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MOLS WITH HOLES

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We consider sets of MOLS (mutually orthogonal Latin squares) having holes, corresponding to missing sub-MOLS, which are disjoint and spanning. We give several constructions for sets of MOLS with holes.

1. Introduction

Let $P=\{S_1,\ldots,S_n\}$ be a partition of a set S ($n\geq 2$). A *partitioned incomplete Latin square*, (or PILS), having partition P, is an $|S|$ by $|S|$ array L, indexed by S, satisfying the following properties:

(0) a cell of L either contains an element of S or is empty,

(1) the subarrays indexed by $S_i\times S_i$ are empty, for $1\leq i\leq n$ (we will refer to these subarrays as *holes*),

(2) the elements occurring in row (or column) s of L are precisely those in $S\backslash S_i$, where $s\in S_i$.

We will say that the *type* of L is the multiset $\{|S_1|,\ldots,|S_n|\}$. We will use the notation $t_1^{u_1}\cdots t_k^{u_k}$ to describe the type of a PILS, where there are precisely u_i S_i's of cardinality t_i, for $1\leq i\leq k$.

Suppose L and M are both PILS having partition P. We say that L and M are *orthogonal* if their superposition yields every ordered pair in $S^2\backslash\bigcup_{i=1}^n S_i^2$. Several PILS, each having partition P, are said to be orthogonal if each pair is. We abbreviate the term orthogonal PILS to OPILS.

We give an example of two OPILS of type 2^4 in Fig. 1. (See Lemma 3.10.) Here the symbol set $S=\mathbb{Z}_8$, and the partition is formed by $S_1=\{0,4\}$, $S_2=\{1,5\}$, $S_3=\{2,6\}$, and $S_4=\{3,7\}$.

Just as for ordinary Latin squares, one can investigate the largest set of OPILS of a specified type T. We will denote the size of such a set by $N(T)$.

In this paper we primarily investigate $N(T)$ where T is of the form 2^n, n a positive integer. We show that $N(2^n)\geq 2$ if and only if $n\geq 4$, and that $N(2^q)\geq 6$

	7	1	5		6	3	2
3		0	2	6		7	4
5	4		1	3	7		0
1	6	5		2	4	0	
	2	7	6		3	5	1
2		3	0	7		4	6
7	3		4	1	0		5
6	0	4		5	2	1	

	2	7	6		3	5	1
2		3	0	7		4	6
7	3		4	1	0		5
6	0	4		5	2	1	
	7	1	5		6	3	2
3		0	2	6		7	4
5	4		1	3	7		0
1	6	5		2	4	0	

Fig. 1. Two OPILS of type 2^4.

for several primes q. This is accomplished by a direct and a recursive construction, described in the next section.

Two OPILS have been used (see [9]) to show the existence of the class of Howell designs $H(s, 2s-2)$, s odd. As well OPILS can be used in recursive constructions to construct new examples of MOLS (mutually orthogonal Latin squares) with certain sub-MOLS (see Section 5).

We end this section by indicating a trivial upper bound.

Lemma 1.1. *Let $T = t_1^{u_1} \cdots t_k^{u_k}$ and $n = \sum_{i=1}^{k} u_i$. Then $N(T) \leq n-2$.*

Proof. Let $P = \{S_1, \ldots, S_n\}$ be a partition of type T of a set S. Let $L_1, \ldots, L_m$ be a set of OPILS having partition P. Consider a cell $C = (s, t)$ with $s \in S_1$, $t \in S_2$. The set $\{L_i(C): 1 \leq i \leq m\}$ is disjoint from $S_1 \cup S_2$ and meets any S_i $(i \geq 3)$ in at most one symbol; thus $m \leq n-2$. □

2. Two constructions

The constructions in this section closely resemble constructions for MOLS and other designs. Therefore we will not give proofs, but we will indicate appropriate references.

The first construction resembles [12; Lemma 1]. It uses difference methods. Let G be an abelian group, H a subgroup of G, X any set disjoint from G, and $k \geq 2$ an integer. For a vector $b \in (G \cup X)^k$, we denote the ith coordinate of b by b_i.

