On the Irregularity Strength of the $m \times n$ Grid

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ABSTRACT

Given a graph G with weighting w: $E(G) \rightarrow Z^+$, the *strength* of $G(w)$ is the maximum weight on any edge. The *sum* of a vertex in $G(w)$ is the sum of the weights of all its incident edges. The network $G(w)$ is *irregular* if the vertex sums are distinct. The *irregularity strength* of G is the minimum strength of the graph under all irregular weightings. In this paper we determine the irregularity strength of the $m \times n$ grid for certain m and n. In particular, for every positive integer d we find the irregularity strength for all but a finite number of $m \times n$ grids where $n - m = d$. In addition, we present a general lower bound for the irregularity strength of graphs. © 1992 John Wiley & Sons, Inc.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with no K_2 component and at most one isolated vertex. A network $G(w)$ consists of the graph G together with an assignment w: $E(G) \rightarrow Z^+$. The *strength* s of $G(w)$ is defined by $s(G(w)) = \max\{w(e): e \in E(G)\}$. For each vertex $v \in V(G)$, define the *sum* (or *weight*) $w(v)$of v in $G(w)$ by $\Sigma_{e \text{ incident to } v} w(e)$ and call $G(w)$ *irregular* if for all distinct $u, v \in V(G)$, $w(u) \neq w(v)$. The irregularity strength $I(G)$ is defined to be

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min{$s(G(w))$: $G(w)$ is irregular}. Thus the irregularity strength of a graph G is the smallest strength of all irregular weightings of G; we call an irregular weighting *minimal* if it has the smallest possible strength. The problem can also be described as that of choosing positive weights for the nonzero entries in a symmetric adjacency matrix such that the row sums are distinct [6].

The study of $I(G)$ was proposed in [1]. There it was shown that $(3p - 2q)/3 \leq I(G) \leq 2p - 3$ for a graph G with p vertices and q edges. In [10] the following stronger lower bound was obtained (here $\lceil\ \rceil$ represents the ceiling function):

Theorem 1.1. Let d_k be the number of vertices of degree k in $V(G)$, then

$$I(G) \geq \lambda(G) = \left\lceil \max\left\{\left(\left(\sum_{k=i}^{j} d_k\right) + i - 1\right)\Big/ j \colon i \leq j\right\}\right\rceil .$$

The problem of studying irregularity strengths of graphs has proven to be difficult; there are not many graphs for which the irregularity strength is known. In [1] it was shown that $I(K_n) = 3$ and $I(K_{2n,2n}) = 3$; $I(P_n)$ was also determined. That $I(K_{2n+1,2n+1}) = 4$ was proven in [8]. Work has also been done on binary trees, dense graphs, and the disjoint unions of paths, cycles, and complete graphs [2,5,9,11]. Recently, the irregularity strengths of wheels, k-cubes, and $2 \times n$ grids has also been determined [4]. In each of these cases it was found that $I(G) = \lambda(G)$ or $\lambda(G) + 1$, and it is conjectured that if T is a tree, then $I(T) = \lambda(T)$ or $\lambda(T) + 1$. Results on irregularity strengths of graphs are surveyed in [12].

In this paper we extend the work of [4] by finding the irregularity strength of the $m \times n$ grid $X_{m,n}$. Figure 1 gives an irregular weighting of strength $\lambda(X_{4,5}) = 6$ on $X_{4,5}$. In this figure, the edge weights are given, while the vertex sums are circled. It is easy to compute that if $m, n \geq 3$ (and $\{m,n\} \neq \{3,5\}$), then

$$\lambda(X_{m,n}) = \lceil (mn + 1)/4 \rceil .$$

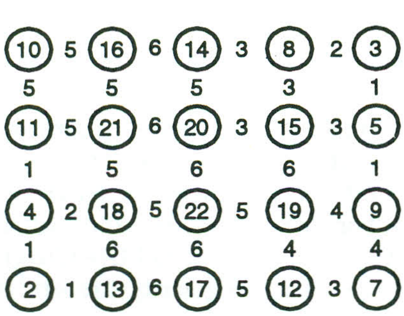

FIGURE 1. A minimal irregular weighting of $X_{4,5}$.

This can be simplified depending on the congruences of m and n modulo 4:

$$\lambda(X_{m,n}) = \begin{cases} mn/4 + 1 & \text{if } m \text{ and } n \text{ are both even} \\ mn/4 + 1 & \text{if } n \text{ is odd and } m \equiv 0 \pmod 4 \\ (mn+2)/4 & \text{if } n \text{ is odd and } m \equiv 2 \pmod 4 \\ (mn+3)/4 & \text{if } m \text{ and } n \text{ both odd and } m \equiv n \pmod 4 \\ (mn+1)/4 & \text{if } m \text{ and } n \text{ both odd and } m \not\equiv n \pmod 4. \end{cases}$$

We will show for every positive integer d that $I(X_{m,n}) = \lambda(X_{m,n})$ or $\lambda(X_{m,n}) + 1$ for all but a finite number of $m \times n$ grids where $n - m = d$. This finite number is easy to compute and is at most $d + 13$. As a special case of the results of this paper we prove the conjecture from [7] that $I(X_{n,n}) = \lambda(X_{n,n})$ for all $n \geq 3$. Next, we sketch a proof of this conjecture, as it illustrates well the techniques we use to prove the more general result.

We provide a recursive construction that, given a minimal irregular weighting of $X_{2k,2k}$, yields a minimal irregular weighting of $X_{2k+2,2k+2}$. The construction (Construction 2 in Section 2) proceeds by ringing a $2k \times 2k$ grid G with $4(2k+1)$ vertices to create a $(2k+2) \times (2k+2)$ grid G'. The external edges, connecting the $4(2k+1)$ vertices on the borders of G' to each other and to the internal (nonborder) vertices in G', are weighted so that the border vertices of G' will have distinct sums ranging from 2 to $4(2k+1)+1$. The edges in G' that correspond to edges in G are given weights based on the weights of the corresponding edges in G, but in such a way that all of the internal vertices in G' will have weights that are $4(2k+1)$ greater than the weights of the corresponding vertices in G. Thus, since the vertex sums in G were distinct, the vertices in G' corresponding to vertices in G will also have distinct sums. Further, the smallest sum on any such vertex in G' will be $4(2k+1)+2$, which is greater than the largest sum on any external vertex ringing the internal $(2k)^2$ vertices.

The construction requires that the weighting of G meets certain constraints; however, since the construction preserves those constraints, it can be applied recursively. Thus, given a minimal weighting of $X_{2k,2k}$ that has the essential properties, the construction provides minimal weightings of all $X_{n,n}$ where n is even and $n \geq 2k$. In Section 2 we provide a minimal weighting of $X_{16,16}$ that has the necessary ingredients for the construction.

