3.4. The details of how the difference sets are chosen are given in Lemma 3.5.

Lemma 3.2 *Let m and n be positive integers with $m \equiv 0 \pmod 4$, n odd, and $4 \le m < n$. If $D = \{d_1, d_2, \dots, d_{m/2}\}$, where $d_1, d_2, \dots, d_{m/2}$ are positive integers satisfying $d_1 < d_2 < \cdots < d_{m/2} \le (n-1)/2$, then $C_m \mid X(n; \pm D)$.*

Proof. Label the vertices of $X(n; \pm D)$ with $u_0, u_1, \dots, u_{n-1}$ and $v_0, v_1, \dots, v_{n-1}$. We have $u_iv_j \in E(X(n; \pm D))$ if and only if $j - i \in \pm D$. Let ρ denote the permutation

$$(u_0\ u_1\ \cdots\ u_{n-1})(v_0\ v_1\ \cdots\ v_{n-1}).$$

Observe that $\rho \in \mathrm{Aut}(X(n; \pm D))$, so for any subgraph L of $X(n; \pm D)$, $\rho(L)$ is also a subgraph. Similarly, let τ denote the permutation $(u_0\ v_0)(u_1\ v_1) \cdots (u_{n-1}\ v_{n-1})$. Let $e_k = \sum_{i=1}^{k} (-1)^{i+1} d_i$, and let P be the trail of length $(m-2)/2$ given by

$$P \quad : \quad u_{e_1}, v_{e_2}, u_{e_3}, v_{e_4}, \dots, u_{e_{(m-2)/2}}, v_{e_{m/2}}.$$

Now, the lengths of the edges of P, in the order that they are encountered, are $-d_2, -d_3, \dots, -d_{m/2}$. Since $e_1, e_3, \dots, e_{(m-2)/2}$ is a strictly increasing sequence while $n + e_2, n + e_4, \dots, n +$

$e_{m/2}$ is a strictly decreasing sequence, it follows that the vertices of P are distinct so that P is a path. Let $P' = \rho^{-d_1}(\tau(P))$ so that P' begins at v_0 and ends at $u_{e_{m/2}-d_1}$ and the edges of P' have lengths $d_2, d_3, \ldots, d_{m/2}$. Since $d_1, d_{m/2} \leq (n-1)/2$, we see that $u_{e_{(m-2)/2}} \neq u_{e_{m/2}-d_1}$ and $v_{e_{(m-2)/2}} \neq v_{e_{m/2}-d_1}$. Therefore, the vertices of P' are distinct from the vertices of P.

Next, we form a cycle C of length m by taking

$$C \;=\; \{u_{e_1}v_0, u_{e_{m/2}-d_1}v_{e_{m/2}}\} \cup\; P \;\cup\; P'.$$

Observe that these two additional edges have difference $\pm d_1$. From the above remarks, it follows that

$$C, \rho(C), \rho^2(C), \ldots, \rho^{n-1}(C)$$

is a partition of the edge set of $X(n; \pm D)$ into m-cycles. ■

Suppose n is odd, $m \equiv 0 (\bmod 4)$ with $4 \leq m < n$ and $D = \{d_1, d_2, \ldots, d_{m/2}\}$ is a set of positive integers with $n-1 \geq d_1 > d_2 > \cdots > d_{m/2} > (n-1)/2$. Then, applying Lemma 3.2 to $-D$, we find a decomposition of $X(n; \pm D)$ into m-cycles. Another consequence of Lemma 3.2 is the following. Suppose that A is a set of $mq/2$ distinct positive integers for some positive integer q, such that all elements of A are either at most $(n-1)/2$ or at least $(n+1)/2$. Then, applying Lemma 3.2 q times, we have that $X(n; \pm A)$ decomposes into m-cycles.

In Lemma 3.2, we found a cycle with m distinct differences, and used ρ to create n cycles that collectively covered all edges with those differences. We now consider cycles that have repeated differences.