Lemma 2.1. *Suppose $B \subseteq (G \cup X)^k$ satisfies the following properties:*

(1) *For each i, $1 \leq i \leq k$, and each $x \in X$, there is a unique $b \in B$ with $b_i = x$.*

(2) *No $b \in B$ has two coordinates in X.*

(3) *For each i, j $(1 \leqslant i < j \leqslant k)$ and each $a \in G \setminus H$ there is a unique $b \in B$ with $b_i, b_j \in G$ and $b_i - b_j = a$.*

Then there is a set of $k-2$ OPILS of type $h^{g/h}|X|^1$, having partition $\{Hg: g \in G\} \cup \{X\}$, where $g = |G|$ and $h = |H|$.

We remark that the above construction is valid with $X = \emptyset$; conditions (1) and (2) are then satisfied trivially.

The following recursive construction uses GDDs. A *group-divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$ satisfying:

(1) $\mathcal{G}$ is a partition of X into subsets (called *groups*),

(2) $\mathcal{A}$ is a set of subsets of X (called *blocks*), each having size at least two,

(3) a group and a block have at most one common element (or *point*) of X,

(4) every unordered pair of points not contained in a group is contained in a unique block.

The *group-type* of $(X, \mathcal{G}, \mathcal{A})$ will be the multiset $\{|G|: G \in \mathcal{G}\}$.

A *weighting* of a GDD is a mapping $w: X \to \mathbb{Z}^+ \cup \{0\}$. For $(X, \mathcal{G}, \mathcal{A})$ a GDD, w a weighting, and $Y \subseteq X$, let $w(Y)$ denote the multiset $\{w(x): x \in Y\}$.

The following construction is similar to [10, Construction 2.2].

Lemma 2.2. *Suppose that $(X, \mathcal{G}, \mathcal{A})$ is a GDD, w is a weighting, and let $k \geqslant 1$. Further, suppose that, for every block $A \in \mathcal{A}$, there are k OPILS of type $w(A)$. Then there are k OPILS of type $\{\sum_{x \in G} w(x): G \in \mathcal{G}\}$.*

3. Two OPILS

We shall show in this section that $N(2^n) \geqslant 2$ for $n \geqslant 4$. Note that if $n \leqslant 3$, then $N(2^n) < 2$ by Lemma 1.1. First, we reduce the problem to a finite one.

A *pairwise balanced design* (or PBD) is a pair $(X, \mathcal{A})$ such that $(X, \{\{x\}: x \in X\}, \mathcal{A})$ is a GDD. We say that $(X, \mathcal{A})$ is a (v, K)-PBD provided that $|X| = v$ and $|A| \in K$ for all $A \in \mathcal{A}$. (K is a set of integers, each exceeding one.) A set J of positive integers is said to be *PBD-closed* provided that $v \in J$ whenever a (v, J)-PBD exists. Define $N_2 = \{n: N(2^n) \geqslant 2\}$

Lemma 3.1. *N_2 is PBD-closed.*

Proof. Apply Lemma 2.2 with $k = 2$, giving every point weight two. □

Let $K_2 = \{4, 5, \ldots, 12, 14, 15, 18, 19, 23, 27\}$.

Lemma 3.2. *Suppose $K_2 \subseteq N_2$. Then $N_2 = \{n: n \geqslant 4\}$.*

Proof. If $n < 4$, then we have noted that $n \notin N_2$. It has been shown [5] that a (v, K_2)-PBD exists for all $v \geqslant 4$. Apply Lemma 3.1. □

Lemma 3.3. *Suppose* $q \equiv 3 \pmod 4$ *is a prime power exceeding* 3. *Then* $q \in N_2$.

Proof. We will apply Lemma 2.1 with $G = \mathrm{GF}(q)\times\mathbb{Z}_2$, $H = \{0\}\times\mathbb{Z}_2$, $X = \emptyset$ and $k = 4$. Let R denote the non-zero squares in $\mathrm{GF}(q)$; let N denote the (non-zero) non-squares. Choose any non-zero $c \in \mathrm{GF}(q)$ such that $c^2 - 1 \in N$ (this is where we require $q > 3$). Now let

$$B = \{(y_0, cy_1, (c+1)y_1, 0_0),\ (y_1, -cy_1, -(c-1)y_0, 0_0),$$
$$(-y_0, -cy_0, -(c+1)y_0, 0_0),\ (-y_1, cy_0, (c-1)y_1, 0_0)\colon\ y \in R\}.$$

(We are writing the ordered pair (x, i) as x_i for simplicity.) It is straightforward to verify property (3) in Lemma 2.1. Two OPILS of type 2^q result. □

If $q \equiv 1 \pmod 4$, the above construction does not work because -1 is a square. However we have another construction.

