PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 1, September 1983

ON NONISOMORPHIC ROOM SQUARES

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ABSTRACT. Let $\mathrm{NR}(s)$ denote the number of nonisomorphic Room squares of side s. We prove that for s sufficiently large, $\mathrm{NR}(s) \geq \exp(cs^2)$ for some absolute constant c. More precisely, $\mathrm{NR}(s) \geq .19\exp(.04s^2)$ for $s \geq 153$ odd; and $\mathrm{NR}(s) \geq .19\exp(.09s^2)$ for $s \geq 1001$ odd.

1. Introduction. Let s be a positive odd integer, and let S be a set of size $s+1$. A *Room square* of *side s* is an s by s array, R, which satisfies the following properties:

(0) each cell of R either is empty or contains an unordered pair of elements (*symbols*) of S,

(1) every symbol occurs in precisely one cell of each row and column of R,

(2) every unordered pair of symbols occurs in precisely one cell of R.

Room squares have appeared in the literature as early as 1850 (see Kirkman [**5**]). They have been studied extensively since the 1960s, and the existence question was solved in 1974 by Mullin and Wallis [**8**]. We state their result as

THEOREM 1.1. *There exists a Room square of side s if and only if s is an odd positive integer other than* 3 *or* 5.

For $i = 1, 2$, let R_i be a Room square of side s based on the symbol set S_i, and let $\phi\colon S_1 \to S_2$ be a bijection. R_1^ϕ is defined to be the Room square based on symbol set S_2, in which x is replaced by $\phi(x)$ for all $x \in S_1$. We say that R_1 and R_2 are *isomorphic* Room squares if there exists some ϕ such that R_2 can be obtained from R_1^ϕ by permutations of rows and columns. It is clear that isomorphism is an equivalence relation. It is thus natural to ask how many nonisomorphic Room squares exist for each side s. We will denote this quantity by $\mathrm{NR}(s)$. We are able to show that $\mathrm{NR}(s) \geq \exp(cs^2)$ for some absolute constant c, so that clearly $\mathrm{NR}(s)$ grows extremely rapidly.

The *transpose* of a Room square R is the Room square R^T obtained by interchanging the roles of the rows and columns of R. Two Room squares R_1 and R_2 are *equivalent* if R_1 is isomorphic to either of R_2 or R_2^T (this also clearly yields an equivalence relation). The number of inequivalent Room squares will be denoted by $\mathrm{IR}(s)$. The following lemma is immediate.

LEMMA 1.2. $\mathrm{IR}(s) \geq \frac{1}{2}\mathrm{NR}(s)$

Received by the editors September 30, 1982 and, in revised form, January 26, 1983.
1980 *Mathematics Subject Classification*. Primary 05B15.

We note that strict inequality will hold if there exists a Room square of side s which is isomorphic to its transpose. However, our lower bounds for $\mathrm{NR}(s)$ provide bounds for $\mathrm{IR}(s)$ as well.

The question of determining the number of nonisomorphic designs of a given type has been studied for several types of designs: Steiner triple systems (see [**15**]), Steiner quadruple systems [**4**], and one-factorizations of complete graphs [**1** and **7**]. In [**6**], Lindner establishes that $\mathrm{IR}(s)$ is very large for certain s, but our paper provides the first proof that $\mathrm{NR}(s) > 1$ for all but a finite number of sides s.

The number of inequivalent and nonisomorphic Room squares of side 7 was determined exactly in [**3** and **14**].

2. A recursive construction. We obtain our lower bounds on $\mathrm{NR}(s)$ by using a recursive construction for Room squares based on group divisible designs. A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$ which satisfies:

(0) X is a finite set (elements of which are called *points*), and $\mathcal{G}$ and $\mathcal{A}$ are sets of nonempty subsets of X (elements of $\mathcal{G}$ are called *groups* and elements of $\mathcal{A}$ are called *blocks*),

(1) $\mathcal{G}$ is a partition of X,

(2) every block has size at least two, and the groups and blocks together contain every unordered pair of points exactly once.

A *transversal design* (or $\mathrm{TD}(k, n)$) is a GDD with $|X| = kn$, having k groups of size n, and n^2 blocks of size k. It is well known that a $\mathrm{TD}(k, n)$ is equivalent to a set of $k - 2$ mutually orthogonal Latin squares of order n.

Our recursive construction starts with a suitable transversal design, replaces each block by an array called a frame, and then replaces each group by a Room square. Let X be a set, and let $\{S_1, S_2, \ldots, S_r\}$ be a partition of X. An $\{S_1, S_2, \ldots, S_r\}$*-frame* is a square array F of size $|X|$, with rows and columns indexed by X, which satisfies:

(0) every cell of F either is empty or contains an unordered pair of symbols of X,

(1) for every i, $1 \leqslant i \leqslant r$, the cells of $F(s, s')$, with $\{s, s'\} \subseteq S_i$, are empty (these empty cells are called *holes*),

(2) each symbol of $X \setminus S_i$ occurs in precisely one cell of each row and each column $s \in S_i$ of F,

(3) the pairs occurring in F are precisely those $\{s, s'\}$ with $(s, s') \in X^2 \setminus \bigcup_{i=1}^{r} S_i^2$.

Suppose R is a Room square of side s on symbol set $S \cup \{\infty\}$ ($|S| = s$). Index the rows and columns of R so that $\{\infty, S\}$ occurs in cell (s, s) for each $s \in S$. If the contents of these cells are deleted, an $\{\{s\}: s \in S\}$-frame is formed. Conversely, given such a frame, one can "complete" it to a Room square by filling in the holes (i.e. cells (s, s)), appropriately.

