

THE EXISTENCE OF N_2 RESOLVABLE LATIN SQUARES*

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Abstract. An N_2 resolvable Latin square is a Latin square with no 2×2 subsquares that also has an orthogonal mate. In this paper we show that N_2 resolvable Latin squares exist for all orders n with $n \neq 2, 4, 6, 8$.

Key words. orthogonal Latin squares, resolvable designs

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1. Introduction. A *Latin square of order n* is an $n \times n$ matrix with entries from an n -set V , where every row and every column is a permutation of V . Labeling the rows and columns by V , it is convenient to view a Latin square as a pair (V, B) , where B is a set of ordered triples on V such that $(x, y, z) \in B$ if and only if the cell at row x and column y contains the entry z for $x, y, z \in V$. The six *conjugates* of a Latin square are obtained by permuting the coordinates of B . General information about Latin squares can be found in [4].

A Latin square (V, B) is said to be N_2 if it contains no subsquare of order 2, that is, if there do not exist four distinct triples in B of the form

$$(x, y, a), (x, z, b), (w, y, b), (w, z, a).$$

Restricting the Latin square to rows x and w and columns y and z , the subsquare of order 2 looks like

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Notice that a Latin square is N_2 if and only if a conjugate of the square is N_2 . In 1991, Heinrich [7] proved that N_2 Latin squares exist for all orders $n \neq 2$ or 4. See [7] for an excellent discussion of the history up to that point.

A *transversal T* of a Latin square (V, B) is a subset of B consisting of n triples such that any two distinct triples (x, y, z) and (x', y', z') of T have $x \neq x'$, $y \neq y'$, and $z \neq z'$. A Latin square (V, B) of order n is said to be *resolvable* if B can be partitioned into n transversals. We say that a Latin square is an N_2 *resolvable Latin square* if it is both N_2 and resolvable. In this paper we will abbreviate an N_2 resolvable Latin square as N_2 RLS and say that there exists an N_2 RLS(n) when there is an N_2 resolvable Latin square of order n . There is obviously no N_2 RLS for orders $n = 2, 4, 6$. In [8] it was stated that there is no N_2 RLS(8). We reaffirmed this fact by checking the three nonisomorphic N_2 Latin squares of order 8 [5]. After several hours of computing time we found that none was resolvable.

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The motivation for finding N_2 resolvable Latin squares comes from a search for anti-Pasch Kirkman triple systems. An *anti-Pasch Kirkman triple system of order n* , abbreviated anti-Pasch KTS(n), is a pair (V, B) , where V is an n -set and B is a set of unordered triples of V such that the following hold:

1. (V, B) is a Steiner triple system; i.e., every distinct pair of elements of V lies in exactly one triple of B .
2. B is resolvable; i.e., B can be partitioned into parallel classes: $B = \cup\{C_i\}$, where each C_i is a partition of V (i.e., (V, B) is a Kirkman triple system).
3. B does not contain any Pasch configurations. A *Pasch configuration* is a set of four distinct unordered triples of the form

$$\{x, y, z\}, \{x, a, b\}, \{w, y, b\}, \{w, a, z\}.$$

Anti-Pasch Steiner triple systems were shown to exist for all orders $n \equiv 1$ or $3 \pmod{6}$ ($n \neq 7$ or 13) by Grannell, Griggs, and Whitehead in 2000 [6]. However, the existence question for anti-Pasch Kirkman triple systems is made more difficult by the resolvability requirement. We believe that anti-Pasch Kirkman triple systems are interesting in their own right, but additionally these systems play an important role in low density parity check codes. In fact, anti-Pasch KTS(n)'s are equivalent to $(n(n-1)/6, n, 3, 4)$ -error correcting codes with optimal check disk overhead. For more information on these codes, see [1].

It is well known that a necessary condition for an anti-Pasch KTS(n) is that $n \equiv 3 \pmod{6}$. In [1], it was shown that anti-Pasch KTS(n)'s exist for orders $n \equiv 9 \pmod{18}$. Finding such systems of the remaining orders $n \equiv 3, 15 \pmod{18}$ seems difficult. Many orders of these remaining classes will realize anti-Pasch Kirkman triple systems through a recursive construction given in [1] which produces an anti-Pasch KTS($2vn+1$) from an anti-Pasch KTS($2v+1$) and an anti-Pasch KTS($2n+1$). However, for this construction to work, it also requires the use of an N_2 resolvable Latin square of order n . This motivates the question, For which n do N_2 RLS(n) exist?

In the sections that follow, it is convenient to use the language of transversal designs when describing the constructions of N_2 resolvable Latin squares. For this reason, it is important to be familiar with the relationship between transversal designs and N_2 resolvable Latin squares.

A *transversal design of order n with blocksize k* , denoted by TD(k, n), is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$, where

1. \mathcal{V} is a set of kn elements;
2. \mathcal{G} is a partition of V into k groups, each of size n ;
3. \mathcal{B} is a set of k -subsets of B called *blocks*;
4. every unordered pair of elements of V lies in exactly one group or exactly one block but not both.

Note that a TD(k, n) has n^2 blocks.

Conventionally, we write the elements of a TD(k, n) as $V \times K$, where V is an n -set and K is a k -set and the groups are the sets of the form $V \times \{a\}$ for $a \in K$. Blocks of the design are of the form $\{(v_1, a_1), (v_2, a_2), \dots, (v_k, a_k)\}$, where $K = \{a_1, a_2, \dots, a_k\}$. One can identify a block with an ordered k -tuple of V by transforming $\{(v_1, a_1), (v_2, a_2), \dots, (v_k, a_k)\}$ to (v_1, v_2, \dots, v_k) . Writing these tuples as an array of column vectors, we form what is termed an *orthogonal array*, OA(k, n). Throughout this paper we periodically identify blocks of a TD as columns of the corresponding OA.

The following well-known lemma gives the relationship between Latin squares and TDs.

LEMMA 1.1. A $\text{TD}(3, n)$ is equivalent to a Latin square of order n .

Proof. Let V be an n -set. The identification of a Latin square of order n on V and a $\text{TD}(3, n)$ on $V \times \{1, 2, 3\}$ is the following:

$$\begin{array}{ll} \text{triple from Latin square:} & \text{block from } \text{TD}(3, n): \\ (x, y, z) & \iff \{(x, 1), (y, 2), (z, 3)\}. \end{array} \quad \square$$

In the above identification, we could have instead made a correspondence between any of the conjugates of the Latin square and the $\text{TD}(3, n)$ by permuting the groups of the TD. For convention, a Latin square created from a TD as above is referred to as *the* corresponding Latin square for the TD. We now investigate notions of N_2 and resolvability for TDs.

An N_2 - $\text{TD}(k, n)$ is a $\text{TD}(k, n)$ on $V \times \{1, 2, \dots, k\}$ such that for every three distinct groups, say $V \times \{i\}$, $V \times \{j\}$, and $V \times \{k\}$, there do not exist four distinct subblocks of the form

$$\{(x, i), (y, j), (a, k)\}, \{(x, i), (z, j), (b, k)\}, \{(w, i), (y, j), (b, k)\}, \{(w, i), (z, j), (a, k)\}.$$

Such configurations are termed *intercalates*.

An intercalate in a TD corresponds to an intercalate in any of the conjugates of a corresponding Latin square. Thus we have the following.

LEMMA 1.2. A $\text{TD}(3, n)$ is N_2 if and only if any of the corresponding Latin squares is N_2 .

Let V be an n -set and K a k -set. A *resolvable* $\text{TD}(k, n)$ on $V \times K$ is a TD where the blocks can be partitioned into n parallel classes such that each class is a partition of $V \times K$. The following lemma states an equivalence between resolvable Latin squares and resolvable TDs. The proof is clear, so we omit it.

LEMMA 1.3. A *resolvable Latin square of order n* is equivalent to a *resolvable $\text{TD}(3, n)$* .

The following is a well-known fact about resolvable TDs (see [4]).

LEMMA 1.4. A *resolvable $\text{TD}(k, n)$* is equivalent to a $\text{TD}(k + 1, n)$.

Combining Lemmas 1.2, 1.3, and 1.4, we can understand N_2 resolvable Latin squares as a special kind of transversal design.

LEMMA 1.5. An N_2 *resolvable Latin square of order n* is equivalent to a $\text{TD}(4, n)$ which can be truncated to an N_2 - $\text{TD}(3, n)$.

If T is a $\text{TD}(4, n)$, then let $\text{Trunc}(T, 4)$ denote the transversal design $\text{TD}(3, n)$ obtained from T by truncating the last (fourth) group.

In this paper we will show that $N_2\text{RLS}(n)$'s exist for all orders n with $n \neq 2, 4, 6, 8$. More specifically we will give $\text{TD}(4, n)$, which can be truncated to N_2 - $\text{TD}(3, n)$ for all these orders of n which then, by Lemma 1.5, give us $N_2\text{RLS}(n)$. In section 2 we will cover the case when either n is odd. In section 3 we give some product-type constructions that are particularly useful in the case when $n \equiv 0 \pmod{4}$. Unfortunately the nonexistence of $N_2\text{RLS}(4)$ and $N_2\text{RLS}(8)$ makes this case much less straightforward than if they had existed. Section 4 contains a very powerful Wilson-type recursive construction that will be used for the remainder of the orders. Finally, in section 5 we put the constructions together to complete the spectrum of $N_2\text{RLS}(n)$. The appendix contains constructions for some small orders that were constructed by the use of a computer.

2. N_2 resolvable Latin squares of odd order. In [1], it is shown that $N_2\text{RLS}(n)$'s exist for all odd n . Although this fact is elementary, we provide a proof to emphasize the flexibility in choosing the parameters of the construction.

LEMMA 2.1. *Let n be any odd positive integer. Then there exists an N_2 resolvable Latin square of order n .*

Proof. Let n be any odd number, and let \mathbb{U}_n be the multiplicative group of units of the ring \mathbb{Z}_n . Choose $s_1, s_2, t_1, t_2 \in \mathbb{U}_n$, where (s_1, s_2) and (t_1, t_2) are linearly independent over \mathbb{Z}_n , and choose distinct elements $a, b \in \mathbb{Z}_n$. Construct a $\text{TD}(4, n)$ on $\mathbb{Z}_n \times \{1, 2, 3, 4\}$ where the blocks are given by $\{(x, 1), (y, 2), (s_1x + s_2y + a, 3), (t_1x + t_2y + b, 4)\}$ over $x, y \in \mathbb{Z}_n$.