Lemma 3.4. *If* $q \equiv 1 \pmod 4$ *is a prime power, then* $q \in N_2$.

Proof. Let ω be primitive in $\mathrm{GF}(q)$ and let c be any non-square. Let $t = \frac{1}{4}(q-1)$, and define $Q = \{1, \omega^2, \ldots, \omega^{2t-2}\}$. Note that

$$Q \cup -Q \cup cQ \cup -cQ = \mathrm{GF}(q)\setminus\{0\}.$$

We apply Lemma 2.1 with the same G, H, X and k as in Lemma 3.3. Define

$$B = \{\pm(y_1, cy_1, (1+c)y_0, 0_0),\ \pm(cy_0, y_1, (1+c)y_1, 0_0),$$
$$\pm(c^2y_0, cy_0, c(1+c)y_0, 0_0),\ \pm(cy_1, c^2y_0, c(1+c)y_1, 0_0)\colon\ y \in Q\}.$$

It can be verified that property (3) in Lemma 2.1 is satisfied; hence two OPILS of type 2^q result. □

We can construct two OPILS of type 2^n, n even, be altering slightly those of type 2^{n-1} constructed by Lemmata 3.3 and 3.4. (This is essentially the method of 'sum composition', as used by Horton [6, Theorem 3] and Ruiz and Seiden [8]). Suppose B satisfies the hypotheses of Lemma 2.1 with $k = 4$, and that there are four vectors (in B) $b^i = (r_i, s_i, t_i, 0)$, $1 \leq i \leq 4$, which satisfy the following properties:

(1) r_i, s_i and t_i are all in G, $1 \leq i \leq 4$.

(2) $\{s_i - r_i\colon 1 \leq i \leq 4\} = \{s_2 - r_1,\ s_4 - r_3,\ s_2 - r_1 - (t_2 - t_1),\ s_4 - r_3 - (t_4 - t_3)\}$.

Let $\{\infty_1, \infty_2\}$ be disjoint from $G \cup X$, and define

$$B' = B\setminus\{b^i\colon 1 \leq i \leq 4\} \cup \{(r_1, s_2, \infty_1, 0),\ (\infty_1, s_1, t_1, 0),\ (r_2, \infty_1, t_2, 0),$$
$$(r_1 - t_1, s_2 - t_2, 0, \infty_1),\ (r_3, s_4, \infty_2, 0),\ (\infty_2, s_3, t_4, 0),$$
$$(r_4, \infty_2, t_4, 0),\ (r_3 - t_3, s_4 - t_4, 0, \infty_2)\}.$$

The following result can be easily verified.

Lemma 3.5 (Projection Lemma). *B′, described above, satisfies the hypothesis of*

Lemma 2.1 *with respect to* G, H, *and* $X' = X \cup \{\infty_1, \infty_2\}$, *with* $k = 4$. *Hence two* OPILS *of the relevant type exist.*

Now, consider the B constructed in Lemma 3.3. Choose any $y \in R$, and let $b^1 = (y_0, cy_1, (c+1)y_1, 0_0)$, $b^2 = (-y_0, -cy_0, -(c+1)y_0, 0_0)$, $b^3 = (-y_1, cy_0, (c-1)y_1, 0_0)$ and $b^4 = (y_1, -cy_1, -(c-1)y_0, 0_0)$. These four vectors satisfy the two properties preceding Lemma 3.5, hence they may be 'projected' as described above. This process can be carried out successively for various elements of R. There are $\frac{1}{2}(q-1)$ residues, so we obtain the following.

Lemma 3.6. *For* $q \equiv 3 \pmod 4$ *a prime power exceeding* 3, *and* $0 \leq i \leq \frac{1}{2}(q-1)$, *there are two* OPILS *of type* $2^q(2i)^1$.