To obtain minimal irregular weightings for $X_{n,n}$, where n is odd, we use a technique similar to Construction 2. Construction 4a begins with G, a minimal irregular weighting of $X_{2k,2k}$, and creates a $(2k+1) \times (2k+1)$ grid G' by adding a row and a column of vertices. The added $4k+1$ vertices have weights ranging from 2 to $4k+2$, and the remaining $(2k)^2$ vertices in G' have weights that are $4k+1$ greater than the weights of their corresponding vertices in G. Construction 4a cannot be applied recursively, and requires G to have certain properties; however, all grids created by Construction 2, as well as the weighting of $X_{16,16}$ provided in Section 2, have the necessary properties. Therefore, the weighting of $X_{16,16}$, together with Constructions 2 and 4a, prove that $I(X_{n,n}) = \lambda(X_{n,n})$ for all $n \geq 16$.

In addition to the above constructions, Section 2 provides three other constructions that we use to prove a more general result. We will first present the minimal irregular weighting of $X_{16,16}$ that possesses the special properties essential in starting the induction. Construction 1 begins with a special minimal irregular weighting of $X_{m,n}$ and produces a minimal irregular weighting of $X_{m+2,n+4}$. This construction can be used inductively. Construction 2 also begins with a special minimal irregular weighting of $X_{m,n}$ and produces a minimal irregular weighting of $X_{m+2,n+2}$. This construction can also be used inductively or can extend a weighting resulting from Construction 1.

Constructions 3, 4a, and 4b can each only be used once, and follow the use of either Construction 1 or 2. These three constructions begin with a minimal irregular weighting of $X_{m,n}$ (derived from either Construction 1 or 2); Construction 3 gives a minimal weighting of $X_{m,n+1}$, while Constructions 4a and 4b produce minimal weightings of $X_{m+1,n+1}$. These five constructions when applied in the proper order produce minimal irregular weightings for all $X_{m,n}$ where $2\lfloor (m+2)/4 \rfloor + 8 \leq n \leq 4\lfloor m/2 \rfloor - 15$. (Note that $\lfloor\ \rfloor$ is the greatest integer function.)

Additionally, we present a general lower bound that identifies a large class of graphs for which the lower bound on the irregularity strength is greater than the lower bound λ.

In Section 3 we will discuss some small cases of m and n, and will summarize what is known about the irregularity strength of grids.

2. MAIN RESULT

We begin this section with some terminology. Certain edges in $X_{m,n}$ will have special names. We call the outermost edges the *border* edges; so all edges (u, v) in $E(X_{m,n})$, where the degrees of u and v are both less than 4, are border edges. All nonborder edges are called *internal.* The border edges are further described as *left, right, top,* or *bottom* edges. When m and n are both even and if these border edges are labeled $1, 2, 3, \ldots$ starting from a corner vertex, then the edges with odd labels are termed *heavy edges.*

In Figure 2 we give a minimal irregular weighting w of the edges of $X_{16,16}$ ($\lambda(X_{16,16}) = 65$), which has the further property that $w(e) \leq 39$ for any heavy border edge e. Also, w has the property that there exists a one-factor f in $X_{16,16}$ where, for every edge $e \in f$, $w(e)$ is no greater than 3 less than the above constraints ($w(e) \leq 36$ for heavy border edges, $w(e) \leq 62$ for other edges in f); such a one-factor consists of the alternate horizontal edges in each row. This weighting was found by the hill-climbing algorithm IS-1 from [7]. It was found in 151.5 minutes on a DECstation 5000 rated at 24 MIPS. This is the weighting that we will use to begin our induction;

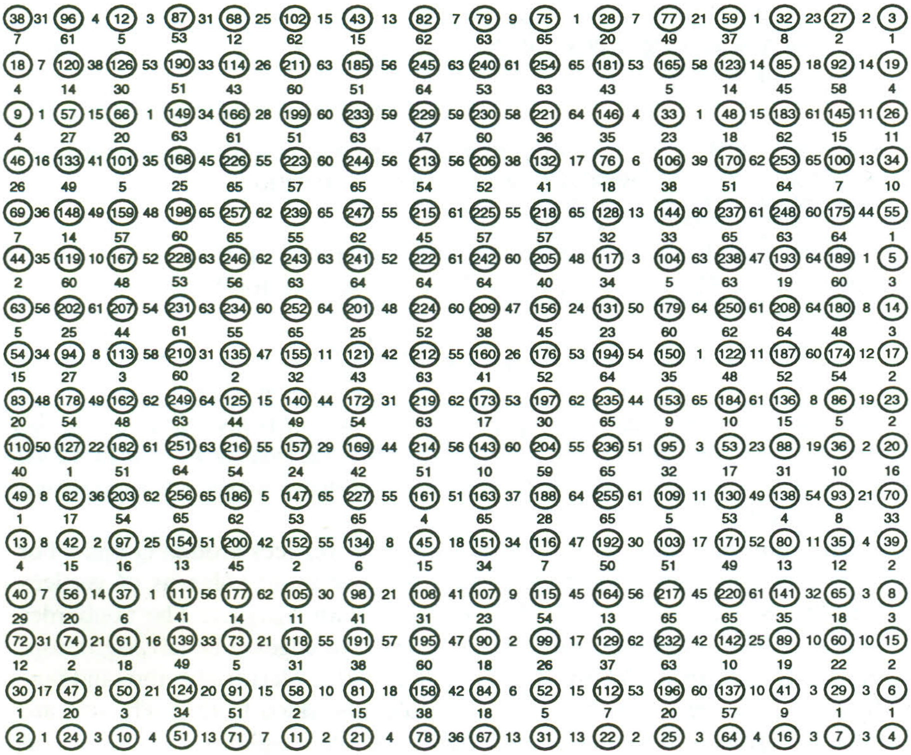

FIGURE 2. An irregular weighting of $X_{16,16}$.

$X_{16,16}$ is the smallest square grid that meets all of the special requirements of each construction. Note that the inductive approach collapses without such a starting point.

We will now define the special properties that are needed in a weighting in order for the induction to work.

Definition. If m and n are both even, then an irregular weighting w of $X_{m,n}$ is *Type 1* if for every edge e, $w(e) \leq (mn)/4 + 1$ $(= \lambda(X_{m,n}))$. Furthermore:

1. If e is a top or bottom heavy edge, then $w(e) \leq (m + 2)(n + 4)/4 - (m + n + 1) - (m + 2)$.
2. If e is a left heavy edge, then $w(e) \leq (m + 2)(n + 4)/4 - 2m - 3$.
3. If e is a right heavy edge, then $w(e) \leq (m + 2)(n + 4)/4 - m - n - 3$.
4. There is a one-factor f in $X_{m,n}$ that has the property that for all $e \in f$, $w(e)$ is less than or equal to the previous constraints minus three. Thus:

   (a) If e is internal or a nonheavy border edge, then $w(e) \leq (mm)/4 - 2$.