By the choice of s_1, s_2, t_1 , and t_2 , it is clear that we indeed have a $\text{TD}(4, n)$. Now truncate the TD by removing the group $\mathbb{Z}_n \times \{4\}$ and consider the induced resolvable Latin square of order n on \mathbb{Z}_n with triples B defined by

$$B = \{(x, y, s_1x + s_2y + a) : x, y \in \mathbb{Z}_n\}.$$

We claim that this $\text{TD}(3, n)$ is N_2 . Suppose, on the contrary, that there is an intercalate in B . Then there are four distinct triples of B of the form

$$(x, y, s_1x + s_2y + a), (x, z, s_1x + s_2z + a), (w, y, s_1w + s_2y + a), (w, z, s_1w + s_2z + a),$$

where $s_1x + s_2z + a = s_1w + s_2y + a$ and $s_1x + s_2y + a = s_1w + s_2z + a$.

Subtracting the second equation from the first gives $s_2(z - y) = s_2(y - z)$ and thus $2(z - y) = 0$. Since n is odd, it must be that $z = y$, a contradiction. Thus the resolvable Latin square is N_2 . \square

3. Three product constructions. To prove that an N_2 Latin square of order $4n$ exists, we will make use of the following standard product construction of TDs.

LEMMA 3.1. *Let V be an n -set, and let W be an m -set. Suppose we are given a $\text{TD}(k, n)$ on $V \times \{1, 2, \dots, k\}$ with block set B . Further suppose that for each $b \in B$, we have a $\text{TD}(k, m)$ on $W \times \{1, 2, \dots, k\}$ with block set C_b . Then there exists a $\text{TD}(k, mn)$ on $V \times W \times \{1, 2, \dots, k\}$.*

Proof. Beginning with the $\text{TD}(k, n)$ on $V \times \{1, 2, \dots, k\}$, we will construct a $\text{TD}(k, mn)$ on $V \times W \times \{1, 2, \dots, k\}$. Form the set of blocks D as follows.

For each $b = \{(b_1, 1), (b_2, 2), \dots, (b_k, k)\} \in B$, we *inflate* b by C_b ; i.e., for each $c = \{c_1, 1), (c_2, 2), \dots, (c_k, k)\} \in C_b$, we include in D blocks of the form

$$\{(b_1, c_1, 1), (b_2, c_2, 2), \dots, (b_k, c_k, k)\}.$$

It is clear that D forms the blocks of a $\text{TD}(k, mn)$ on $v \times W \times \{1, 2, \dots, k\}$. Also, it is interesting to note that this TD contains a copy of each C_b : For $b = \{(b_1, 1), (b_2, 2), \dots, (b_k, k)\} \in B$ consider the copy of C_b obtained by replacing each element $(w, i) \in W \times \{1, 2, \dots, k\}$ with (b_i, w, i) . The blocks of this copy are contained in D . \square

For future reference, in the product construction in Lemma 3.1, the $\text{TD}(k, n)$ is referred to as the *master design* and the $\text{TD}(k, m)$'s are referred to as the *ingredients*. The construction in Lemma 3.1 allows us to create an $N_2\text{RLS}(mn)$ from one of order m and one of order n . We give this multiplication next.

LEMMA 3.2. *Suppose there exist an N_2 resolvable Latin square of order n and an N_2 resolvable Latin square of order m . Then there exists an N_2 resolvable Latin square of order mn .*

Proof. Let V be an n -set and let W be an m set. Let B be the blocks of a $\text{TD}(4, n)$ on $V \times \{1, 2, 3, 4\}$ such that $\text{Trunc}(B, 4)$ yields an N_2 -TD, and let C be the blocks of a $\text{TD}(4, m)$ on $W \times \{1, 2, 3, 4\}$ such that $\text{Trunc}(C, 4)$ yields an N_2 -TD. Apply Lemma 3.1 with B as the master and with m^2 copies of C as the ingredients

to create a $TD(4, mn)$ which we call M . Let $T = \text{Trunc}(M, 4)$. It remains to show that T has no intercalates. Define two blocks of T to be on the same level if they were derived from the same block of B , i.e., if the projection of the blocks to the first and third coordinates are equal. Otherwise, we say that they are on different levels.

Now suppose that there exists an intercalate Q of T . We have two cases to consider:

1. *There are two blocks of Q at the same level.* It follows that all the blocks are at the same level as each other. But then projecting the blocks to the second and third coordinates yields an intercalate in $\text{Trunc}(C, 4)$, which cannot happen.
2. *Every block of Q is at a different level.* Then projecting Q to the first and third coordinates yields an intercalate of B , which cannot happen.

Thus T is an $N_2\text{RLS}(mn)$. □

We next show that there is an $N_2\text{RLS}(4n)$ when n is odd. It is well known that no N_2 Latin square of order 4 exists, so this proof is not as simple as applying Lemma 3.2. However, by a careful selection of a set of 16 $TD(4, n)$'s and a $TD(4, 4)$ we can implement the product construction given in Lemma 3.1 to produce an $N_2\text{RLS}(4n)$.

LEMMA 3.3. *Suppose n is an odd number with $n > 1$. Then there exists an N_2 resolvable square of order $4n$.*

Proof. Let n be an odd number with $n > 1$. Let M be the blocks of the following $TD(4, 4)$ on $\{0, 1, 2, 3\} \times \{1, 2, 3, 4\}$, written as the following $OA(4, 4)$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 \end{pmatrix}.$$

For each block $b \in M$, we define a $TD(4, n)$ on $\mathbb{Z}_n \times \{1, 2, 3, 4\}$. The blocks of the new TD , viewed as an OA , are generated by developing over all $x, y \in \mathbb{Z}_n$ the (i, j) position of the square S below, where (i, j) is the unique pair such that $(i, 1), (j, 2) \in b$. The $TD(4, n)$ constructed in this manner is denoted $B_{(i,j)}$ or, equivalently, B_b .

$i \setminus j$	0	1	2	3
0	$\begin{pmatrix} x \\ y \\ -x+y+1 \\ -x-y \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ x+y \\ -x+y \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ -x-y \\ x-y \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ -x+y \\ x+y \end{pmatrix}$
1	$\begin{pmatrix} x \\ y \\ -x+y \\ -x-y \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ x+y \\ x-y \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ -x+y+1 \\ x+y \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ -x-y \\ -x+y \end{pmatrix}$
2	$\begin{pmatrix} x \\ y \\ -x-y \\ x-y \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ x+y \\ -x+y \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ -x+y \\ x+y \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ -x+y+1 \\ x+y \end{pmatrix}$
3	$\begin{pmatrix} x \\ y \\ x-y \\ x+y \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ -x+y+1 \\ x+y \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ x-y \\ -x-y \end{pmatrix}$	$\begin{pmatrix} x \\ y \\ x-y \\ x+y \end{pmatrix}$

Notice that each B_b is N_2 (by the proof of Lemma 2.1). Now create a $TD(4, 4n)$ on $\{0, 1, 2, 3\} \times \mathbb{Z}_n \times \{1, 2, 3, 4\}$ by applying Lemma 3.1 to M and $\{B_b\}$. Define T as the block set of this TD. We will show that $Trunc(T, 4)$ is N_2 .

As in the proof of Lemma 3.2, define two blocks of $Trunc(T, 4)$ to be on the same level if they were derived from the same block of $Trunc(M, 4)$. Otherwise, we again say that they are on different levels. So assume, on the contrary, that there is an intercalate Q in $Trunc(T, 4)$. Thus we have two cases to consider: either there are two blocks in Q at the same level or no two blocks in Q are at the same level.

Assume there are two blocks in Q at the same level. It follows that all blocks must be at the same level and all were derived from, say, $b \in M$. But then projecting to the second coordinates gives us an intercalate from B_b , which cannot happen.

Now assume that no two blocks in Q are at the same level. Thus projecting to the first coordinates yields an intercalate in $Trunc(M, 4)$. Consider the corresponding Latin square for $Trunc(M, 4)$:

$$L = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

Note that there are 12 intercalates in L , and so we have 12 cases to consider depending on which intercalate from L contributed to the blocks of Q . Given the intercalate from L , we can determine which cells of S were used to determine the blocks B_b that contributed to Q . The union of the blocks of Q contains elements of the form $(-, x_1, 1)$, $(-, x_2, 1)$, $(-, y_1, 2)$, and $(-, y_2, 2)$. For each case, we consider the relations between x_1, x_2, y_1 , and y_2 that must hold and show that there is no solution. We illustrate this with one of the 12 intercalates of L .

Suppose Q was derived from the intercalate of L in the upper left-hand corner

$$\begin{pmatrix} \boxed{0} & \boxed{1} & 2 & 3 \\ \boxed{1} & \boxed{0} & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

Then the TDs $B_{(0,0)}$, $B_{(0,1)}$, $B_{(1,0)}$, and $B_{(1,1)}$ contributed to Q , and thus Q looks like

$$\begin{aligned} & \{ \{ (0, x_1, 1), (0, y_1, 2), (0, -x_1 + y_1 + 1, 3) \}, \{ (0, x_1, 1), (1, y_2, 2), (1, x_1 + y_2, 3) \}, \\ & \{ (1, x_2, 1), (0, y_1, 2), (1, -x_2 + y_1, 3) \}, \{ (1, x_2, 1), (1, y_2, 2), (0, x_2 + y_2, 3) \} \}, \text{ where} \\ & -x_1 + y_1 + 1 = x_2 + y_2 \quad \text{and} \quad x_1 + y_2 = -x_2 + y_1. \end{aligned}$$

But this is an inconsistent system of equations, and so Q could not have existed in this case.

In the following table we list the relations of x_1, x_2, y_1 , and y_2 that must be satisfied for Q to exist for each of the 12 intercalates of L . In each case it is straightforward to check that the resulting equations are inconsistent. Thus there are no intercalates in $Trunc(T, 4)$, and hence there exists an N_2 resolvable Latin square of order $4n$.