Lemma 3.3 *Let m and n be positive integers with $m \equiv 0 \pmod 4$, n odd, $4 \leq m < n$, and let $g = gcd(m, n) > 1$. Let $D = \{d_1, d_2, \ldots, d_{m/(2g)}\}$ be a set of $m/(2g)$ positive integers, and let $\bar{d}_i \equiv d_i \pmod{(n/g)}$. Suppose either*

(1) $0 < d_1 < d_2 < \cdots < d_{m/(2g)} \leq (n-1)/2 - n/g$ *and* $0 < \bar{d}_1 < \bar{d}_2 < \cdots < \bar{d}_{m/(2g)} \leq (n-g)/(2g)$,

or

(2) $(n-1)/2 - n/g \geq d_1 > d_2 > \cdots > d_{m/(2g)} > 0$ *and* $n/g - 1 \geq \bar{d}_1 > \bar{d}_2 > \cdots > \bar{d}_{m/(2g)} > (n-g)/(2g)$.

Then $C_m \mid X(n; \pm(D \cup (D + n/g)))$.

Proof. Label the vertices of $X(n; \pm(D \cup (D + n/g))$ as in Lemma 3.2 and let ρ, τ be as defined in Lemma 3.2. Suppose first $0 < d_1 < d_2 < \cdots < d_{m/(2g)} \leq (n-1)/2 - n/g$ and $0 < \bar{d}_1 < \bar{d}_2 < \cdots < \bar{d}_{m/(2g)} \leq (n-g)/(2g)$. Let $e_k = \displaystyle\sum_{i=1}^{k}(-1)^{i+1}d_i$. Let P_1 be the trail of length $m/(2g) - 1$ given by

$$P_1 \;:\; u_{e_1}, v_{e_2}, u_{e_3}, v_{e_4}, \ldots, u_{e_{m/(2g)-1}}, v_{e_{m/(2g)}}.$$

Letting $\bar{e}_k = \sum_{i=1}^{k}(-1)^{i+1}\bar{d}_i$, we have that $\bar{e}_1, \bar{e}_3, \ldots, \bar{e}_{m/(2g)-1}$ is a strictly increasing sequence while $n/g + \bar{e}_2, n/g + \bar{e}_4, \ldots, n/g + \bar{e}_{m/(2g)}$ is a strictly decreasing sequence. Hence, the subscripts of vertices in P_1 lie in different nonzero congruence classes modulo n/g so that P_1 is a path. Let $P_1' = \rho^{-d_1}(\tau(P_1))$ and note that the vertices of P_1' are distinct from P_1 as in the proof of Lemma 3.2.

Form a path W_1 of length m/g by taking

$$W_1 \quad = \quad \{u_{e_1}v_{-n/g}, u_{e_{m/(2g)}-d_1}v_{e_{m/(2g)}}\} \cup \; P_1 \; \cup \; P_1'.$$

Observe that these two additional edges have differences d_1 and $-(d_1 + n/g)$, so W_1 is a path from v_0 to $v_{-n/g}$. Moreover, the first and last vertices are the only ones whose subscripts are congruent modulo n/g. It follows that

$$C_1 = W_1 \cup \rho^{n/g}(W_1) \cup \rho^{2n/g}(W_1) \cup \cdots \cup \rho^{(g-1)n/g}(W_1)$$

is a cycle of length m. Each difference occurs exactly g times, and the subscripts of the u_i's incident with edges of difference k are all congruent modulo n/g.

From the above remarks, it follows that

$$C_1, \rho(C_1), \rho^2(C_1), \ldots, \rho^{n/g-1}(C_1)$$

is a partition of the edge set of $X(n; \pm D \cup \{-(d_1 + n/g)\} \setminus \{-d_1\})$ into m-cycles.