   (b) If e is a top or bottom heavy edge, then $w(e) \leq (m + 2)(n + 4)/4 - (m + n + 1) - (m + 2) - 3$.

(c) If e is a left heavy edge, then $w(e) \leq (m + 2)(n + 4)/4 - 2m - 6$.
(d) If e is a right heavy edge, then $w(e) \leq (m + 2)(n + 4)/4 - m - n - 6$.

One can see that the weighting given in Figure 2 is a Type 1 weighting of $X_{16,16}$. We can now give the first recursive construction.

Construction 1. If $m, n \geq 16$ are both even and if there is a Type 1 weighting for $X_{m,n}$, then there is a Type 1 weighting for $X_{m+2,n+4}$.

Proof. Assume that w is a Type 1 weighting for $G = X_{m,n}$. We will make a new weighting w' for $X_{m+2,n+4}$ and will show that it also is Type 1. The construction will be described presently. The reader is also directed to Figure 3, where this construction is pictured for the case of $m \equiv 0 \pmod 4$; the edges are labeled with their weights, and heavy edges are represented with thick lines.

To begin the construction, first add a ring of vertices around G and all of the necessary connecting edges. Then add two more columns of vertices (and necessary edges) to the right side to obtain $X_{m+2,n+4}$. The nonborder edges of the original $X_{m,n}$ will be termed *internal* edges. The left, right, top, and bottom edges in the original $X_{m,n}$ are called *old* left, right, top, and bottom edges, respectively. The final two columns added to the right side are called columns $n + 3$ and $n + 4$. We define the weighting function w' on $X_{m+2,n+4}$ as follows:

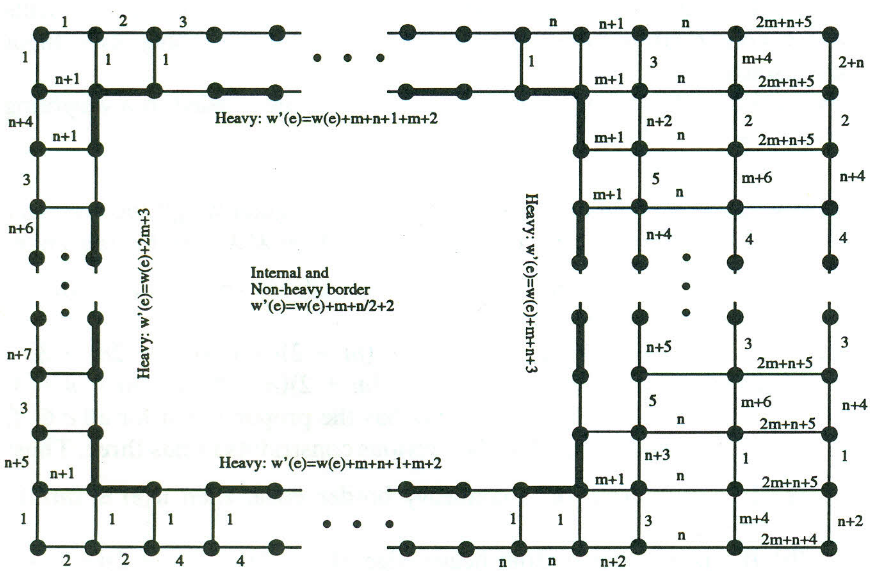

FIGURE 3. A labeling for $X_{m+2,n+4}$ when $m \equiv 0 \pmod 4$.

1. $w'(e) = w(e) + ((m + n)/2 + 1) + (m/2 + 1)$, if e is an internal edge or a nonheavy old border edge.
2. $w'(e) = w(e) + (m + n + 1) + (m + 2)$, if e is an old top or bottom heavy edge.
3. $w'(e) = w(e) + (2m + 3)$, if e is an old left heavy edge.
4. $w'(e) = w(e) + (m + n + 3)$, if e is an old right heavy edge.
5. $w'(e) = n + 1$, if e is a horizontal edge from the old left to the new left.
6. $w'(e) = 1$, if e is a vertical edge from the old top or bottom to the new top or bottom.
7. $w'(e) = m + 1$, if e is a horizontal edge from the old right to column $n + 2$.
8. $w'(e) = n$, if e is a horizontal edge from column $n + 2$ to column $n + 3$.
9. $w'(e) = 2m + n + 5$, if e is a horizontal edge from column $n + 3$ to column $n + 4$; except if $m \equiv 0 \pmod 4$, then the bottom edge gets weight $2m + n + 4$.
10. The $n + 1$ edges along the new bottom have weights (from left to right):

$$m \equiv 0 \pmod 4\text{: } 2, 2, 4, 4, 6, 6, \ldots, n, n, n + 2\,;$$
$$m \equiv 2 \pmod 4\text{: } 1, 3, 3, 5, 5, \ldots, n - 1, n - 1, n + 1, n + 1\,.$$

11. The $m + 1$ edges in the new left border have weights (from bottom to top):

$$m \equiv 0 \pmod 4\text{: } 1, n + 5, 3, n + 7, 5, \ldots, n + m/2 + 3,\ m/2 + 1, n + m/2 + 2, \ldots, 5, n + 6, n + 4, 1\,;$$
$$m \equiv 2 \pmod 4\text{: } 2, n + 4, 4, n + 6, \ldots, m/2 + 1, n + m/2 + 3,\ m/2, \ldots, 5, n + 6, 3, n + 4, 1\,.$$

12. The $n + 1$ edges in the path forming the new top are assigned weights (going left to right): $1, 2, 3, 4, \ldots, n, n + 1$.
13. The $m + 1$ edges in column $n + 2$ have weights (from bottom to top):

$$m \equiv 0 \pmod 4\text{: } 3, n + 3, 5, n + 5, \ldots, m/2 + 1, n + m/2 + 1,\ m/2 + 3, n + m/2, \ldots, n + 4, 5, n + 2, 3\,;$$
$$m \equiv 2 \pmod 4\text{: } 4, n + 2, 6, n + 4, \ldots, m/2 + 3, n + m/2 + 1,\ m/2 + 2, \ldots, n + 4, 5, n + 2, 3\,.$$

14. The $m + 1$ edges in column $n + 3$ have weights (from bottom to top):

$$m \equiv 0 \pmod 4)\text{: } m + 4, 1, m + 6, 3, \ldots, m + m/2 + 2, m/2 - 1, m + m/2 + 4, m/2, m + m/2 + 2, \ldots, 4, m + 6, 2, m + 4;$$

$$m \equiv 2 \pmod 4)\text{: } m + 4, 2, m + 6, 4, \ldots, m + m/2 + 1, m/2 - 1, m + m/2 + 3, m/2 + 1, m + m/2 + 2, m/2 - 1, \ldots, 4, m + 5, 2, m + 3.$$