Number	Intercalate	Equations
1	$\begin{pmatrix} \boxed{0} & \boxed{1} & 2 & 3 \\ \boxed{1} & \boxed{0} & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$	$-x_1 + y_1 + 1 = x_2 + y_2, x_1 + y_2 = -x_2 + y_1.$
2	$\begin{pmatrix} 0 & 1 & \boxed{2} & \boxed{3} \\ 1 & 0 & \boxed{3} & \boxed{2} \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$	$-x_1 - y_1 = -x_2 - y_2, -x_1 + y_2 = -x_2 + y_1 + 1$
3	$\begin{pmatrix} 0 & 1 & 2 & 3 \\ \boxed{1} & \boxed{0} & 3 & 2 \\ \boxed{2} & \boxed{3} & 0 & 1 \\ \boxed{3} & \boxed{2} & 1 & 0 \end{pmatrix}$	$-x_1 - y_1 = -x_2 + y_2 + 1, x_1 + y_2 = x_2 - y_1$
4	$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & \boxed{0} & \boxed{1} \\ 3 & 2 & \boxed{1} & \boxed{0} \end{pmatrix}$	$-x_1 + y_1 = x_2 - y_2, -x_1 + y_2 + 1 = x_2 - y_1$
5	$\begin{pmatrix} \boxed{0} & 1 & \boxed{2} & 3 \\ \boxed{1} & 0 & 3 & 2 \\ \boxed{2} & 3 & \boxed{0} & 1 \\ \boxed{3} & 2 & 1 & 0 \end{pmatrix}$	$-x_1 + y_1 + 1 = -x_2 + y_2, -x_1 - y_2 = -x_2 - y_1$
6	$\begin{pmatrix} 0 & \boxed{1} & 2 & \boxed{3} \\ 1 & 0 & 3 & 2 \\ 2 & \boxed{3} & 0 & \boxed{1} \\ 3 & 2 & 1 & 0 \end{pmatrix}$	$x_1 + y_1 = -x_2 + y_2 + 1, -x_1 + y_2 = x_2 + y_1$
7	$\begin{pmatrix} 0 & 1 & 2 & 3 \\ \boxed{1} & 0 & \boxed{3} & 2 \\ \boxed{2} & 3 & \boxed{0} & 1 \\ \boxed{3} & 2 & \boxed{1} & 0 \end{pmatrix}$	$-x_1 + y_1 = x_2 - y_2 + 1, -x_1 + y_2 + 1 = x_2 - y_1$
8	$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & \boxed{0} & 3 & \boxed{2} \\ 2 & 3 & 0 & \boxed{1} \\ 3 & \boxed{2} & 1 & \boxed{0} \end{pmatrix}$	$x_1 + y_1 = x_2 - y_2, -x_1 - y_2 = -x_2 + y_1 + 1$
9	$\begin{pmatrix} \boxed{0} & 1 & 2 & \boxed{3} \\ \boxed{1} & 0 & 3 & 2 \\ \boxed{2} & 3 & 0 & 1 \\ \boxed{3} & 2 & 1 & \boxed{0} \end{pmatrix}$	$-x_1 + y_1 + 1 = x_2 - y_2, -x_1 + y_2 = x_2 - y_1$
10	$\begin{pmatrix} 0 & 1 & 2 & 3 \\ \boxed{1} & 0 & 3 & \boxed{2} \\ \boxed{2} & 3 & 0 & \boxed{1} \\ \boxed{3} & 2 & 1 & 0 \end{pmatrix}$	$-x_1 + y_1 = -x_2 + y_2 + 1, -x_1 - y_2 = -x_2 - y_1$
11	$\begin{pmatrix} 0 & \boxed{1} & \boxed{2} & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & \boxed{2} & \boxed{1} & 0 \end{pmatrix}$	$x_1 + y_1 = x_2 - y_2, -x_1 - y_2 = -x_2 + y_1 + 1$
12	$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & \boxed{0} & \boxed{3} & 2 \\ 2 & \boxed{3} & \boxed{0} & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$	$x_1 + y_1 = -x_2 + y_2, -x_1 + y_2 + 1 = x_2 + y_1$

□

In order to show that N_2 resolvable Latin squares exist for all multiples of 4, the next step is to show that they exist for order $8n$ when n is odd. To do this, we will use a TD(4, 8) that is obtained from a quasi-difference matrix.

A *quasi-difference matrix* on $\mathbb{Z}_n \cup \{\infty_1, \infty_2, \dots, \infty_m\}$ with r rows, denoted by QDM(n, r, m), is an $r \times (n + 2m)$ matrix $M = (m_{i,j})$ on $\mathbb{Z}_n \cup \{\infty_1, \infty_2, \dots, \infty_m\}$ such that the following hold:

1. For any two rows r_1, r_2 and for any $x \in \mathbb{Z}_n$ there is exactly one column c such that $m_{r_1,c} - m_{r_2,c} \equiv x \pmod{n}$.
2. For every row r and for every $i \in \{1, 2, \dots, m\}$ there is exactly one column c such that $m_{r,c} = \infty_i$.
3. Each column has at most one entry from $\{\infty_1, \infty_2, \dots, \infty_m\}$.

Note: A more general definition of QDMs can be found in [2]. QDMs can be very useful in constructing TDs. In particular, we will use the following standard construction.

LEMMA 3.4. *If there exist a QDM(n, r, m) and a TD(r, m), then there exists a TD($r, m + n$).*

Proof. Suppose Q is a QDM(n, r, m), and let M be an orthogonal array OA(r, m) on $\{\infty_1, \infty_2, \dots, \infty_m\}$ corresponding to the posited TD(r, m). Extend the group operation of \mathbb{Z}_n to $\{\infty_1, \infty_2, \dots, \infty_m\}$ by defining

$$\infty_i + x = x + \infty_i = \infty_i$$

for each $x \in \mathbb{Z}_n$. Now develop the columns of Q by replacing each column v of the matrix with $v + (x, x, x, x)^t$ for $x \in \mathbb{Z}_n$. This gives a partial OA($r, m + n$). Append M to the partial OA to produce an OA($r, m + n$) which is equivalent to a TD($r, m + n$). \square

We are now ready to show that N_2 RLS($8n$)’s exist for all odd $n > 1$. Again, this would be trivial if there existed an N_2 RLS(8); however, as noted earlier, no such square exists.

LEMMA 3.5. *Suppose n is an odd number with $n > 1$. Then there exists an N_2 resolvable Latin square of order $8n$.*

Proof. Let Q be the following QDM(7, 4, 1):

$$Q = \begin{pmatrix} \infty & 0 & 0 & 5 & 0 & 1 & 0 & 3 & 0 \\ 0 & \infty & 5 & 0 & 1 & 0 & 3 & 0 & 0 \\ 3 & 1 & \infty & 4 & 3 & 6 & 4 & 5 & 0 \\ 1 & 3 & 4 & \infty & 6 & 3 & 5 & 4 & 0 \end{pmatrix}.$$

Define M as the blocks of the TD(4, 8) obtained from Q by appending to the developed Q the column $(\infty, \infty, \infty, \infty)^t$. We will construct a TD(4, $8n$) in a manner similar to the way we created a TD(4, $4n$) in the proof of Lemma 3.3. Let b_1 and b_2 be blocks in M . Define an equivalence relation \sim on M by

$b_1 \sim b_2$ if and only if b_1 and b_2 were developed from the same column of Q .

Let $\langle b \rangle$ denote the equivalence class of M containing block b . It is clear that the 10 equivalence classes of M are given by

$$\{\langle c_i \rangle \mid i \in \{1, 2, \dots, 9\}\} \cup \{(\infty, \infty, \infty, \infty)^t\},$$

where c_i is the i th column of Q .

For each block b in the equivalence class $\langle c \rangle$ define a TD(4, n) on $\mathbb{Z}_n \times \{1, 2, 3, 4\}$ with block set B_b defined as the set of blocks developed by \mathbb{Z}_n from the generator block given in following table.

Equivalence class	Generator block
$\langle c_1 \rangle$	$\{(x, 1), (y, 2), (x - y, 3), (x + y, 4)\}$
$\langle c_2 \rangle$	$\{(x, 1), (y, 2), (x - y, 3), (x + y, 4)\}$
$\langle c_3 \rangle$	$\{(x, 1), (y, 2), (x + y, 3), (x - y, 4)\}$
$\langle c_4 \rangle$	$\{(x, 1), (y, 2), (x - y, 3), (x + y, 4)\}$
$\langle c_5 \rangle$	$\{(x, 1), (y, 2), (x - y + 1, 3), (x + y, 4)\}$
$\langle c_6 \rangle$	$\{(x, 1), (y, 2), (x + y, 3), (x - y, 4)\}$
$\langle c_7 \rangle$	$\{(x, 1), (y, 2), (-x - y, 3), (x - y, 4)\}$
$\langle c_8 \rangle$	$\{(x, 1), (y, 2), (-x + y, 3), (x + y, 4)\}$
$\langle c_9 \rangle$	$\{(x, 1), (y, 2), (x - y + 1, 3), (x + y, 4)\}$
$(\infty, \infty, \infty, \infty)^t$	$\{(x, 1), (y, 2), (x + y + 1, 3), (x - y, 4)\}$

Create a TD(4, $8n$) on $(\mathbb{Z}_7 \cup \{\infty\}) \times \mathbb{Z}_n \times \{1, 2, 3, 4\}$ by applying Lemma 3.1 with master design M and ingredients $\{B_b\}$ for $b \in M$. Define T as the block set of this TD. We must show that $\text{Trunc}(T, 4)$ contains no intercalates.

Suppose that there is an intercalate Q in $\text{Trunc}(T, 4)$. Then, as in the proof of Lemma 3.3, it is clear that the blocks of Q must be derived from an intercalate of $\text{Trunc}(M, 4)$. It is easy to check that there are 21 intercalates of $\text{Trunc}(M, 4)$. Based on the action of \mathbb{Z}_7 on the blocks of T , without loss of generality, we need only consider a small set of intercalates of $\text{Trunc}(M, 4)$ that generate all 21 intercalates. There are three generating intercalates. As such we consider the following three cases.