Setting $i = 1$, we obtain

Corollary 3.7. *If* $q \equiv 3 \pmod 4$ *is a prime power exceeding* 3, *then* $q+1 \in N_2$.

The case $q \equiv 1 \pmod 4$ may be dealt with similarly, by altering the B produced in Lemma 3.4. Choose any $y \in Q$, and consider $b^1 = (y_1, \omega y_1, (1+\omega)y_0, 0_0)$, $b^2 = (\omega^2 y_0, \omega y_0, \omega(1+\omega)y_0, 0_0)$, $b^3 = (-\omega y_0, -y_1, -(1+\omega)y_1, 0_0)$, and $b^4 = (-\omega y_1, -\omega^2 y_0, -\omega(1+\omega)y_1, 0_0)$. These vectors may be 'projected', for any $y \in Q$. Also, the vectors $-b^i$ $(1 \leq i \leq 4)$ may be 'projected', for any $y \in Q$, independently of the vectors b^i $(1 \leq i \leq 4)$. There are $\frac{1}{4}(q-1)$ choices for $y \in Q$, so we obtain a result similar to Lemma 3.6.

Lemma 3.8. *For* $q \equiv 1 \pmod 4$ *a prime power, and* $0 \leq i \leq \frac{1}{2}(q-1)$, *there are two* OPILS *of type* $2^q(2i)^1$.

Corollary 3.9. *If* $q \equiv 1 \pmod 4$ *is a prime power, then* $q+1 \in N_2$.

Now, from Lemmata 3.3 and 3.4, and Corollaries 3.7 and 3.9, we have $\{5, 6, 7, 8, 9, 10, 11, 12, 14, 18, 19, 23, 27\} \subseteq N_2$. In view of Lemma 3.2, we need only show that $\{4, 15\} \subseteq N_2$ in order to determine N_2.

Lemma 3.10. $4 \in N_2$.

Proof. We apply Lemma 2.1 with $G = \mathbb{Z}_8$, $H = \{0, 4\}$, $X = \emptyset$, and $k = 4$. Define $B = \{(1,2,7,0), (2,7,1,0), (7,1,2,0), (3,6,5,0), (6,5,3,0), (5,3,6,0)\}$. □

Lemma 3.11. $15 \in N_2$.

Proof. Apply Lemma 3.6 with $q = 11$ and $i = 4$, to obtain two OPILS of type $2^{11}8^1$. Now, fill in the hole of size 8 with two OPILS of type 2^4 on the relevant symbol set. This produces two OPILS of type 2^{15}. □

Thus we have proved

Theorem 3.12. *There are two* OPILS *of type* 2^n *if and only if* $n \geq 4$.

4. Six OPILS

In this section, by means of a computer, we construct six OPILS of type 2^q for several prime powers $q \equiv 1 \pmod 4$.

Let $q \equiv 1 \pmod 4$ be a prime power, let $t = \frac{1}{4}(q-1)$, and let ω be primitive in GF(q). Define $C_0 = \{1 = \omega^0, \omega^4, \ldots, \omega^{4t-4}\}$, and for $1 \leq i \leq 3$ define $C_i = \omega^i C_0$. C_0 is the (multiplicative) subgroup of index four in GF$(q)\backslash\{0\}$, and the C_i's are its cosets.

We will apply Lemma 2.1 with $G = \mathrm{GF}(q) \times \mathbb{Z}_2$, $H = \{0\} \times \mathbb{Z}_2$, $X = \emptyset$, and $k = 8$, making use of the cyclotomic classes C_i defined above.