15. The $m + 1$ edges in the path forming column $n + 4$ (the new right edges) are assigned weights (going bottom to top):

$$m \equiv 0 \pmod 4)\text{: } 2 + n, 1, 4 + n, 3, \ldots, m/2 + n, m/2 - 1, m/2 + 2 + n, m/2, m/2 + n, \ldots, 4, 4 + n, 2, 2 + n;$$

$$m \equiv 2 \pmod 4)\text{: } 2 + n, 2, 4 + n, 4, \ldots, n + m/2 - 1, m/2 - 1, n + m/2 + 1, m/2 + 1, n + m/2, m/2 - 1, \ldots, 4, 3 + n, 2, 1 + n.$$

There are three main conditions that now must be checked: that the edge weightings satisfy the conditions of Type 1, that the vertex sums are all distinct, and that there is a one-factor f in the new graph satisfying the definition in Type 1. For all edges e one can check that the following four conditions are all met under the hypothesis that $m, n \geq 16$:

1. If e is a nonheavy edge, then $w'(e) \leq (m + 2)(n + 4)/4 + 1$.
2. If e is a top or bottom heavy edge, then $w'(e) \leq mn/4 + 5$.
3. If e is a left heavy edge, then $w'(e) \leq mn/4 + n + 5$.
4. If e is a right heavy edge, then $w'(e) \leq mn/4 + m + 5$.

To prove that w' is an irregular weighting of $X_{m+2,n+4}$, it must be shown that all of the vertex sums are distinct. Note that for each old vertex v in the original $X_{m,n}$, $w'(v) = w(v) + 4m + 2n + 8$. So for all old vertices, $w'(v) \geq 4m + 2n + 10$. Since these were all distinct under the weighting w, they are all distinct under the weighting w'. It can be checked that the vertex sums of the new vertices are all distinct, and less than $4m + 2n + 10$, under the new weighting w'. Thus w' is an irregular weighting of the grid $X_{m+2,n+4}$.

We finally check that there is a one-factor f in $X_{m+2,n+4}$ satisfying conditions 4(a)–4(d) of the definition of Type 1. Construct the one-factor f as follows: beginning from the bottom edge on the left side, take every other edge up the left side, across the top until column $n + 1$, down column $n + 2$ to the bottom, then across the bottom to the left. In columns $n + 3$ and $n + 4$ take every other vertical edge beginning with the bottom one. In the old $X_{m,n}$ take the edges that were in the special one-factor that exists by the inductive hypothesis. It is not difficult to check that all of these edges have weights satisfying conditions 4(a)–4(d) of the definition of Type 1. This completes the proof of this construction. ∎

Our next construction is very similar to Construction 1; however, this construction extends $X_{m,n}$ to $X_{m+2,n+2}$ by putting a ring around the smaller grid. As an ingredient in this construction one can use a weighting of $X_{m,n}$ resulting from the application of Construction 1 or this construction can be used inductively. We must again begin with a definition.

Definition. If m and n are both even, then an irregular weighting w of $X_{m,n}$ is *Type 2* if for every edge e, $w(e) \leq (mn)/4 + 1$ $(= \lambda(X_{m,n}))$. Furthermore:

1. If e is a top or bottom heavy edge, then $w(e) \leq (m + 2)(n + 2)/4 - (m + n + 1)$.
2. If e is a left heavy edge, then $w(e) \leq (m + 2)(n + 2)/4 - m - 1$.
3. If e is a right heavy edge, then $w(e) \leq (m + 2)(n + 2)/4 - n - 1$.
4. There is a one-factor f in $X_{m,n}$ that has the property that for all $e \in f$, $w(e)$ is less than or equal to the previous constraints minus three. Thus,

   (a) If e is internal or a nonheavy border edge, then $w(e) \leq (mn)/4 - 2$.
   (b) If e is a top or bottom heavy edge, then $w(e) \leq (m + 2)(n + 2)/4 - (m + n + 1) - 3$.
   (c) If e is a left heavy edge, then $w(e) \leq (m + 2)(n + 2)/4 - m - 4$.
   (d) If e is a right heavy edge, then $w(e) \leq (m + 2)(n + 2)/4 - n - 4$.

It is easy to prove the following proposition:

Proposition 2.1. If $m, n \geq 16$, then any Type 1 weighting of $X_{m,n}$ is also a Type 2 weighting of $X_{m,n}$.

Construction 2. If there is a Type 2 weighting of $X_{m,n}$, then there is a Type 2 weighting for $X_{m+2,n+2}$.

Proof. Assume that w is a Type 2 weighting for $G = X_{m,n}$. We will make a new weighting w' for $X_{m+2,n+2}$ that we will also show to be Type 2. The reader is directed to Figure 4 where this construction is pictured. To begin the construction add a ring of vertices around G and all of the necessary connecting edges. We use the same terminology as in Construction 1. Define the weighting function w' on $X_{m+2,n+2}$ as follows:

1. $w'(e) = w(e) + (m + n)/2 + 1$, if e is an internal edge or a nonheavy border edge.
2. $w'(e) = w(e) + (m + n + 1)$, if e is an old top or bottom heavy edge.
3. $w'(e) = w(e) + (m + 1)$, if e is an old left heavy edge.
4. $w'(e) = w(e) + (n + 1)$, if e is an old right heavy edge.
5. $w'(e) = n + 1$, if e is a horizontal edge from the old left to the new left.

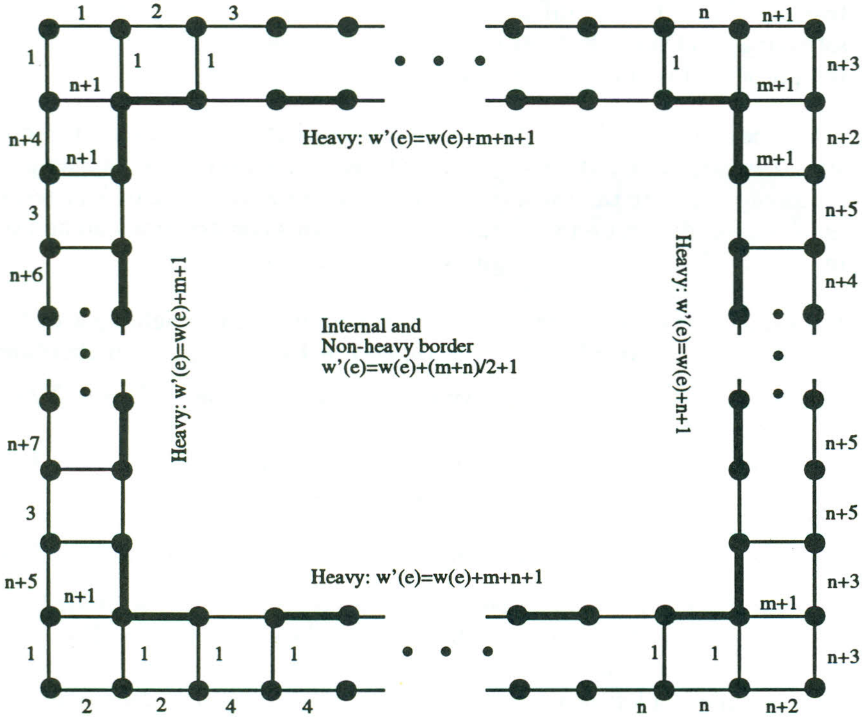

FIGURE 4. A labeling for $X_{m+2,n+2}$ when $m \equiv 0 \pmod 4$.