Case 1: Q was derived from the intercalate of $\text{Trunc}(M, 4)$

$$\begin{aligned} & \{(0, 1), (5, 2), (\infty, 3)\}, \\ & \{(0, 1), (\infty, 2), (1, 3)\}, \\ & \{(\infty, 1), (5, 2), (1, 3)\}, \\ & \{(\infty, 1), (\infty, 2), (\infty, 3)\}. \end{aligned}$$

Then Q projected to the second and third coordinates looks like

$$\begin{aligned} & \{(x_1, 1), (y_1, 2), (x_1 + y_1, 3)\}, \\ & \{(x_1, 1), (y_2, 2), (x_1 - y_2, 3)\}, \\ & \{(x_2, 1), (y_1, 2), (x_2 - y_1, 3)\}, \\ & \{(x_2, 1), (y_2, 2), (x_2 + y_2 + 1, 3)\} \end{aligned}$$

for some $x_1, y_1, x_2, y_2 \in \mathbb{Z}_n$, where

$$x_1 + y_1 = x_2 + y_2 + 1 \quad \text{and} \quad x_1 - y_2 = x_2 - y_1.$$

But since these equations are inconsistent, Q cannot exist in this case.

Case 2: Q was derived from the intercalate of $\text{Trunc}(M, 4)$

$$\begin{aligned} & \{(0, 1), (0, 2), (0, 3)\}, \\ & \{(0, 1), (6, 2), (5, 3)\}, \\ & \{(3, 1), (0, 2), (5, 3)\}, \\ & \{(3, 1), (6, 2), (0, 3)\}. \end{aligned}$$

Then Q projected to the second and third coordinates looks like

$$\begin{aligned} & \{(x_1, 1), (y_1, 2), (x_1 - y_1 + 1, 3)\}, \\ & \{(x_1, 1), (y_2, 2), (x_1 + y_2, 3)\}, \\ & \{(x_2, 1), (y_1, 2), (-x_2 + y_1, 3)\}, \\ & \{(x_2, 1), (y_2, 2), (-x_2 - y_2, 3)\} \end{aligned}$$

for some $x_1, y_1, x_2, y_2 \in \mathbb{Z}_n$, where

$$x_1 - y_1 + 1 = -x_2 - y_2 \quad \text{and} \quad x_1 + y_2 = -x_2 + y_1.$$

But these equations are inconsistent. Thus again Q cannot exist.

Case 3: Q was derived from the intercalate of $\text{Trunc}(M, 4)$

$$\begin{aligned} & \{(0, 1), (1, 2), (3, 3)\}, \\ & \{(0, 1), (4, 2), (2, 3)\}, \\ & \{(5, 1), (1, 2), (2, 3)\}, \\ & \{(5, 1), (4, 2), (3, 3)\}. \end{aligned}$$

Then Q projected to the second and third coordinates looks like

$$\begin{aligned} & \{(x_1, 1), (y_1, 2), (x_1 - y_1 + 1, 3)\}, \\ & \{(x_1, 1), (y_2, 2), (-x_1 + y_2, 3)\}, \\ & \{(x_2, 1), (y_1, 2), (-x_2 - y_1, 3)\}, \\ & \{(x_2, 1), (y_2, 2), (x_2 + y_2, 3)\} \end{aligned}$$

for some $x_1, y_1, x_2, y_2 \in \mathbb{Z}_n$, where

$$x_1 - y_1 + 1 = x_2 + y_2 \quad \text{and} \quad -x_1 + y_2 = -x_2 - y_1.$$

These equations are again inconsistent. Thus Q cannot exist.

Hence there are no intercalates of $\text{Trunc}(T, 4)$, and thus it is an N_2 -TD(3, 8n). This implies that there exists an N_2 resolvable Latin square of order $8n$. \square

4. Main recursive construction. In this section we give our main recursive construction. This construction is essentially a ‘‘spike Wilson’’ construction (see [3, section 5]), where great care is taken to ensure the resulting TD is N_2 .

LEMMA 4.1. *Let m and n be positive numbers, and let q be an odd prime power. Suppose there exist an N_2 resolvable Latin square of order $n + m$, an N_2 resolvable Latin square of order $n + 1$, and an N_2 resolvable Latin square of order n . Further suppose that there are m elements $\{\alpha_1, \alpha_2, \dots, \alpha_m\} \subseteq F_q$ such that no α_i is contained in $\{0, 1, -1, -2, -1/2\}$ and for any i, j (not necessarily distinct) none of the following relations holds:*

$$\begin{aligned} \frac{1 + \alpha_i}{\alpha_i} &= -\frac{1 + \alpha_j}{\alpha_j}, & 1 + \alpha_i &= -(1 + \alpha_j), & \alpha_i &= -\alpha_j, \\ 1 + \alpha_i &= \alpha_j, & 1 + \alpha_i &= -\frac{1 + \alpha_j}{\alpha_j}, & \alpha_i &= \frac{\alpha_j}{1 + \alpha_j}. \end{aligned}$$

Then there exists an N_2 resolvable Latin square of order $qn + m$.

Proof. Let V be an n -set. We will construct an $N_2\text{RLS}(qn + m)$ in the form of a TD(4, $qn + m$) on $(\mathbb{F}_q \times V \cup \{\infty_1, \infty_2, \dots, \infty_m\}) \times \{1, 2, 3, 4\}$. Set the master design D as a TD(4, q) having blocks

$$\{(x, 1), (y, 2), (-x - y, 3), (x - y, 4)\} : x, y \in \mathbb{F}_q\}.$$

Using the m elements $\{\alpha_1, \alpha_2, \dots, \alpha_m\} \subseteq F_q$ from the hypothesis, one can define m transversals of the points of D , namely T_1, T_2, \dots, T_m , as

$$T_i = \{(x, 1), (\alpha_i x, 2), (-x - \alpha_i x, 3), (x - \alpha_i x, 4)\} : x \in \mathbb{F}_q\}.$$

Note that these transversals are mutually disjoint except that they all contain the block $\{(0, 1), (0, 2), (0, 3), (0, 4)\}$.

Let A be a $\text{TD}(4, n + m)$ on $(V \cup \{\infty_1, \infty_2, \dots, \infty_m\}) \times \{1, 2, 3, 4\}$ such that $\text{Trunc}(A, 4)$ is N_2 .

Let B be a $\text{TD}(4, n + 1)$ on $(V \cup \{\infty\}) \times \{1, 2, 3, 4\}$ (where ∞ is an element not in $V \cup \{\infty_1, \infty_2, \dots, \infty_m\}$) such that $\text{Trunc}(B, 4)$ is N_2 , and by permuting the elements as necessary, assume B has a block of the form $\{(\infty, 1), (\infty, 2), (\infty, 3), (\infty, 4)\}$; i.e., B has a sub-TD on $\{\infty\}$. For each $i \in \{1, 2, \dots, m\}$, define B_i as the $\text{TD}(4, n + 1)$ obtained from B by replacing ∞ with ∞_i .

Let C be a $\text{TD}(4, n)$ on $V \times \{1, 2, 3, 4\}$ such that $\text{Trunc}(C, 4)$ is N_2 .

We construct a $\text{TD}(4, qn + m)$, W , by inflating the following blocks of the master design D :

- Inflate the block $\{(0, 1), (0, 2), (0, 3), (0, 4)\}$ by A .
- Inflate each transversal $T_i \setminus \{(0, 1), (0, 2), (0, 3), (0, 4)\}$ by B_i for each $i \in \{1, 2, \dots, m\}$.
- For the remaining blocks of D , inflate each block by C .

Two blocks of W are said to be *on the same level* if those blocks were derived from the same block of D .

Since none of the α_i 's is 0, -1 , or 1, it is clear that D is indeed a $\text{TD}(4, q)$ and that the T_i 's are m transversals sharing one common block, so by Wilson's fundamental construction with a spike (see [3]) we have that W is a $\text{TD}(4, qn + m)$. It remains to show that $\text{Trunc}(W, 4)$ is N_2 . Any intercalate of $\text{Trunc}(W, 4)$ cannot have two (or more) blocks derived from the same block of D , since otherwise all the blocks of the intercalate would be at the same level and this would imply that there is an intercalate in either $\text{Trunc}(A, 4)$, $\text{Trunc}(B_i, 4)$, or $\text{Trunc}(C, 4)$, a contradiction. In addition to this, an intercalate must also contain infinite points, since otherwise the form of the intercalate would be an intercalate from $\text{Trunc}(D, 4)$, and $\text{Trunc}(D, 4)$ has no intercalates. So let us proceed by assuming, on the contrary, that there is an intercalate Q of $\text{Trunc}(W, 4)$. With the above restrictions in mind, there are four cases for the structure of the blocks of Q .

Case (1). Q contains a block with three infinite points. That block must have been derived from the block $\{(0, 1), (0, 2), (0, 3)\}$ of $\text{Trunc}(D, 4)$. It follows that the remaining three blocks of the intercalate must contain exactly one infinite point each and must be derived from blocks of various B_i . But then if we project each of the blocks of Q to its corresponding block in B and replace each infinite point with ∞ , we arrive at an intercalate in $\text{Trunc}(B, 4)$, which cannot happen.

Case (2). Q contains a block b with exactly two infinite points. That block must have been derived from the block $\{(0, 1), (0, 2), (0, 3)\}$ of $\text{Trunc}(D, 4)$, and so from the above we can assume that no other block from Q was derived from $\text{Trunc}(A, 4)$. We have three cases to consider based on the form of b .

Subcase (a). b is the block $\{(\infty_j, 1), (\infty_i, 2), (z, 3)\}$ for some i, j and with $z \in V$. Then again b must have been derived from the block $\{(0, 1), (0, 2), (0, 3)\}$ of $\text{Trunc}(D, 4)$. Also, one of the three other blocks in Q must come from a block in transversal T_j of D that has been inflated by B_j and another must come from a block in transversal T_i that was inflated by B_i . The following are the blocks of the intercalate Q as well as the blocks of $\text{Trunc}(D, 4)$ that produce this intercalate.

	intercalate Q	corresponding blocks in $\text{Trunc}(D, 4)$			
$\{(\infty_j, 1),$	$(\infty_i, 2),$	$(z, 3)\}$	$\{(0, 1),$	$(0, 2),$	$(0, 3)\}$
$\{(\infty_j, 1),$	$(\alpha_j x, 2),$	$(-x - \alpha_j x, 3)\}$	$\{(x, 1),$	$(\alpha_j x, 2),$	$(-x - \alpha_j x, 3)\}$
$\{(y, 1),$	$(\infty_i, 2),$	$(-y - \alpha_i y, 3)\}$	$\{(y, 1),$	$(\alpha_i y, 2),$	$(-y - \alpha_i y, 3)\}$
$\{(y, 1),$	$(\alpha_j x, 2),$	$(-y - \alpha_j x, 3)\}$	$\{(y, 1),$	$(\alpha_j x, 2),$	$(-y - \alpha_j x, 3)\}$

Here $x, y \in \mathbb{F}_q \setminus \{0\}$. Furthermore, $0 = -y - \alpha_j x$ and $-x - \alpha_j x = -y - \alpha_i y$. This implies that $1 + \alpha_i = -\frac{1+\alpha_j}{\alpha_j}$, which cannot happen by the hypothesis.