We form a second set of cycles in a similar manner. We define P_2 analogously to P_1, except that, d_i is replaced by $d_i + n/g$ and $-d_i$ by $-(d_i + n/g)$ in e_k. Let $P_2' = \rho^{-(d_1+n/g)}(\tau(P_2))$. Form W_2 by adding the edges $u_{e_1+n/g}v_{n/g}$ and $u_{e_{m/(2g)}-(d_1+n/g)}v_{e_{m/(2g)}}$ with differences $-d_1$ and $d_1 + n/g$.

The cycles

$$C_2, \rho(C_2), \rho^2(C_2), \ldots, \rho^{n/g-1}(C_2)$$

are a partition of the edge set of $X(n; \pm(D + n/g) \cup \{-d_1\} \setminus \{-(d_1 + n/g)\})$ into m-cycles. Taken with the first set of cycles, we have our desired partition of $X(n; \pm(D \cup (D + n/g)))$ into m-cycles.

Now suppose $(n-1)/2 - n/g \geq d_1 > d_2 > \cdots > d_{m/(2g)} > 0$ and $n/g - 1 \geq \bar{d}_1 > \bar{d}_2 > \cdots > \bar{d}_{m/(2g)} > (n-g)/(2g)$. In this case, let $e_k = \sum_{i=1}^{k}(-1)^i d_i$. Let P_1 be as defined above and note that if $\bar{e}_k = \sum_{i=1}^{k}(-1)^i \bar{d}_i$, again $\bar{e}_1, \bar{e}_3, \ldots, \bar{e}_{m/(2g)-1}$ is a strictly increasing sequence while $n/g + \bar{e}_2, n/g + \bar{e}_4, \ldots, n/g + \bar{e}_{m/(2g)}$ is a strictly decreasing sequence. Hence, the subscripts of vertices in P_1 lie in different nonzero congruence classes modulo n/g so that P_1 is a path. Let $P_1' = \rho^{d_1}(\tau(P_1))$ and note that the vertices of P_1' are distinct from P_1 as in the proof of Lemma 3.2.

Form a path W_1 of length m/g by taking

$$W_1 \;=\; \{u_{e_1}v_{n/g}, u_{e_{m/(2g)}+d_1}v_{e_{m/(2g)}}\} \cup\; P_1 \;\cup\; P_1',$$

where these two additional edges have differences $-d_1$ and d_1+n/g, so W_1 is a path from v_0 to $v_{n/g}$. Again, the first and last vertices are the only ones whose subscripts are congruent modulo n/g so that

$$C_1 = W_1 \cup \rho^{n/g}(W_1) \cup \rho^{2n/g}(W_1) \cup \cdots \cup \rho^{(g-1)n/g}(W_1)$$

is a cycle of length m and

$$C_1, \rho(C_1), \rho^2(C_1), \ldots, \rho^{n/g-1}(C_1)$$

is a partition of the edge set of $X(n; \pm D \cup \{d_1+n/g\} \setminus \{d_1\})$ into m-cycles.

Form a second set of cycles as before, defining P_2 analogously to P_1 by replacing d_i with d_i+n/g and $-d_i$ with $-(d_i+n/g)$ in e_k. Let $P_2' = \rho^{d_1+n/g}(\tau(P_2))$. Form W_2 by adding the edges $u_{e_1-n/g}v_{-n/g}$ and $u_{e_{m/(2g)}+d_1+n/g}v_{e_{m/(2g)}}$ with differences d_1 and $-(d_1+n/g)$.

The cycles

$$C_2, \rho(C_2), \rho^2(C_2), \ldots, \rho^{n/g-1}(C_2)$$

are a partition of the edge set of $X(n; \pm(D+n/g) \cup \{d_1\} \setminus \{d_1+n/g\})$ into m-cycles. As in the previous case, we have our desired partition of $X(n; \pm(D \cup (D+n/g)))$ into m-cycles. ■

The previous lemma used $2m/g$ differences. The following lemma will use m/g differences and will give a 1-factor.