For any $u \in (\mathrm{GF}(q))^8$, say $u = (u_i\colon\ 1 \leq i \leq 8)$, and any $v \in (\mathbb{Z}_2)^8$, say $v = (v_i\colon$ $1 \leq i \leq 8)$, define $u \circ v = ((u_i, v_i)\colon\ 1 \leq i \leq 8)$. Let v^i, $1 \leq i \leq 8$, denote the ith row of the matrix

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus $v^i \in (\mathbb{Z}_2)^8$, $1 \leq i \leq 8$. Suppose u and u' are two vectors in $(\mathrm{GF}(q))^8$, with $u = (u_i\colon\ 1 \leq i \leq 8)$, and $u' = (u'_i\colon\ 1 \leq i \leq 8)$. Define

$$B = B(u, u', V) = \{xu \circ v^1, \omega xu \circ v^2, \omega^2 xu \circ v^3, \omega^3 xu \circ v^4, xu' \circ v^5,$$
$$\omega xu' \circ v^6, \omega^2 xu' \circ v^7, \omega^3 xu' \circ v^8\colon\ x \in C_0\}.$$

One can ask when B will satisfy the hypotheses of Lemma 2.1. A set of conditions is obtained, each of the form $(u_j - u_i)/(u'_j - u'_i) \in X$ where X is a union of some or all of the cyclotomic classes. For example, when $i = 1$ and $j = 2$, we obtain the differences $(u_2 - u_1)x_0$, $(u_2 - u_1)\omega x_0$, $(u_2 - u_1)\omega^2 x_1$, $(u_2 - u_1)\omega^3 x_1$, $(u'_2 - u'_1)x_1$, $(u'_2 - u'_1)\omega x_1$, $(u'_2 - u'_1)\omega^2 x_0$, $(u'_2 - u'_1)\omega^3 x_0$, for all $x \in C_0$. These differences are distinct if and only if $(u_2 - u_1)/(u'_2 - u'_1) \in C_0$. We record all of these conditions in Table 1 below. The ij-entry (or entries) indicate the permissible cyclotomic class(es) for $(u_j - u_i)/(u'_j - u'_i)$, where $1 \leq i < j \leq 8$.

Table 1

$i \backslash j$	2	3	4	5	6	7	8
1	0	0, 2	0	2	1, 3	2	0, 1, 2, 3
2		2	1, 3	0, 1, 2, 3	0	0, 2	2
3			2	0	0, 1, 2, 3	0	1, 3
4				0, 2	0	0, 1, 2, 3	2
5					2	1, 3	0
6						2	0, 2
7							0

By computer, we have found vectors u and u' so that these conditions are satisfied, for the primes 29, 37, 41, 53, 61, 73, 89, and 97. Larger primes (congruent to 1 modulo 4) undoubtedly could also be handled by this method. So too could prime powers (such as 25 or 81); the programming would be more difficult, however. For the prime 17 we have only four OPILS by this method. We list permissible vectors u and u', in Table 2.

We record our results.

Theorem 4.1. *$N(2^{17}) \geq 4$; $N(2^q) \geq 6$ for $q = 29, 37, 41, 53, 61, 73, 89$, and 97.*

Table 2

q	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u'_1	u'_2	u'_3	u'_4	u'_5	u'_6	u'_7	u'_8
17	1	3	7	4	6			0	1	10	8	15	7			0
29	1	2	3	27	9	26	20	0	1	8	13	19	22	3	16	0
37	1	2	3	4	8	31	10	0	1	8	19	12	5	23	26	0
41	1	3	2	9	4	24	22	0	1	6	11	37	16	15	19	0
53	1	2	3	4	7	8	18	0	1	14	4	42	11	51	50	0
61	1	2	3	4	5	6	24	0	1	10	24	15	14	55	32	0
73	1	2	3	4	6	13	5	0	1	3	26	50	23	34	28	0
89	1	2	3	6	5	7	13	0	1	5	10	70	21	26	52	0
97	1	2	3	4	5	10	16	0	1	36	5	49	13	84	81	0

5. Further results

In this section we indicate several further results concerning MOLS with holes. First, we briefly consider OPILS of type 1^n. The following is well known.

Lemma 5.1. *There exist k OPILS of type 1^n if and only if there exist k MOLS of order n having a common transversal.*

Denote the maximum number of MOLS of order n by $N(n)$.

Corollary 5.2. $N(1^n) \geq N(n) - 1$.

Lower bounds for $N(n)$ have been extensively tabulated (see for example, [3]), so the above corollary yields lower bounds which are sometimes quite good.

Corollary 5.3. *If q is a prime power, then* $N(1^q) = q - 2$.