Subcase (b). b is the block $\{(\infty_j, 1), (z, 2), (\infty_i, 3)\}$ for some i, j with $z \in V$. Then as in subcase (a) this implies the following blocks in the intercalate Q and in $\text{Trunc}(D, 4)$.

intercalate Q	corresponding blocks in $\text{Trunc}(D, 4)$
$\{\infty_j, 1), (z, 2), (\infty_i, 3)\}$	$\{(0, 1), (0, 2), (0, 3)\},$
$\{(\infty_j, 1), (\alpha_j x, 2), (-x - \alpha_j x, 3)\}$	$\{(x, 1), (\alpha_j x, 2), (-x - \alpha_j x, 3)\}$
$\{(x + \alpha_j x, 1), (z, 2), (-x - \alpha_j x, 3)\}$	$\{(x + \alpha_j x, 1), (0, 2), (-x - \alpha_j x, 3)\}$
$\{(x + \alpha_j x, 1), (\alpha_j x, 2), (\infty_i, 3)\}$	$\{(x + \alpha_j x, 1), (\alpha_j x, 2), (-x - 2\alpha_j x, 3)\}$

Here $x \in \mathbb{F}_q \setminus \{0\}$. Looking at the fourth block, since it is a block in the parallel class T_i we see that necessarily $\alpha_j x = \alpha_i(x + \alpha_j x)$. It now follows that $\alpha_i = \frac{\alpha_j}{1+\alpha_j}$, which cannot happen by hypothesis.

Subcase (c). b is the block $\{(z, 1), (\infty_i, 2), (\infty_j, 3)\}$ for some i, j with $z \in V$. This implies the following blocks are in the intercalate Q and in $\text{Trunc}(D, 4)$.

intercalate Q	corresponding blocks in $\text{Trunc}(D, 4)$
$\{(z, 1), (\infty_i, 2), (\infty_j, 3)\}$	$\{(0, 1), (0, 2), (0, 3)\}$
$\{(z, 1), (x, 2), (-x, 3)\}$	$\{(0, 1), (\alpha_j x, 2), (-x - \alpha_j x, 3)\}$
$\{(x, 1), (\infty_i, 2), (-x, 3)\}$	$\{(x, 1), (\alpha_i x, 2), (-x - \alpha_i x, 3)\}$
$\{(x, 1), (x, 2), (\infty_j, 3)\}$	$\{(x, 1), (\alpha_j x, 2), (-x - \alpha_j x, 3)\}$

Here $x \in \mathbb{F}_q \setminus \{0\}$. Looking at the second block we see that it must be the case that $-\alpha_j x = -x - \alpha_i x$. This implies that $1 + \alpha_i = \alpha_j$, a contradiction to the hypothesis.

Case (3). Q contains exactly two blocks b_1 and b_2 with one point of infinity and two blocks with no points of infinity. In this case necessarily there is only one point of infinity (say ∞_i) in Q . It is possible that (without loss of generality) b_1 arises from A (the $\text{TD}(4, n + m)$) and b_2 comes from a B_i (a $\text{TD}(4, n + 1)$) placed on blocks in parallel class T_i . It is not difficult to check that because the α_i 's avoid the values $-2, -1/2$, and 1 this configuration never arises. We leave it to the interested reader to check this fact. So, for the remainder of this case we can assume that both b_1 and b_2 come from a B_i . We have three subcases to consider.

Subcase (a). b_1 and b_2 both contain $(\infty_i, 1)$ for some i . Then the following blocks must be in $\text{Trunc}(D, 4)$, where the first block is implied by the existence of b_1 and the second from b_2 :

$$\begin{aligned} &\{(x, 1), (\alpha_i x, 2), (-x - \alpha_i x, 3)\}, \\ &\{(y, 1), (\alpha_i y, 2), (-y - \alpha_i y, 3)\}, \\ &\{(z, 1), (\alpha_i x, 2), (-y - \alpha_i y, 3)\}, \\ &\{(z, 1), (\alpha_i y, 2), (-x - \alpha_i x, 3)\}, \end{aligned}$$

where $x, y, z \in \mathbb{F}_q$ and x and y are distinct. From the third and fourth blocks we have that $z + \alpha_i x = y + \alpha_i y$ and $z + \alpha_i y = x + \alpha_i x$. It follows that $\alpha_i = -1/2$, which cannot happen by hypothesis.

Subcase (b). b_1 and b_2 both contain $(\infty_i, 2)$ for some i . Then as in subcase (a) the following blocks must be in $\text{Trunc}(D, 4)$ (where now the first block is implied by

the existence of b_1 and the third from b_2):

$$\begin{aligned} &\{(x, 1), (\alpha_i x, 2), (-x - \alpha_i x, 3)\}, \\ &\{(x, 1), (y, 2), (-x - y, 3)\}, \\ &\{(z, 1), (\alpha_i z, 2), (-x - y, 3)\}, \\ &\{(z, 1), (y, 2), (-x - \alpha_i x, 3)\}, \end{aligned}$$

where $x, y, z \in \mathbb{F}_q$ and x and z are distinct. From looking at the point in the third group in the third and fourth blocks we get that $-x - y = -z - \alpha_i z$ and $-z - y = -x - \alpha_i x$. It follows that $\alpha_i = -2$, which is assumed to not hold.

Subcase (c). b_1 and b_2 both contain $(\infty_i, 3)$ for some i . Then the following blocks must be in $\text{Trunc}(D, 4)$ (the first block is implied by the existence of b_1 and the fourth from b_2):

$$\begin{aligned} &\{(x, 1), (\alpha_i x, 2), (-x - \alpha_i x, 3)\}, \\ &\{(x, 1), (y, 2), (-x - y, 3)\}, \\ &\{(z, 1), (\alpha_i x, 2), (-x - y, 3)\}, \\ &\{(z, 1), (\alpha_i z, 2), (-z - \alpha_i z, 3)\}, \end{aligned}$$

where $y, z \in \mathbb{F}_q$ and x and z are distinct. Looking at the point y in the second group and the point $-x - y$ in the third group we get the two equations $-z - \alpha_i x = -x - y$ and $y = \alpha_i z$. These equations imply $\alpha_i = 1$, contradicting the hypothesis.

Case (4). Each block of Q contains exactly one point of infinity. It must then be the case that Q contains exactly two infinity elements and they must be in the same group of $\text{Trunc}(D, 4)$. Again it is possible that one block in Q arises from A and the other blocks all come from the B_i 's. It is straightforward to check that this configuration never arises. So, for the remainder of this case we assume all blocks arise from the B_i TDs. There are again three subcases to consider.

Subcase (a). The two infinity elements occur in the first group of $\text{Trunc}(D, 4)$. Assuming that the infinity elements are ∞_i and ∞_j , then the following blocks from $\text{Trunc}(D, 4)$ must be the ones which when inflated yield Q . Note that the first and second blocks are in the parallel class T_i and the third and fourth blocks are in the parallel class T_j :

$$\begin{aligned} &\{(x, 1), (\alpha_i x, 2), (-x - \alpha_i x, 3)\}, \\ &\{(y, 1), (\alpha_i y, 2), (-y - \alpha_i y, 3)\}, \\ &\{(w, 1), (\alpha_i x, 2), (-y - \alpha_i y, 3)\}, \\ &\{(z, 1), (\alpha_i y, 2), (-x - \alpha_i x, 3)\}, \end{aligned}$$

where $\{x, y, w, z\} \subseteq \mathbb{F}_q$, x and y are distinct, and w and z are distinct. Additionally the following conditions must hold:

$$\alpha_i x = \alpha_j w, \quad y - \alpha_i y = -w - \alpha_j w, \quad \alpha_i y = \alpha_j z, \quad -x - \alpha_i x = -z - \alpha_j z.$$

It follows that $\frac{1+\alpha_i}{\alpha_i} = -\frac{1+\alpha_j}{\alpha_j}$, which cannot happen by hypothesis.

Subcase (b). The two infinity elements occur in the second group of $\text{Trunc}(D, 4)$. Assuming that the infinity elements are ∞_i and ∞_j , then the following blocks from

$\text{Trunc}(D, 4)$ must be the ones which when inflated yield Q . Note that the first and third blocks are in the parallel class T_i and the second and fourth blocks are in the parallel class T_j :

$$\begin{aligned} &\{(x, 1), (\alpha_i x, 2), (-x - \alpha_i x, 3)\}, \\ &\{(x, 1), (\alpha_j x, 2), (-x - \alpha_j x, 3)\}, \\ &\{(z, 1), (\alpha_i x, 2), (-x - \alpha_j x, 3)\}, \\ &\{(z, 1), (\alpha_j x, 2), (-x - \alpha_i x, 3)\}, \end{aligned}$$

where $x, z \in \mathbb{F}_q$ and x and z are distinct. Also $-x - \alpha_j x = -z - \alpha_i z$ and $-x - \alpha_i x = -z - \alpha_j z$. This implies that $1 + \alpha_i = -(1 + \alpha_j)$, contradicting the hypothesis.

Subcase (c). The two infinity elements occur in the third group of $\text{Trunc}(D, 4)$. Assuming that the infinity elements are ∞_i and ∞_j , then the following blocks from $\text{Trunc}(D, 4)$ must be the ones which when inflated yield Q . The first and fourth blocks are in the parallel class T_i and the second and third blocks are in the parallel class T_j :

$$\begin{aligned} &\{(x, 1), (\alpha_i x, 2), (-x - \alpha_i x, 3)\}, \\ &\{(x, 1), (\alpha_j x, 2), (-x - \alpha_j x, 3)\}, \\ &\{(y, 1), (\alpha_i x, 2), (-x - \alpha_j x, 3)\}, \\ &\{(y, 1), (\alpha_j x, 2), (-x - \alpha_i x, 3)\}, \end{aligned}$$

where $y \in \mathbb{F}_q$ and x and y are distinct. Also $\alpha_i x = \alpha_j y$ and $\alpha_j x = \alpha_i y$. This implies that $\alpha_i = -\alpha_j$, which contradicts by hypothesis.