6. $w'(e) = 1$, if e is a vertical edge from the old top or bottom to the new top or bottom.
7. $w'(e) = m + 1$, if e is a horizontal edge from the old right to the new right.
8. The $n + 1$ edges along the new bottom have weights (from left to right):

$$m \equiv 0 \pmod 4\text{: } 2, 2, 4, 4, 6, 6, \ldots, n, n, n + 2;$$

$$m \equiv 2 \pmod 4\text{: } 1, 3, 3, 5, 5, \ldots, n - 1, n - 1, n + 1, n + 1.$$

9. The $m + 1$ edges in the new left border have weights (from bottom to top):

$$m \equiv 0 \pmod 4\text{: } 1, n + 5, 3, n + 7, 5, \ldots, n + m/2 + 3,$$
$$m/2 + 1, n + m/2 + 2, \ldots, 5, n + 6, 3, n + 4, 1;$$

$$m \equiv 2 \pmod 4\text{: } 2, n + 4, 4, n + 6, \ldots, m/2 + 1, n + m/2 + 3,$$
$$m/2, \ldots, 5, n + 6, 3, n + 4, 1.$$

10. The $n + 1$ edges in the path forming the new top are assigned weights (going left to right): $1, 2, 3, 4, \ldots, n, n + 1$.

11. The $m + 1$ edges in the new right column have weights (from bottom to top):

$$m \equiv 0 \pmod 4: n + 3, n + 3, n + 5, n + 5, \ldots, n + m/2 + 1, n + m/2 + 1, n + m/2 + 3, n + m/2, \ldots, n + 4, n + 5, n + 2, n + 3;$$

$$m \equiv 2 \pmod 4: n + 4, n + 2, n + 6, n + 4, \ldots, n + m/2 + 3, n + m/2 + 1, n + m/2 + 2, \ldots, n + 4, n + 5, n + 2, n + 3.$$

It is straightforward to check that for every edge e, $w'(e)$ is less than the number prescribed in the definition of Type 2. In checking the vertex sums, note that the new border vertices all have distinct sums that are less than $2n + 2m + 6$. One can also see that for each internal vertex v, $w'(v) = w(v) + 2n + 2m + 4$; so they are also all distinct and $w'(v) \geq 2n + 2m + 6$. Finally, the one-factor f of edges with weights satisfying the conditions stated in the definition of Type 2 is composed of the edges in the one-factor in $X_{m,n}$ (which exists by the inductive hypothesis) and every other edge on the new border. ∎

We are now in a position to give the irregularity strength for a larger class of grids. Define the *feasible region* to be

$$\mathcal{F} = \{(x, y) \in Z \times Z: 2\lfloor (x + 2)/4 \rfloor + 8 \leq y \leq 4\lfloor x/2 \rfloor - 15\}.$$

This is the infinite region in the first quadrant that is approximately bounded by the two lines through (16, 16) with slopes 1/2 and 2; it is the shaded region in Figure 5. In the main theorem of this paper (Theorem 2.7) we will give the irregularity strength for all $m \times n$ grids where (m, n) is in the feasible region. Here, from Constructions 1 and 2 along with Theorem 1.1 and the weighting of the 16×16 grid displayed in Figure 2, we have the following theorem:

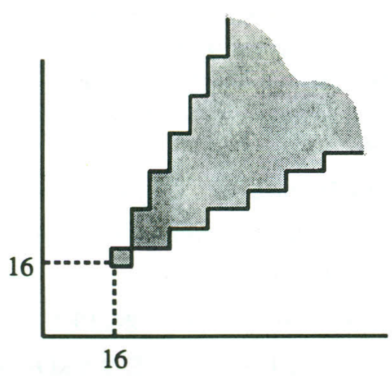

FIGURE 5. The feasible region.

Theorem 2.2. For every even m, n in the feasible region, $I(X_{m,n}) = \lambda(X_{m,n}) = mm/4 + 1$.

Next we present a construction that begins with a weighting w (with certain properties) on $G = X_{m,n}$ for even m and n, and yields an irregular weighting on $X_{m,n+1}$ of strength $\lambda(X_{m,n+1})$. We first define *Type 3* weightings that exhibit these necessary properties.

Definition. If m and n are both even, then an irregular weighting w on $X_{m,n}$ is *Type 3* if for every edge e, $w(e) \leq (mn)/4 + 1$ $(= \lambda(X_{m,n}))$. Furthermore:

1. If e is a border heavy edge, then $w(e) \leq \lceil (m(n+1)+1)/4 \rceil - m/2$.
2. There is one-factor f in $X_{m,n}$ that has the property that for all $e \in f$,
   (a) if e is not a heavy edge, then $w(e) \leq (mn)/4 - 2$;
   (b) if e is a heavy edge, then $w(e) \leq \lceil (m(n+1)+1)/4 \rceil - m/2 - 3$.

The following proposition enables us to input weightings of the grids from the prior two constructions into Construction 3; it is easily verified.

Proposition 2.3. The following three classes of weightings are all Type 3 when $m, n \geq 16$:

1. the weighting of $X_{16,16}$ in Figure 2;
2. the weighting of $X_{m+2,n+4}$ produced by applying Construction 1 to a Type 1 weighting of $X_{m,n}$;
3. the weighting of $X_{m+2,n+2}$ produced by applying Construction 2 to a Type 2 weighting of $X_{m,n}$.

Construction 3. If there is a Type 3 weighting of $X_{m,n}$, then there is an irregular weighting of $X_{m,n+1}$ of strength $\lambda(X_{m,n+1})$ and an irregular weighting of $X_{m+1,n}$ of strength $\lambda(X_{m+1,n})$.

Proof. Assume that w is a Type 3 weighting for $G = X_{m,n}$. We will make a new weighting w' for $X_{m,n+1}$ that will be irregular and have strength of $\lambda(X_{m,n+1})$. The construction begins by adding a new column of m vertices to the right side of G. We use the same terminology as in the previous constructions. As before, the weighting function w' on $X_{m,n+1}$ is dependent on whether $m \equiv 0 \pmod 4$ or $m \equiv 2 \pmod 4$.