Proof. It is well known that $N(q) = q - 1$ for q a prime power, so $N(1^q) \geq q - 2$. Lemma 1.1 shows that $N(1^q) \leq q - 2$. □

It has been determined [1] precisely when $N(1^n) \geq 2$.

Theorem 5.4. $N(1^n) \geq 2$ *if and only if* $n \geq 4$ *is an integer,* $n \neq 6$.

It is also known, with a few exceptions, when $N(1^n) \geq 3$.

Theorem 5.5. $N(1^n) \geq 3$ *if* $n \geq 5$, $n \neq 6, 10, 14$.

Proof. It is known (see [3]) that $N(n) \geq 4$ if $n \geq 5$, $n \notin \{6, 10, 14, 18, 20, 22, 24, 26, 28, 30, 33, 34, 38, 42, 44, 52\}$; so $N(1^n) \geq 3$ for all other n. Also, in [11] it is shown that $N(1^n) \geq 3$ for all the above possible exceptions other than 6, 10, and 14. □

Next, we record an improvement to Lemma 3.4, which is proved in [7].

Theorem 5.6. *If* $q \equiv 1 \pmod 4$ *is a prime power, then* $N(2^q) \geq 3$.

Although we have already established that $N(2^n) \geq 2$ for all $n \geq 4$, we give an alternate construction, valid for $n \equiv 1 \pmod 3$.

Theorem 5.7. *For* $n \equiv 1 \pmod 3$, $n \geq 7$, $N(2^n) \geq 2$.

Proof. Let $n \equiv 1 \pmod 3$, $n \geq 7$. There is a GDD having group-type 2^n and blocks of size 4 by [4]. Apply Lemma 2.2, with $k = 2$, giving every point weight 1. By Theorem 5.4 there are two OPILS of type 1^4. Thus two OPILS of type 2^n are constructed. □

The following result can be established in a similar manner.

Theorem 5.8. *If* $n \equiv 0 \pmod 3$ *and* $n \geq 9$, *then* $N(2^n 5^1) \geq 2$.

Proof. There is a GDD having group-type $2^n 5^1$ and blocks of size 4 by [2]. Proceed as in Lemma 5.7. □

Finally we would like to indicate how MOLS with holes can be useful in constructing MOLS with sub-MOLS. The following singular direct product construction is useful.

Theorem 5.9. *Suppose there exist:*
(1) k OPILS *of type* t^u,
(2) k MOLS *of order* $(v-w)/t$,
(3) k MOLS *of order* v *containing* k *sub*-MOLS *of order* w.
Then there are k MOLS *of order* $u(v-w)+w$, *containing* k *sub*-MOLS *of orders* v *and* w.

Let us consider an example. Suppose one wished to construct three MOLS of order 51 containing three sub-MOLS of order 11. The equation $51=5(11-1)+1$ comes to mind. In applying Lemma 5.9, one might try $u=5$, $v=11$, $w=1$ (and $k=3$, of course). If $t=1$, the construction fails since three MOLS of order 10 are not known. However, we can set $t=2$: we have $N(2^5)\geqslant 3$ by Lemma 5.6, $N(5)\geqslant 3$, and there are three MOLS of order 11 with three sub-MOLS of order 1. Thus we have shown

Theorem 5.10. *There are three* MOLS *of order* 51 *containing three sub*-MOLS *of order* 11.

The authors know of no other method of obtaining the above result.

As another application of Lemma 5.9, we prove the following.

Theorem 5.11. *If* $n\geqslant 9$ *is odd, then there exist two* MOLS *of order* n *containing two sub*-MOLS *of order* 3.

Proof. Apply Lemma 5.9 with $k=2$, $t=2$, $u=\frac{1}{2}(n-1)$, $v=3$ and $w=1$. We have $u\geqslant 4$, so the required OPILS exist by Theorem 3.12. □

6. Summary and comments

MOLS are of widespread and basic importance in the construction of combinatorial designs. We feel that MOLS with holes will also prove useful, and we have given some indication of ways in which they can be used (see also [7] and [10, Construction 2.4]).

What is needed now are more direct constructions for MOLS with holes. One question that comes to mind is: When is $N(2^n)=n-2$ the case? Perhaps equality holds for n a prime power.

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