Thus, Q cannot exist, and so $\text{Trunc}(W, 4)$ is N_2 . □

5. The spectrum. We begin this section by proving that there exist N_2 resolvable Latin squares of order $4n$ for all $n > 2$.

LEMMA 5.1. *For any $r \geq 4$ there exists an N_2 resolvable Latin square of order 2^r .*

Proof. N_2 resolvable Latin squares of orders 16 and 32 can be found in Appendices A.3 and A.7, respectively. Writing $64 = 7 \times 9 + 1$, let $n = 9, q = 7$, and $m = 1$ in Lemma 4.1 to produce an $N_2\text{RLS}(64)$. Since $128 = 18 \times 7 + 2$ we can again apply Lemma 4.1 with $n = 18, q = 7$, and $m = 2$. Note that an $N_2\text{RLS}(18)$ is given in Appendix A.4 and an $N_2\text{RLS}(20)$ can be produced from Lemma 3.3. Also, for $q = 7, \alpha_1 = 2$ and $\alpha_2 = 4$ can be used to satisfy the hypothesis of Lemma 4.1.

If $r \geq 8$, then recursively apply the product construction, Lemma 3.1, on the aforementioned Latin squares. □

We are now in position to prove that for any $n > 2$ there exists an $N_2\text{RLS}(4n)$.

THEOREM 5.2. *There exists an N_2 resolvable Latin square of order $4n$ if and only if $n > 2$.*

Proof. In light of Lemmas 3.3 and 3.5 we can assume that $n \equiv 0 \pmod{4}$. So write $4n = 2^r k$ with k odd and $r \geq 4$. By Lemma 5.1 there exists an $N_2\text{RLS}(2^r)$, and by Lemma 2.1 there exists an $N_2\text{RLS}(k)$. Thus simply apply Lemma 3.2 on these squares to produce an $N_2\text{RLS}(4n)$. The fact that there is no N_2 resolvable Latin square of orders either 4 or 8 completes the proof. □

We have shown that $N_2\text{RLS}(n)$'s exist for odd n and for $n \equiv 0 \pmod{4}$ ($n \neq 4$ or 8). It now remains to show that there exist $N_2\text{RLS}(n)$'s when $n \equiv 2 \pmod{4}$. We will show that Lemma 4.1 can be applied to cover nearly all of these cases inductively. To see this, we first investigate when the conditions of Lemma 4.1 can be satisfied.

We begin with a definition. Let q be a prime power. Define $R(q)$ as the maximum number m such that there exist m numbers $\alpha_1, \alpha_2, \dots, \alpha_m$ in \mathbb{F}_q such that each $\alpha_i \notin \{0, 1, -1, -\frac{1}{2}, -2\}$ and none of the following relations holds for any i, j :

$$\frac{1 + \alpha_i}{\alpha_i} = -\frac{1 + \alpha_j}{\alpha_j}, \quad 1 + \alpha_i = -(1 + \alpha_j), \quad \alpha_i = -\alpha_j,$$

$$1 + \alpha_i = \alpha_j, \quad 1 + \alpha_i = -\frac{1 + \alpha_j}{\alpha_j}, \quad \alpha_i = \frac{\alpha_j}{1 + \alpha_j}.$$

LEMMA 5.3. *Let $R(q)$ be as above; then $R(q) \geq \lceil \frac{q-5}{9} \rceil$. Also, $R(q) \geq \frac{q+1}{4}$ if $q \equiv 7 \pmod{8}$ and $R(q) \geq \frac{q-11}{4}$ if $q \equiv 3 \pmod{8}$. For some small orders q the following is a list of lower bounds on $R(q)$.*

q	7	9	11	13	17	19
Lower bounds for $R(q)$	2	1	2	2	6	6

Proof. Since $\alpha_i \notin \{0, 1, -1, -\frac{1}{2}, -2\}$ there are five elements in F_q which can never be used for any α_i . Furthermore, the first three of the six restrictions on the α_i 's are symmetric, so any α_i chosen will make at most nine other values unavailable as a choice for an α . Thus, given k α_i 's chosen that avoid the restrictions, there are at most $9k + 5$ other values of \mathbb{F}_q that cannot be chosen to extend the list of α 's. Thus $R(q) \geq \lceil \frac{q-5}{9} \rceil$.

If $q \equiv 3 \pmod{4}$, then choose the α_i 's as the set of elements x in $F_q \setminus \{0, 1, -1, -\frac{1}{2}, -2\}$ such that x is a quadratic residue of F_q and $x + 1$ is a nonresidue. There are $(q + 1)/4$ such elements in F_q . Now, when $q \equiv 7 \pmod{8}$, 2 is a quadratic residue, so none of the values of $x \in \{0, 1, -1, -1/2, -2\}$ has the property that x is a quadratic residue and $x + 1$ is a nonresidue. Thus if $q \equiv 7 \pmod{8}$, then $R(q) \geq \frac{q+1}{4}$. If $q \equiv 3 \pmod{8}$, then the numbers $x = 1, -2, -\frac{1}{2}$ have the property that x is a quadratic residue and $x + 1$ is a nonresidue, and so in this case $R(q) \geq \frac{q+1}{4} - 3 = \frac{q-11}{4}$.

Finally, for each of the small orders q in the hypothesis, we realize the lower bounds listed by providing a list of α_i 's that satisfy the conditions in the definition of $R(q)$:

q	Lower bounds for $R(q)$	α_i 's
7	2	2,4
9	1	x
11	2	2,6
13	3	4, 8
17	6	2,5,7,9,11,14
19	6	2,6,8,10,12,16

The x in the table above is any element of \mathbb{F}_9 of order 4. □

To get enough "base cases," to recursively construct $N_2\text{RLS}(n)$ for all $n \neq \{2, 4, 6, 8\}$ through the application of Lemma 4.1 we will use the following number theoretic result, the proof of which was greatly aided by the results in [9].

LEMMA 5.4. *Every integer $x \geq 17756$ can be written as $x = 9p + m$ with p a prime and $m \leq p/10$.*

Proof. In [9] it was shown that if $x \geq 2010760$, then there is a prime number p between x and $x + x/16587$. For our purposes the implication of this fact is that for $x \geq 18297916$ there is a prime p such that $\frac{10}{91}x \leq p \leq \frac{1}{9}x$. When $17756 \leq x \leq 2010760$ it was easily checked on a computer that there is a prime p such that $\frac{10}{91}x \leq p \leq \frac{1}{9}x$. Hence for all $x \geq 17756$ there is a prime p such that $\frac{10}{91}x \leq p \leq \frac{1}{9}x$.

Now assume that $x \geq 17756$, and let p be a prime with $\frac{10}{91}x \leq p \leq \frac{1}{9}x$. Thus $x = 9p + m$, where $m \leq \frac{1}{91}x \leq \frac{1}{91} \times \frac{91}{10}p = \frac{p}{10}$ as desired. \square

COROLLARY 5.5. *Any number $x \notin \{4, 6, 10, 14, 22, 26, 34, 38, 46, 58, 62, 74, 94, 146, 202\}$ such that x is twice a prime can be written as $x = nq + m$, where $n \geq 9$, q is an odd prime power with $q \geq 7$, and $m \leq R(q)$.*

Proof. If $x \geq 17756$, then apply Lemma 5.4 to write $x = 9p + m$ with $m \leq p/10$ and p a prime. Now from Lemma 5.3 we know that $R(p) \geq p/10$, so $x = 9p + m$ with $m \leq R(p)$ as required. The cases of $x \leq 17755$ were verified via a computer program. \square

We are now ready to prove the main result of this paper.

THEOREM 5.6. *There exists an N_2 resolvable Latin square of order n if and only if $n \neq 2, 4, 6, 8$.*

Proof. We will prove this theorem by induction. Note that there exists $N_2\text{RLS}(n)$ for $n \neq 2, 4, 6, 8$ and $n \leq 9$. Proceeding inductively, we assume that there exist N_2 resolvable Latin squares of order $n \neq 2, 4, 6, 8$ for all $n < k$, where $k \geq 10$. We will show that there exists an $N_2\text{RLS}(k)$. The cases when k is odd or $k \equiv 0 \pmod{4}$ are covered in Lemma 2.1 and Theorem 5.2, respectively.

Assume that $k = 2t$ for some odd t . Direct constructions for the cases of $k \in \{10, 14, 18, 22, 26, 34, 38, 46, 58, 62, 74, 94, 146, 202\}$ are given in the appendix. Now if t is a prime, and $k \notin \{10, 14, 18, 22, 26, 34, 38, 46, 58, 62, 74, 94, 146, 202\}$, then by Corollary 5.5 one can write $k = nq + m$ with $n \geq 9$, $q \geq 7$ an odd prime power and $m \leq R(q)$. Applying Lemma 4.1 gives an $N_2\text{RLS}(k)$. If t is not a prime, and $t \neq 3^s$ for some $s > 1$, then $t = p \times q$, where $p \neq 3$ is an odd prime and q is odd. Since there are an $N_2\text{RLS}(2p)$ and an $N_2\text{RLS}(q)$, by the multiplication theorem, Lemma 3.2, there is an $N_2\text{RLS}(k)$. Finally, if $t = 3^s$ for some $s > 1$, then use the $N_2\text{RLS}(18)$ (in Appendix A.4) and one of order 3^{s-2} to obtain an N_2 resolvable Latin square of order $k = 2t = 18 \times 3^{s-2}$, completing the proof.

As stated earlier, there is no N_2 resolvable Latin square of order n when $n = 2, 4, 6$, or 8 . \square

Appendix. We give constructions for N_2 resolvable Latin squares for some small orders. Most of the constructions involve developing a QDM and appending the result with a particular OA on the points $\{\infty_i\}$. For convenience, we define $\text{In}(m)$ as the OA(4, m) on $\{\infty_0, \infty_1, \dots, \infty_{m-1}\}$ with columns given by $(\infty_i, \infty_j, \infty_{i+j}, \infty_{2i+j})^t$ over $i, j \in \mathbb{Z}_m$.