1. If e is an internal edge or a nonheavy border edge, then
   (a) $w'(e) = w(e) + \lfloor m/4 \rfloor + 3$, if $e \in f$ and $m \equiv 2 \pmod 4$;
   (b) $w'(e) = w(e) + \lfloor m/4 \rfloor$, otherwise.
2. If e is a top, left, or bottom heavy edge, then
   (a) $w'(e) = w(e) + 2\lfloor m/4 \rfloor + 3$, if $e \in f$ and $m \equiv 2 \pmod 4$;
   (b) $w'(e) = w(e) + 2\lfloor m/4 \rfloor$, otherwise.

3. If e is an old right heavy edge, then

   (a) $w'(e) = w(e) + 2\lfloor m/4 \rfloor + 2$, if $e \in f$ and $m \equiv 2 \pmod 4$;
   (b) $w'(e) = w(e) + 2\lfloor m/4 \rfloor - 1$, otherwise.

4. $w'(e) = 1$, if e connects an old right border vertex to a new right border vertex.
5. The $m - 1$ edges in the new right border are weighted from bottom to top as follows:

$$m \equiv 0 \pmod 4\text{: } 2, 2, 4, 4, \ldots, m/2 - 2, m/2 - 2, m/2, m/2, m/2 - 1, m/2 - 2, \ldots, 3, 2, 1\,;$$

$$m \equiv 2 \pmod 4\text{: } 2, 1, 4, 3, \ldots, m/2 - 1, m/2 - 2, m/2 + 1, m/2, m/2 - 2, m/2 - 2, \ldots, 5, 5, 3, 3, 1\,.$$

Again, one can check that for an edge e, $w'(e) \leq \lambda(X_{m,n+1})$. To see that the weighting is irregular, note that since the weighting w on G was irregular, then all of the old vertices under weighting w' will also have distinct sums; further, the m vertices in the new right border all have distinct sums that are less than the sum of any of the old vertices under w'. ∎

We have shown that $I(X_{m,n+1}) = \lambda(X_{m,n+1})$. To show that $I(X_{m+1,n}) = \lambda(X_{m+1,n})$, one merely needs to take the transpose of the weighting on $X_{m,n}$, apply the above construction adding the new column (getting a weighting of $X_{m,n+1}$), and then transpose again to get a weighting of $X_{m+1,n}$ with strength $\lambda(X_{m+1,n})$.

Next we will present a construction that begins with a weighting w on $G = X_{m,n}$ for even m and n where $m \equiv n \pmod 4$, and yields an irregular weighting on $X_{m+1,n+1}$ of strength $\lambda(X_{m+1,n+1})$. We first define *Type 4* weightings that exhibit the necessary properties for this construction.

Definition. If m and n are both even, then an irregular weighting w on $X_{m,n}$ is *Type 4* if for every edge e, $w(e) \leq (mn)/4 + 1$ $(= \lambda(X_{m,n}))$. Furthermore:

1. If e is a border heavy edge, then $w(e) \leq \lceil ((m+1)(n+1)+1)/4 \rceil - (m+n)/2$.
2. There is a one-factor f in $X_{m,n}$ that has the property that for all $e \in f$,

   (a) if e is not a heavy edge, then $w(e) \leq (mn)/4 - 2$;
   (b) if e is a heavy edge, then $w(e) \leq \lceil ((m+1)(n+1)+1)/4 \rceil - (m+n)/2 - 3$.

The following proposition enables us to input weightings of the grids from Constructions 1 and 2 into Constructions 4a and 4b; it is easily verified.

Proposition 2.4. The following three classes of weightings are all Type 4 when $m, n \geq 16$:

1. the weighting of $X_{16,16}$ in Figure 2;
2. the weighting of $X_{m+2,n+4}$ produced by applying Construction 1 to a Type 1 weighting of $X_{m,n}$;
3. the weighting of $X_{m+2,n+2}$ produced by applying Construction 2 to a Type 2 weighting of $X_{m,n}$.

Construction 4a. If there is a Type 4 weighting of $X_{m,n}$ where $m \equiv n \pmod 4$, then there is an irregular weighting of $X_{m+1,n+1}$ of strength $\lambda(X_{m+1,n+1})$.

Proof. Assume that w is a Type 4 weighting for $G = X_{m,n}$ and $m \equiv n \pmod 4$. We will make a new weighting w' for $X_{m+1,n+1}$ that will be irregular and have strength of $\lambda(X_{m+1,n+1})$. The construction, which is pictured in Figure 6 (for $m \equiv n \equiv 0 \pmod 4$ and $m < n$), begins by adding a new column of vertices to the right side of G and a new row to the bottom. The weighting function w' on $X_{m+1,n+1}$ is dependent on whether $m \equiv n \equiv 0 \pmod 4$ or $m \equiv n \equiv 2 \pmod 4$.

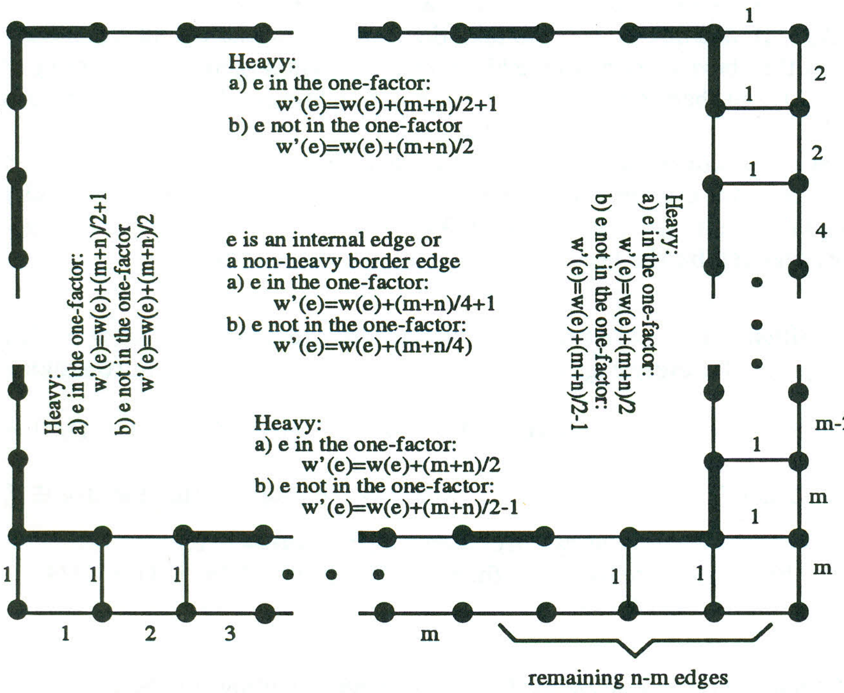

FIGURE 6. A labeling for $X_{m+1,n+1}$ when $m \equiv n \equiv 0 \pmod 4$ and $m < n$.