Lemma 3.4 *Let m and n be positive integers with $m \equiv 0 \pmod 4$, n odd, $4 \le m < n$, and let $g = gcd(m,n) > 1$. Let $D = \{d_1, d_2, \ldots, d_{m/(2g)-1}\}$ be a set of positive integers and let $\bar{d}_i \equiv d_i \pmod{(n/g)}$. Suppose either*

(1) $0 < d_1 < d_2 < \cdots < d_{m/(2g)-1} \le (n-1)/2$ and $0 < \bar{d}_1 < \bar{d}_2 < \cdots < \bar{d}_{m/(2g)-1} \le (n-g)/(2g)$

or

(2) $(n-1)/2 \ge d_1 > d_2 > \cdots > d_{m/(2g)-1} > 0$ and $n/g - 1 \ge \bar{d}_1 > \bar{d}_2 > \cdots > \bar{d}_{m/(2g)-1} > (n-g)/(2g)$.

Then $X(n; \pm D \cup \{0, \pm n/g\})$ decomposes into m-cycles and a 1-factor.

Proof. The proof is similar to that of Lemma 3.3 and uses the same notation. Suppose first that $0 < d_1 < d_2 < \cdots < d_{m/(2g)-1} \le (n-1)/2$ and $0 < \bar{d}_1 < \bar{d}_2 < \cdots < \bar{d}_{m/(2g)-1} \le (n-g)/(2g)$. Let $e_k = \sum_{i=1}^{k} (-1)^i d_i$. Let P be the trail of length $m/(2g)-1$ given by

$$P \;:\; u_0, v_{e_1}, u_{e_2}, v_{e_3}, \ldots, u_{e_{m/(2g)-2}}, v_{e_{m/(2g)-1}}.$$

Clearly, P is a path and the lengths of the edges of P, in the order they are encountered and reduced modulo n/g, are $-\bar{d}_1, -\bar{d}_2, \ldots, -\bar{d}_{m/(2g)-1}$. Hence, as in Lemma 3.3, the subscripts of vertices in P lie in different nonzero congruence classes modulo n/g.

Form a path W of length m/g by taking

$$W \;=\; \{u_0 v_{n/g}, u_{e_{m/(2g)-1}} v_{e_{m/(2g)-1}}\} \cup P \cup \tau(P).$$

Observe that these two additional edges have differences n/g and 0, respectively, so W is a path from v_0 to $v_{n/g}$. Moreover, the first and last vertices are the only ones whose subscripts are congruent modulo n/g. As before,

$$C = W \cup \rho^{n/g}(W) \cup \rho^{2n/g}(W) \cup \cdots \cup \rho^{(g-1)n/g}(W)$$

is a cycle of length m, and

$$C, \rho(C), \rho^2(C), \ldots, \rho^{n/g-1}(C)$$

is a partition of the edge set of $X(n; \pm D \cup \{0, n/g\})$ into m-cycles. The edges with difference $-n/g$ form the 1-factor, completing the construction.

Now suppose $(n-1)/2 \geq d_1 > d_2 > \cdots > d_{m/(2g)-1} > 0$ and $n/g - 1 \geq \bar{d}_1 > \bar{d}_2 > \cdots > \bar{d}_{m/(2g)-1} > (n-g)/(2g)$. Let $e_k = \sum_{i=1}^{k} (-1)^{i+1} d_i$. Let P, W, and C be defined as above so that

$$C, \rho(C), \rho^2(C), \ldots, \rho^{n/g-1}(C)$$

is a partition of the edge set of $X(n; \pm D \cup \{0, n/g\})$ into m-cycles. As before, let the edges with difference $-n/g$ form the 1-factor.

■

We now have all of the constructions needed for the proof of Theorem 1.1 in the case $m \equiv 0 \pmod 4$ and $m < n$.