A.1. Order 10. Develop the columns of the following QDM in \mathbb{Z}_7 to obtain 91 columns:

$$\begin{pmatrix} \infty_0 & 1 & 2 & 6 & \infty_1 & 3 & 0 & 5 & \infty_2 & 1 & 4 & 1 & 5 \\ 3 & \infty_0 & 5 & 0 & 5 & \infty_1 & 5 & 0 & 5 & \infty_2 & 1 & 0 & 5 \\ 4 & 2 & \infty_0 & 2 & 4 & 1 & \infty_1 & 4 & 3 & 5 & \infty_2 & 3 & 5 \\ 4 & 1 & 3 & \infty_0 & 2 & 5 & 4 & \infty_1 & 5 & 6 & 3 & \infty_2 & 1 \end{pmatrix}.$$

Then append $\text{In}(3)$. The first three rows generate an N_2 resolvable Latin square of order 10 as do rows 1, 2, and 4.

A.2. Order 14. Develop the columns of the following QDM in \mathbb{Z}_{11} to obtain 187 columns:

$$\begin{pmatrix} \infty_0 & 8 & 7 & 2 & \infty_1 & 10 & 10 & 8 & \infty_2 & 5 & 4 & 1 & 7 & 7 & 1 & 10 & 0 \\ 3 & \infty_0 & 6 & 4 & 8 & \infty_1 & 10 & 3 & 3 & \infty_2 & 2 & 8 & 0 & 8 & 9 & 4 & 3 \\ 4 & 5 & \infty_0 & 1 & 1 & 10 & \infty_1 & 1 & 10 & 6 & \infty_2 & 3 & 5 & 10 & 8 & 4 & 6 \\ 3 & 2 & 0 & \infty_0 & 10 & 2 & 9 & \infty_1 & 6 & 6 & 10 & \infty_2 & 9 & 4 & 10 & 10 & 7 \end{pmatrix}.$$

Then append $\text{In}(3)$ to obtain an $\text{OA}(4, 14)$. The first three rows generate an N_2 resolvable Latin square of order 14 as do rows 1, 2, and 4.

A.3. Order 16. Develop the 19 columns of the following QDM in \mathbb{Z}_{13} to obtain 247 columns:

$$\begin{pmatrix} \infty_0 & 7 & 1 & 8 & \infty_1 & 8 & 10 & 0 & \infty_2 & 9 & 2 & 6 & 3 & 3 & 6 & 8 & 12 & 11 & 2 \\ 6 & \infty_0 & 6 & 12 & 12 & \infty_1 & 5 & 0 & 7 & \infty_2 & 3 & 8 & 9 & 6 & 5 & 6 & 9 & 7 & 9 \\ 6 & 9 & \infty_0 & 11 & 3 & 8 & \infty_1 & 9 & 10 & 2 & \infty_2 & 5 & 10 & 11 & 3 & 12 & 4 & 9 & 3 \\ 10 & 5 & 6 & \infty_0 & 6 & 1 & 1 & \infty_1 & 8 & 10 & 9 & \infty_2 & 12 & 3 & 3 & 11 & 11 & 6 & 4 \end{pmatrix}.$$

Then append $\text{In}(3)$ to obtain an $\text{OA}(4, 16)$. The first three rows generate an N_2 resolvable Latin square of order 16 as do rows 1, 2, and 4.

A.4. Order 18. Develop the 21 columns of the following QDM in \mathbb{Z}_{15} to obtain 315 columns:

$$\begin{pmatrix} \infty_0 & 3 & 6 & 2 & \infty_1 & 11 & 3 & 6 & \infty_2 & 0 & 13 & 13 & 12 & 0 & 5 & 5 & 7 & 4 & 10 & 7 & 11 \\ 11 & \infty_0 & 6 & 3 & 6 & \infty_1 & 14 & 12 & 11 & \infty_2 & 3 & 10 & 5 & 13 & 8 & 9 & 9 & 13 & 9 & 14 & 6 \\ 7 & 13 & \infty_0 & 4 & 13 & 1 & \infty_1 & 9 & 11 & 4 & \infty_2 & 9 & 10 & 1 & 14 & 11 & 4 & 11 & 3 & 7 & 10 \\ 10 & 8 & 2 & \infty_0 & 10 & 2 & 5 & \infty_1 & 3 & 3 & 8 & \infty_2 & 6 & 7 & 3 & 9 & 6 & 1 & 11 & 7 & 4 \end{pmatrix}.$$

Then append $\text{In}(3)$ to obtain an $\text{OA}(4, 18)$. The first three rows of this OA generate an N_2 resolvable Latin square of order 18 as do rows 1, 2, and 4.

A.5. Order 22. Replace each of the six columns of the matrix below with its four row-cyclic shifts; i.e., replace each column $(a, b, c, d)^t$ with $(a, b, c, d)^t$, $(b, c, d, a)^t$, $(c, d, a, b)^t$, and $(d, a, b, c)^t$:

$$\begin{pmatrix} \infty_0 & \infty_1 & \infty_2 & 9 & 11 & 16 \\ 11 & 15 & 12 & 0 & 0 & 9 \\ 7 & 3 & 4 & 5 & 17 & 15 \\ 9 & 12 & 17 & 8 & 12 & 0 \end{pmatrix}.$$

Then append $(0, 0, 0, 0)^t$ to create a QDM in \mathbb{Z}_{19} and develop the 25 columns to obtain 475 columns. Append $\text{In}(3)$ to obtain an $\text{OA}(4, 22)$. The first three rows generate an N_2 resolvable Latin square of order 22 as do rows 1, 2, and 4.

A.6. Order 26. Replace each of the seven columns of the matrix below with its four row-cyclic shifts:

$$\begin{pmatrix} \infty_0 & \infty_1 & \infty_2 & 1 & 14 & 14 & 1 \\ 0 & 1 & 5 & 21 & 22 & 10 & 8 \\ 13 & 11 & 9 & 16 & 10 & 16 & 0 \\ 14 & 4 & 11 & 10 & 15 & 5 & 21 \end{pmatrix}.$$

Then append the column $(0, 0, 0, 0)^t$ to create a QDM in \mathbb{Z}_{23} with 29 columns. Develop these 29 columns and append $\text{In}(3)$ to obtain an $\text{OA}(4, 26)$. The first three rows of this OA give us an N_2 resolvable Latin square of order 26 as do rows 1, 2, and 4.

A.7. Order 32. Replace each of the nine columns of the matrix below with its four row-cyclic shifts:

$$\begin{pmatrix} \infty_0 & \infty_1 & \infty_2 & \infty_3 & \infty_4 & 10 & 15 & 24 & 15 \\ 15 & 2 & 5 & 23 & 25 & 26 & 5 & 0 & 21 \\ 13 & 20 & 0 & 0 & 21 & 25 & 26 & 14 & 9 \\ 23 & 4 & 12 & 9 & 22 & 3 & 23 & 22 & 22 \end{pmatrix}.$$

Then append $(0, 0, 0, 0)^t$ to form a QDM in \mathbb{Z}_{27} . Develop the 37 columns of the QDM and append $\text{In}(5)$ to obtain an $\text{OA}(4, 32)$. Rows 1, 2, and 3 of this OA generate an N_2 resolvable Latin square of order 32 as do rows 1, 2, and 4.

A.8. Order 34. Replace each of the nine columns of the matrix below with its four row-cyclic shifts:

$$\begin{pmatrix} \infty_0 & \infty_1 & \infty_2 & 30 & 6 & 24 & 23 & 18 & 18 \\ 3 & 13 & 12 & 5 & 20 & 18 & 10 & 29 & 4 \\ 16 & 22 & 7 & 2 & 23 & 14 & 25 & 14 & 27 \\ 4 & 24 & 0 & 10 & 27 & 26 & 16 & 13 & 17 \end{pmatrix}.$$

Append $(0, 0, 0, 0)^t$, develop the 37 columns of the QDM, and append $\text{In}(3)$ to obtain an $\text{OA}(4, 34)$. Rows 1, 2, and 3 of this OA generate an N_2 resolvable Latin square of order 37 as do rows 1, 2, and 4.

A.9. Order 38. Replace each of the 12 columns of the matrix below with its four row-cyclic shifts:

$$\begin{pmatrix} \infty_0 & \infty_1 & \infty_2 & \infty_3 & \infty_4 & \infty_5 & \infty_6 & \infty_7 & \infty_8 & \infty_9 & \infty_{10} & 14 \\ 9 & 4 & 1 & 0 & 6 & 9 & 20 & 12 & 16 & 17 & 9 & 16 \\ 10 & 22 & 26 & 24 & 14 & 23 & 14 & 25 & 1 & 26 & 26 & 15 \\ 21 & 25 & 6 & 20 & 2 & 6 & 9 & 14 & 6 & 3 & 18 & 8 \end{pmatrix}.$$

Then append $(0, 0, 0, 0)^t$ to create a QDM with 37 columns in \mathbb{Z}_{27} . Develop the QDM and append $\text{In}(11)$ to obtain an $\text{OA}(4, 38)$. Rows 1, 2, and 3 of this OA generate an N_2 resolvable Latin square of order 38 as do rows 1, 2, and 4.

A.10. Order 46. Replace each of the 14 columns of the matrix below with its four row-cyclic shifts:

$$\begin{pmatrix} \infty_0 & \infty_1 & \infty_2 & \infty_3 & \infty_4 & \infty_5 & \infty_6 & \infty_7 & \infty_8 & \infty_9 & \infty_{10} & 30 & 21 & 6 \\ 12 & 24 & 33 & 20 & 19 & 13 & 17 & 27 & 21 & 31 & 2 & 11 & 20 & 3 \\ 25 & 25 & 24 & 15 & 22 & 27 & 32 & 20 & 3 & 20 & 12 & 13 & 24 & 25 \\ 13 & 31 & 10 & 34 & 34 & 21 & 28 & 31 & 8 & 5 & 4 & 21 & 14 & 8 \end{pmatrix}.$$

Then append $(0, 0, 0, 0)^t$ to create a QDM in \mathbb{Z}_{35} with 57 columns. Develop the QDM and append $\text{In}(11)$ to form an $\text{OA}(4, 46)$. Rows 1, 2, and 3 of this OA generate an N_2 resolvable Latin square of order 46.