1. If e is an internal edge or a nonheavy border edge, then
   (a) $w'(e) = w(e) + (m + n)/4 + 1$, if $e \in f$;
   (b) $w'(e) = w(e) + (m + n)/4$, otherwise.
2. If e is a top or left heavy edge, then
   (a) $w'(e) = w(e) + (m + n)/2 + 1$, if $e \in f$;
   (b) $w'(e) = w(e) + (m + n)/2$, otherwise.
3. If e is an old bottom heavy edge, then
   (a) $w'(e) = w(e) + (m + n)/2$, if $e \in f$;
   (b) $w'(e) = w(e) + (m + n)/2 - 1$, otherwise.
4. If e is an old right heavy edge, then
   (a) $w'(e) = w(e) + (m + n)/2 - 1$, if $m \equiv 0 \pmod 4$ and $e \notin f$;
   (b) $w'(e) = w(e) + (m + n)/2$, if $m \equiv 0 \pmod 4$ and $e \in f$;
   (c) $w'(e) = w(e) + m/2 - 1$, if $m \equiv 2 \pmod 4$ and $e \notin f$;
   (d) $w'(e) = w(e) + m/2$, if $m \equiv 2 \pmod 4$ and $e \in f$.
5. $w'(e) = 1$, if e connects an old bottom vertex to a new bottom vertex.
6. If e connects an old right vertex to a new right vertex, then
   (a) $w'(e) = 1$, if $m \equiv 0 \pmod 4$;
   (b) $w'(e) = n/2 + 1$, if $m \equiv 2 \pmod 4$.
7. If $m \equiv 2 \pmod 4$, then the weights on the new border edges are as follows:
   (a) The m edges in the new right border are weighted from top to bottom:
   $$\begin{gathered} n/2 + 2, 2, n/2 + 4, 4, \ldots, n/2 + m/2 - 3, m/2 - 3, \\ n/2 + m/2 - 1, m/2 - 1, n/2 + m/2 + 1, m/2, \\ n/2 + m/2 - 1, \ldots, 5, n/2 + 4, 3, n/2 + 2, 1\,; \end{gathered}$$
   (b) The n edges along the new bottom are weighted from left to right:
   $$\begin{gathered} 3, 2, 5, 4, \ldots, n/2 - 3, n/2, n/2 - 1, n/2 + 1, n/2, \\ n/2 - 2, n/2 - 2, \ldots, 3, 3, 1, 1\,. \end{gathered}$$
8. If $m \equiv 0 \pmod 4$ and $m = n$, then weights on the new border edges are as follows:
   (a) The m edges in the path forming the new right border are assigned weights from top to bottom: $2, 2, 4, 4, \ldots, m - 2, m - 2, m, m + 1$;
   (b) The m edges in the path forming the new bottom border are assigned weights from left to right: $1, 2, 3, \ldots, m - 2, m - 1, m$.

9. If $m \equiv 0 \pmod 4$ and $m \neq n$ (we may assume $m < n$), then the weights on the new border edges are as follows:
   (a) The m edges in the path forming the new right border are assigned weights from top to bottom: $2, 2, 4, 4, \ldots, m-2, m-2, m, m$;
   (b) The first m edges, starting from the left, in the path forming the new bottom border are assigned weights: $1, 2, 3, \ldots, m-2, m-1, m$;
   (c) The remaining $n - m$ edges in the bottom border are assigned weights: First note that since $m \equiv n \equiv 0 \pmod 4$ and $m < n$, then $n - m = 4k$ for some positive integer k. When $k = 1$, the four $(= n - m)$ remaining edges in the bottom border are, from left to right, $m + 2k, m + 2k + 1, m + 2k - 1, m + 2k$. We call these four edges the *center* edges. When $k > 1$, there are $2k - 2$ edges on either side of the center edges. The weights for the $n - m$ remaining edges (where the center edges are enclosed in the box) are $m + 2, m + 2, m + 4, m + 4, \ldots, m + 2k - 2, m + 2k - 2,$

$$\boxed{m + 2k, m + 2k + 1, m + 2k - 1, m + 2k,}$$

$$m + 2k - 3, m + 2k - 2, \ldots, m + 3, m + 4, m + 1, m + 2.$$

One can verify that for every edge e, $w'(e) \leq \lambda(X_{m+1,n+1})$. Given that the weighting w on G was irregular and that all old vertices have had their sums increased by $m + n + 1$, then all of the old vertices under weighting w' will have distinct sums greater than $m + n + 2$. It is easy to verify that the vertices on the new borders all have distinct sums in the range of 2 to $m + n + 2$. ∎

We now handle the construction of a weighting for $X_{m+1,n+1}$ from a Type 4 weighting for $X_{m,n}$ when m and n are both even and $m \not\equiv n \pmod 4$. We first observe that in this case $I(X_{m,n}) > \lambda(X_{m,n})$. This observation is based on the following theorem, which states that $I(G) > \lambda(G)$ whenever weightings of strength $\lambda(G)$ are *tight* (all possible sums are used) and the sum of all vertex sums is odd. (In the following theorem, $d(G)$ and $D(G)$ are the minimum and maximum degrees respectively of a graph G.)

Theorem 2.5. $I(G) > \lambda(G)$ whenever $D(G)\lambda(G) - d(G) + 1 = |V(G)|$ and any one of the following three conditions hold:

1. $|V(G)| \equiv 1 \pmod 4$ and $d(G), D(G)\lambda(G)$ are both odd;
2. $|V(G)| \equiv 2 \pmod 4$;
3. $|V(G)| \equiv 3 \pmod 4$ and $d(G), D(G)\lambda(G)$ are both even.

Proof. Let $d = d(G), D = D(G)$, and $\lambda = \lambda(G)$. In any irregular weighting of a graph G with strength λ, the smallest vertex sum will be at least

d and the greatest vertex sum will be at most $D\lambda$. When $D\lambda - d + 1 = |V(G)|$, then all of the sums from d to $D\lambda$ inclusive will be assigned to vertices. Since each edge weight contributes to two vertex sums, the sum of all vertex sums must be even. But when $D\lambda - d + 1 = |V(G)|$, there are three cases, which are stated in the theorem, where $\Sigma_{i=d}^{D\lambda} i$ is odd; in these cases, $I(G) > \lambda(G)$. ∎

Theorem 2.5 is a general statement of a technique used in [1], [4], and [9]. Case 3 of the theorem provides the following corollary:

Corollary 2.6. When m, n are both odd, $m, n \geq 3$, and $m \not\equiv n \pmod 4$, $I(X_{m,n}) > \lambda(X_{m,n})$, with the exception of $X_{3,5}$.

The grid $X_{3,5}$ is an exception to Corollary 2.6 because $\lambda(X_{3,5}) = (mn + 1)/4 + 1 = 5$. So, in this case, $D\lambda - d + 1 \neq |V(G)|$.