Lemma 3.5 *For positive integers m and n with $m \equiv 0 \pmod 4$ and n odd with $4 \leq m < n$, the graph $K_{n,n}$ can be decomposed into m-cycles and a 1-factor if and only if $m \mid n(n-1)$.*

Proof. Let m and n be positive integers with $m \equiv 0 \pmod 4$, n odd, $4 \leq m < n$, and $m \mid n(n-1)$, say $n(n-1) = mt$. If t is even, then $m \mid n(n-1)/2$. Thus, since $m < n$, an m-cycle system $\mathcal{C}$ of K_n exists [9]. We have already noted that $\mathcal{C}$ will give rise to a collection $\mathcal{C}'$ of m-cycles in $K_{n,n}$ so that what remains when $\mathcal{C}'$ is removed from $K_{n,n}$ is a 1-factor. Therefore, it suffices to consider the case when t is odd.

Let $n = qm + r$, where $q \geq 1$ and $0 \leq r < m$ with r odd. Let $S = \{1, 2, \ldots, (n-1)/2\}$ so that $K_{n,n} = X(n; \pm S \cup \{0\})$, and let $g = \gcd(m, n)$. Suppose first that $g = 1$, and observe that this implies that $m \mid (n-1)$ so that $n - 1 = qm$. Thus $|S| = mq/2$, and by Lemma 3.2, the graph $X(n; \pm S)$ decomposes into m-cycles. Since the edges of difference 0 form a 1-factor, this completes the construction when $g = 1$.

We may now assume that $g > 1$ and let $r - 1 = s(m/g)$ for some positive integer s, say $s = 2k + \epsilon$ for some nonnegative integer k and with $\epsilon = 0$ or $\epsilon = 1$. If $s = 1$, then let $D = \{1, 2, \ldots, m/(2g) - 1\}$. Now $X(n; \pm D \cup \{0, \pm n/g\})$ decomposes into m-cycles and 1-factor by Lemma 3.4. Next, the set $A = S \setminus (D \cup \{n/g\})$ consists of $mq/2$ positive integers and thus $X(n; \pm A)$ decomposes into m-cycles by Lemma 3.2. Therefore, we have found the required decomposition of $K_{n,n}$ in this case.

Now suppose that $s > 1$. Let

$$D_1 = \left\{1,\ 2,\ \ldots,\ \tfrac{m}{2g}\right\} \text{ and } D_2 = \tfrac{n}{g} - D_1.$$

For a positive integer i, let

$$D_{2i+1} = D_1 + 2i\left(\tfrac{n}{g}\right) \text{ and } D_{2i+2} = D_2 + 2i\left(\tfrac{n}{g}\right).$$

Suppose first that k is even. Consider the sets $D_1, D_2, \ldots, D_k$ (so $i = 1, \ldots, k/2 - 1$). Note that

- for each $j = 1, 2, \ldots, k$, the set $D_j = \{d_{j,1}, d_{j,2}, \ldots, d_{j,m/(2g)}\}$ consists of $m/(2g)$ positive integers, and if $\bar{d}_{j,i} \equiv d_{j,i} \pmod{(n/g)}$, then either
  (1) $0 < d_{j,1} < d_{j,2} < \cdots < d_{j,m/(2g)}$ and $0 < \bar{d}_{j,1} < \bar{d}_{j,2} < \cdots < \bar{d}_{j,m/(2g)} \leq (n-g)/(2g)$,
  or
  (2) $d_{j,1} > d_{j,2} > \cdots > d_{j,m/(2g)} > 0$ and $n/g - 1 \geq \bar{d}_{j,1} > \bar{d}_{j,2} > \cdots > \bar{d}_{j,m/(2g)} > (n-g)/(2g)$;
- the sets $D_1, D_2, \ldots, D_k$ are pairwise disjoint;
- if $d \in D_1 \cup D_2 \cup \cdots \cup D_k$, then $d + n/g \notin D_1 \cup D_2 \cup \cdots \cup D_k$;
- $(D_1 \cup (D_1 + n/g)) \cup (D_2 \cup (D_2 + n/g)) \cup \cdots \cup (D_k \cup (D_k + n/g)) \subset \{1, 2, \ldots, nk/g\}$.