A.11. Order 58. Replace each of the 19 columns of the matrix below with its four row-cyclic shifts:

$$\begin{pmatrix} \infty_0 & \infty_1 & \infty_2 & \infty_3 & \infty_4 & \infty_5 & \infty_6 & \infty_7 & \infty_8 & \infty_9 & \infty_{10} & \infty_{11} & \infty_{12} & \infty_{13} & \infty_{14} & \infty_{15} & \infty_{16} & \infty_{17} & \infty_{18} \\ 38 & 17 & 25 & 21 & 35 & 4 & 8 & 0 & 33 & 19 & 11 & 23 & 23 & 30 & 33 & 35 & 23 & 38 & 37 \\ 31 & 12 & 28 & 5 & 22 & 25 & 30 & 24 & 8 & 27 & 22 & 17 & 29 & 31 & 7 & 21 & 25 & 27 & 16 \\ 8 & 29 & 1 & 20 & 10 & 30 & 11 & 4 & 0 & 37 & 19 & 7 & 28 & 1 & 5 & 28 & 21 & 18 & 20 \end{pmatrix}.$$

Then append $(0, 0, 0, 0)^t$ to create a QDM in \mathbb{Z}_{39} with 77 columns. Develop the QDM and append $\text{In}(19)$ to form an $\text{OA}(4, 58)$. Rows 1, 2, and 3 of this OA generate an N_2 resolvable Latin square of order 58.

A.12. Order 62. Replace each of the 20 columns of the matrix below with its four row-cyclic shifts:

$$\begin{pmatrix} \infty_0 & \infty_1 & \infty_2 & \infty_3 & \infty_4 & \infty_5 & \infty_6 & \infty_7 & \infty_8 & \infty_9 & \infty_{10} & \infty_{11} & \infty_{12} & \infty_{13} & \infty_{14} & \infty_{15} & \infty_{16} & \infty_{17} & \infty_{18} & 0 \\ 9 & 5 & 0 & 16 & 7 & 19 & 18 & 40 & 7 & 39 & 20 & 0 & 34 & 10 & 23 & 36 & 36 & 18 & 10 & 29 \\ 22 & 36 & 6 & 41 & 30 & 10 & 16 & 0 & 18 & 28 & 7 & 20 & 3 & 2 & 2 & 12 & 38 & 34 & 19 & 19 \\ 19 & 19 & 20 & 3 & 2 & 18 & 10 & 28 & 28 & 24 & 31 & 41 & 30 & 3 & 40 & 11 & 42 & 27 & 37 & 26 \end{pmatrix}.$$

Then append $(0, 0, 0, 0)^t$ to create a QDM in \mathbb{Z}_{43} with 81 columns. Develop the QDM and append $\text{In}(19)$ to form an $\text{OA}(4, 62)$. Rows 1, 2, and 3 of this OA generate an N_2 resolvable Latin square of order 62.

A.13. Order 74. Replace each of the 19 columns of the matrix below with its four row-cyclic shifts:

$$\begin{pmatrix} \infty_0 & \infty_1 & \infty_2 & 68 & 9 & 24 & 23 & 54 & 52 & 27 & 46 & 21 & 15 & 53 & 66 & 18 & 8 & 26 & 69 \\ 62 & 54 & 40 & 4 & 5 & 54 & 2 & 41 & 0 & 16 & 11 & 39 & 30 & 29 & 59 & 56 & 21 & 7 & 6 \\ 34 & 60 & 65 & 8 & 39 & 27 & 19 & 5 & 29 & 7 & 12 & 13 & 61 & 17 & 27 & 6 & 53 & 68 & 63 \\ 45 & 44 & 67 & 56 & 38 & 30 & 1 & 59 & 55 & 47 & 32 & 11 & 23 & 44 & 50 & 43 & 23 & 2 & 20 \end{pmatrix}.$$

Then append $(0, 0, 0, 0)^t$ to create a QDM in \mathbb{Z}_{71} with 77 columns. Develop the QDM and append $\text{In}(19)$ to form an $\text{OA}(4, 62)$. Rows 1, 2, and 3 of this OA generate an N_2 resolvable Latin square of order 74 as do rows 1, 2, and 4.

A.14. Order 94. Let M be the master $\text{TD}(4, 7)$ with blocks of the form $\{(x, 1), (y, 2), (-x - y, 3), (x - y, 4)\}$ for $x, y \in \mathbb{Z}_7$. Let C be the $\text{OA}(4, 16)$ (viewed as a $\text{TD}(4, 16)$) on $\mathbb{Z}_{13} \cup \{\infty_0, \infty_1, \infty_2\}$ defined in Appendix A.3. Let D be the $\text{TD}(4, 13)$ with blocks of the form $\{(x, 1), (y, 2), (-x - y, 3), (x - y, 4)\}$, where $x, y \in \mathbb{Z}_{13}$. Let T be the transversal of M containing blocks of the form $\{(x, 1), (2x, 2), (-3x, 3), (-x, 4)\}$ for $x \in \mathbb{Z}_{13}$. We create a $\text{TD}(4, 94)$, R , by doing the following to the blocks of M :

- Inflate the block $\{(0, 1), (0, 2), (0, 3), (0, 4)\}$ by C .
- Inflate the remaining blocks of T by $C \setminus \text{In}(3)$.
- Inflate the remaining blocks of M by a D .

The reader can verify that $\text{Trunc}(R, 4)$ is N_2 , and so an N_2 -resolvable square of order 94 exists.

A.15. Order 146. Let M be the master $\text{TD}(4, 11)$ with blocks of the form $\{(x, 1), (y, 2), (-x - y, 3), (x - y, 4)\}$ for $x, y \in \mathbb{Z}_{11}$. Let C be the $\text{OA}(4, 16)$ (viewed as a $\text{TD}(4, 16)$) on $\mathbb{Z}_{13} \cup \{\infty_0, \infty_1, \infty_2\}$ defined in Appendix A.3. Let D be the $\text{TD}(4, 13)$ with blocks of the form $\{(x, 1), (y, 2), (-x - y, 3), (x - y, 4)\}$, where $x, y \in \mathbb{Z}_{13}$. Define T as the transversal of M consisting of blocks of the form

$\{(x, 1), (2x, 2), (-3x, 3), (-x, 4)\}$ for $x \in \mathbb{Z}_{13}$. We create a TD(4, 146), R , by doing the following to the blocks of M :

- Inflate the block $\{(0, 1), (0, 2), (0, 3), (0, 4)\}$ by C .
- Inflate the remaining blocks of T by $C \setminus \text{In}(3)$.
- Inflate the remaining blocks of M by a D .

The reader can verify that $\text{Trunc}(R, 4)$ is N_2 , and so an N_2 -resolvable square of order 146 exists.

A.16. Order 202. Let M be the master TD(4, 11) with blocks of the form $\{(x, 1), (y, 2), (-x - y, 3), (x - y, 4)\}$ for $x, y \in \mathbb{Z}_{11}$. Let T be the transversal of M consisting of the blocks of the form $\{(x, 1), (2x, 2), (-3x, 3), (-x, 4)\}$ for $x \in \mathbb{Z}_{11}$. Let C be the partial TD(4, 22) with a hole of size 4 on $\{\infty_0, \dots, \infty_3\}$ given by developing in \mathbb{Z}_{18} the QDM below:

$$\begin{pmatrix} \infty_0 & 0 & 2 & 13 & \infty_1 & 5 & 1 & 13 & \infty_2 & 3 & 0 & 9 & \infty_3 & 5 & 1 & 3 & 11 & 11 & 0 & 1 & 3 & 12 & 0 & 7 & 4 & 0 \\ 8 & \infty_0 & 0 & 1 & 6 & \infty_1 & 0 & 2 & 6 & \infty_2 & 5 & 1 & 2 & \infty_3 & 4 & 0 & 5 & 1 & 4 & 3 & 12 & 5 & 1 & 2 & 0 & 0 \\ 1 & 2 & \infty_0 & 9 & 0 & 14 & \infty_1 & 1 & 3 & 0 & \infty_2 & 1 & 7 & 0 & \infty_3 & 7 & 0 & 10 & 0 & 9 & 4 & 6 & 5 & 0 & 2 & 3 \\ 3 & 17 & 8 & \infty_0 & 10 & 0 & 2 & \infty_1 & 15 & 5 & 12 & \infty_2 & 13 & 3 & 1 & \infty_3 & 1 & 0 & 9 & 4 & 0 & 17 & 4 & 0 & 0 & 10 \end{pmatrix}.$$

Let D be the TD(4, 22) constructed by developing in \mathbb{Z}_{19} the QDM below, appending $\text{In}(3)$, and then relabeling the points so that the pointset is $\mathbb{Z}_{18} \cup \{\infty_0, \dots, \infty_3\}$ with $\{0, 1, 2, 3\}$ mapped to $\{\infty_0, \dots, \infty_3\}$:

$$\begin{pmatrix} \infty_0 & 18 & 17 & 1 & \infty_1 & 6 & 6 & 0 & \infty_2 & 8 & 9 & 10 & 3 & 9 & 4 & 3 & 9 & 11 & 11 & 10 & 8 & 8 & 17 & 2 & 18 \\ 3 & \infty_0 & 12 & 5 & 6 & \infty_1 & 14 & 7 & 8 & \infty_2 & 0 & 6 & 6 & 18 & 9 & 14 & 8 & 4 & 8 & 8 & 9 & 10 & 17 & 8 & 12 \\ 17 & 3 & \infty_0 & 3 & 6 & 12 & \infty_1 & 1 & 1 & 18 & \infty_2 & 8 & 10 & 8 & 1 & 17 & 5 & 5 & 4 & 13 & 0 & 17 & 6 & 7 & 18 \\ 18 & 11 & 7 & \infty_0 & 18 & 8 & 2 & \infty_1 & 11 & 12 & 9 & \infty_2 & 8 & 17 & 7 & 1 & 0 & 12 & 8 & 5 & 0 & 14 & 11 & 9 & 17 \end{pmatrix}.$$

We create a TD(4, 202), R , by doing the following to the blocks of M :

- Inflate the block $\{(0, 1), (0, 2), (0, 3), (0, 4)\}$ by D .
- Inflate the remaining blocks of T by C .
- Inflate the remaining blocks of M by the TD(4, 18) given in Appendix A.4.

The reader can verify that $\text{Trunc}(R, 4)$ is N_2 , and so an N_2 resolvable square of order 202 exists.

Note: In checking all the possible cases for a subsquare of order 2, the reader may find it useful that $\text{Trunc}(D, 4)$ contains no block consisting of only infinite points.

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