We now give a construction that begins with a Type 4 weighting w on $G = X_{m,n}$ (where m and n are both even and $m \not\equiv n \pmod 4$) and yields an irregular weighting on $X_{m+1,n+1}$ of strength $\lambda(X_{m+1,n+1}) + 1$. This, in conjunction with Corollary 2.6, will show that $I(X_{m+1,n+1}) = \lambda(X_{m+1,n+1}) + 1$.

Construction 4b. If there is a Type 4 weighting of $X_{m,n}$ where $m \not\equiv n \pmod 4$, then there is an irregular weighting of $X_{m+1,n+1}$ of strength $\lambda(X_{m+1,n+1}) + 1$.

Proof. Assume that w is a Type 4 weighting for $G = X_{m,n}$, $m \not\equiv n \pmod 4$. We will make a new weighting w' for $X_{m+1,n+1}$ that will be irregular and have strength of $\lambda(X_{m+1,n+1}) + 1$. Without loss of generality we will assume that $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$. The construction begins by adding a new column of vertices to the right side of G, and a new row to the bottom. The weighting function w' on $X_{m+1,n+1}$ is as follows:

1. If e is an internal edge or a nonheavy border edge, then
   (a) $w'(e) = w(e) + (m + n - 2)/4$, if $e \notin f$;
   (b) $w'(e) = w(e) + (m + n - 2)/4 + 4$, if $e \in f$.
2. If e is an old top or left heavy edge, then
   (a) $w'(e) = w(e) + (m + n - 2)/2$, if $e \notin f$;
   (b) $w'(e) = w(e) + (m + n - 2)/2 + 4$, if $e \in f$.
3. If e is an old bottom or right heavy edge, then
   (a) $w'(e) = w(e) + (m + n - 2)/2 - 1$, if $e \notin f$;
   (b) $w'(e) = w(e) + (m + n - 2)/2 + 3$, if $e \in f$.
4. $w'(e) = 1$, if e connects the old border to the new border.
5. The m edges in the new right border receive weights from top to bottom: $2, 2, 4, 4, \ldots, m/2 - 1, m/2 - 1, m/2 + 1, m/2, m/2 - 1, m/2 - 2, \ldots, 4, 3, 2, 1$.

6. The n edges along the new bottom border receive weights from left to right: $m + 2, 2, m + 4, 4, \ldots, m + n/2 - 2, n/2 - 2, m + n/2, n/2 + 1, m + n/2 + 1, n/2 - 1, m + n/2 - 2, n/2 - 3, \ldots, m + 4, 3, m + 2, 1$.

For every edge e, $w'(e) \leq \lambda(X_{m+1,n+1}) + 1$; only some of the edges in the one-factor f will have weights of $\lambda(X_{m+1,n+1}) + 1$. Given that the weighting w on G was irregular and that all old vertices have had their sums increased by $m + n + 2$, all of the old vertices under weighting w' will have distinct sums greater than $m + n + 3$. The vertices on the new borders all have distinct sums in the range of 2 to $m + n + 3$; the sum $m + n$ is not used. ∎

We now state what is known about the irregularity strength of grids for all m, n pairs in the feasible region. From Theorem 2.2, Propositions 2.3 and 2.4, Constructions 3, 4a, and 4b, and Corollary 2.6, we have the following theorem:

Theorem 2.7. For every m, n in the feasible region,

$$I(X_{m,n}) = \begin{cases} \lambda(X_{m,n}) + 1 & \text{if } m, n \text{ are both odd and } m \not\equiv n \pmod 4 \\ \lambda(X_{m,n}) & \text{otherwise}. \end{cases}$$

3. CONCLUSION

In this paper we have extended what is known about the irregularity strength of the $m \times n$ grids, $X_{m,n}$. In [1] the irregularity strength of paths ($X_{1,n}$) was determined, [4] gave the irregularity strength of $X_{2,n}$, and [7] found the irregularity strength of $X_{n,n}$ for $n \leq 23$. Here we determined $I(X_{m,n})$ for all (m, n) inside of the *feasible region* (defined below and in Section 2). Additionally, we have used algorithm IS-1 from [7] to find minimal irregular weightings for all $X_{m,n}$ where $3 \leq m, n \leq 16$; these weightings are listed in [3]. Note that this result, together with Theorem 2.7, proves the conjecture from [7] that $I(X_{n,n}) = \lambda(X_{n,n})$ for all $n \geq 3$. Figure 7 shows the set of (m, n) pairs for which $I(X_{m,n})$ is known.

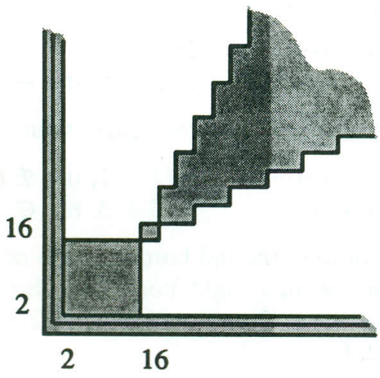

FIGURE 7. Grids for which the irregularity strength is known.

Theorem 3.1 summarizes what is known about the irregularity strength of grids. Recall that we define the feasibility region to be

$$\mathcal{F} = \{(x, y) \in Z \times Z: 2\lfloor (x+2)/4 \rfloor + 8 \le y \le 4\lfloor x/2 \rfloor - 15\}.$$

Theorem 3.1.

$$I(X_{m,n}) = \begin{cases} \lambda(X_{1,n}) & \text{if } n > 2 \text{ and } n \not\equiv 2 \pmod 4 & [1] \\ \lambda(X_{1,n}) + 1 & \text{if } n > 2 \text{ and } n \equiv 2 \pmod 4 & [1] \\ \lambda(X_{2,n}) & \text{if } n \ge 2 \text{ and } n \not\equiv 1 \pmod 6 & [4] \\ \lambda(X_{2,n}) + 1 & \text{if } n \ge 2 \text{ and } n \equiv 1 \pmod 6 & [4] \\ \lambda(X_{m,n}) & \text{if } m = 3 \text{ and } n = 5 & [3] \\ \lambda(X_{m,n}) & \text{if } 3 \le m, n \le 16 \text{ and } mn \not\equiv 3 \pmod 4 & [3] \\ \lambda(X_{m,n}) + 1 & \text{if } 3 \le m, n \le 16,\ mn \ne 15, \text{ and } mn \equiv 3 \pmod 4 & [3] \\ \lambda(X_{m,n}) & \text{if } (m, n) \in \mathcal{F} \text{ and } mn \not\equiv 3 \pmod 4 & \\ \lambda(X_{m,n}) + 1 & \text{if } (m, n) \in F \text{ and } mn \equiv 3 \pmod 4. & \end{cases}$$

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