Let

$$D = \left\{1 + \frac{nk}{g}, 2 + \frac{nk}{g}, \ldots, \frac{m}{2g} - 1 + \frac{nk}{g}\right\},$$

and let

$$T = \left(D_1 \cup \left(D_1 + \frac{n}{g}\right)\right) \cup \left(D_2 \cup \left(D_2 + \frac{n}{g}\right)\right) \cup \cdots \cup \left(D_k \cup \left(D_k + \frac{n}{g}\right)\right).$$

Now $D \cap T = \emptyset$ and the largest difference in $D \cup T$ is $m/(2g) - 1 + nk/g$. We now show $m/(2g) - 1 + nk/g \leq (n-1)/2$ so that these difference sets satisfy the hypotheses of Lemma

3.3 and 3.4. Since $r < m$, we have $r-1 = s(m/g) < g(m/g)-1$, so that $s < g - g/m$. Since s is an integer, it follows that $s \leq g-1$. Hence

$$\begin{aligned} \frac{m}{2g} - 1 + \frac{nk}{g} &\leq \frac{m}{2g} - 1 + \frac{n}{g}\left(\frac{s}{2}\right) \\ &\leq \frac{m}{2g} - 1 + \frac{n}{g}\left(\frac{g-1}{2}\right) \\ &= \frac{n}{2} - \left(\frac{n}{2g} - \frac{m}{2g}\right) - 1 \\ &\leq \frac{n-1}{2}. \end{aligned}$$

For each j with $1 \leq j \leq k$, the graph $X(n; \pm(D_j \cup (D_j + n/g)))$ has a decomposition into m-cycles by Lemma 3.3. If $\epsilon = 1$, then $X(n; \pm D \cup \{0, \pm n/g\})$ decomposes into m-cycles and a 1-factor by Lemma 3.4. Let $A = S \setminus T$ if $\epsilon = 0$ or let $A = S \setminus (D \cup T)$ if $\epsilon = 1$. Then, A consists of $mq/2$ differences and Lemma 3.2 gives a decomposition of $X(n; \pm A)$ into m-cycles, completing the construction in the case that k is even.

Now suppose that k is odd. Consider the sets D_1, D_2, …, D_{k+1} (so $i = 1, \ldots, (k-1)/2$). As before, the sets $D_1, D_2, \ldots, D_{k+1}$ satisfy the same properties as in the case when k is even except that

$$\left(D_1 \cup \left(D_1 + \frac{n}{g}\right)\right) \cup \cdots \cup \left(D_k \cup \left(D_k + \frac{n}{g}\right)\right) \cup D_{k+1} \subset \left\{1, 2, \ldots, \frac{m}{2g} + \frac{nk}{g}\right\}.$$

Let $D = D_{k+1} \setminus \{nk/g - m/(2g)\}$. Let T be defined as above and note that the largest positive integer in $D \cup T$ is $m/(2g) + nk/g$, and we have seen that $m/2g + nk/g < n/2 - (n-m)/(2g)$. Since $m/(2g) + nk/g$ is an integer, it follows that $m/(2g) + nk/g \leq (n-1)/2$. Thus, as was done in the case when k is even, the graph $X(n; \pm(D_j \cup (D_j + n/g)))$ has a decomposition into m-cycles by Lemma 3.3 for each $j = 1, 2, \ldots, k$. If $\epsilon = 1$, then $X(n; \pm D \cup \{0, \pm n/g\})$ decomposes into m-cycles and a 1-factor by Lemma 3.4. Thus, letting A be defined as in the case when k is even, we have that $X(n; \pm A)$ decomposes into m-cycles by Lemma 3.2, completing the construction in the case that k is odd. ■

Theorem 1.1 now follows from Lemmas 3.1 and 3.5, and we have shown that the necessary conditions for an m-cycle system of $K_{n,n} - I$ are sufficient for many values of m and n. The remaining open case is to show that an m-cycle system exists when $m \equiv 0 (\mathrm{mod}\, 4)$ and $n < m < 2n$.

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