The *type* of an $\{S_1, \ldots, S_r\}$-frame is the multiset $\{|S_i|: 1 \leqslant i \leqslant r\}$. We will use the notation $1^{t_1}2^{t_2}\cdots$ to describe the type of frame where there are precisely t_i S_j's of size i $(i \geqslant 1)$. Thus the above discussion demonstrates that a Room square of side s is equivalent to a frame of type 1^s.

The following two frames are essential ingredients in our recursive construction.

LEMMA 2.1. *There exist frames of type* 1^9 *and* $1^8 3^1$.

Proof. A frame of type 1^9 is equivalent to a Room square of side 9, which exists by Theorem 1.1. The frame of type 1^9 in Figure 1 was given by Beaman and Wallis in [**2**]. The frame of type $1^8 3^1$ was found by the first author and is presented in Figure 2. Both frames were found by the use of the computer.

Figure 1. A frame of type 1^9

	1	2	3	4	5	6	7	8	9
1			4,9	3,7	2,8		5,6		
2	8,9					5,7	3,4		1,6
3		5,8			6,9	2,4		1,7	
4		3,6	7,8			1,9		2,5	
5		7,9		1,2		3,8		4,6	
6	4,5						1,8	3,9	2,7
7			2,6	5,9	1,3				4,8
8	6,7	1,4					2,9		3,5
9	2,3		1,5	6,8	4,7				

Figure 2. A frame of type $1^8 3^1$

	1	2	3	4	5	6	7	8	9	10	11
1				5,9			4,8	7,11			6,10
2				7,10			6,11	4,9			5,8
3					4,10	5,11			6,8	7,9	
4	7,8	9,11					1,5	2,6	3,10		
5	6,9	8,10				1,4				3,11	2,7
6	10,11		8,9		1,7			3,5		2,4	
7				1,6	8,11	9,10			2,5		3,4
8		5,7	4,11		3,6		2,10				1,9
9		4,6	5,10	2,11		3,7				1,8	
10	4,5		6,7			2,8	3,9		1,11		
11				3,8	2,9			1,10	4,7	5,6	

Let $(X, \mathcal{G}, \mathcal{A})$ be a TD$(9, n)$, where $\mathcal{G} = \{G_i \colon 1 \leqslant i \leqslant 9\}$, and let $W \subseteq G_9$. Denote $t = |W|$. Define a function S, with domain X, by

$$S(x) = \begin{cases} x & \text{if } x \in X \backslash W, \\ \{x_i \colon 1 \leqslant i \leqslant 3\} & \text{if } x \in W. \end{cases}$$

For every $A \in \mathcal{Q}$, let F_A be any $\{S(x): x \in A\}$-frame. (Such a frame is of type 1^9 or $1^8 3^1$.) Then $F = \bigcup_{A \in \mathcal{Q}} F_A$ is a $\{\bigcup_{x \in G_i} S(x): 1 \leqslant i \leqslant 9\}$-frame of type $n^8(n+2t)^1$ by [**11**, Construction 2.2]. Now, we complete F to a Room square of side $9n + 2t$. Let $\infty \notin \bigcup_{x \in X} S(x)$, and place a Room square, on symbol set $\{\infty\} \cup (\bigcup_{x \in G_i} S(x))$ in the hole of F induced by $\bigcup_{x \in G_i} S(x)$, for $1 \leqslant i \leqslant 9$. The resultant array is a Room square by [**12**, Theorem 3.1]. We summarize this as

LEMMA 2.2. *Suppose there is a* TD(9, n), *with n odd, and* $0 \leqslant t \leqslant n$. *If there exist Room squares of sides n and $n + 2t$, then there exists a Room square of side* $9n + 2t$.

Our basic method is as follows. We construct a large number of distinct Room squares on a specified symbol set. We can obtain a naive upper bound on the number of Room squares isomorphic to a given Room square. The quotient of these two quantities provides a lower bound on the number of nonisomorphic Room squares. This is done in the next section.

3. A bound. We first prove that there are many distinct frames of the types 1^9 and $1^8 3^1$ (on fixed symbol sets).

LEMMA 3.1. *There are at least* 9! *distinct frames of type* 1^9, *on a fixed symbol set.*

PROOF. The frame F_9 of Figure 1 is an $\{\{i\}: 1 \leqslant i \leqslant 9\}$-frame. For Π a permutation of $\{1, 2, \ldots, 9\}$, let F_9^{Π} be the frame defined by $F_9^{\Pi}(\Pi(i), \Pi(j)) = \{\Pi(s), \Pi(t)\}$ where $F_9(i, j) = \{s, t\}$. It is easily seen that F_9^{Π} is also a $\{\{i\}: 1 \leqslant i \leqslant 9\}$-frame. We show that the 9! permutations of $\{1, 2, \ldots, 9\}$ give rise to distinct frames.

Suppose that $F_9^{\Pi} = F_9^{\phi}$. Then, clearly $F_9^{\Pi\phi^{-1}} = F_9$. Thus, it suffices to show that $F_9^{\Pi} \neq F_9$ for all Π. This can be established with only a moderate amount of case-work.

If $F(i, j)$ is nonempty, then $F(\Pi(i), \Pi(j))$ must be nonempty. Thus, there are only 36 possibilities for $\{\Pi(i), \Pi(j)\}$, determined by the filled cells of F_9. We let $i = 1, j = 3$, and consider each case.

All cases are handled similarly. We give an example. Suppose $(\Pi(1), \Pi(3)) = (2, 7)$. Then $\{\Pi(4), \Pi(9)\} = \{3, 4\} = F_9(2, 7)$. If $\Pi(4) = 4$ and $\Pi(9) = 3$, then $F(\Pi(4), \Pi(9)) = F(4, 3)$ must be empty. It is not, so we have a contradiction. Thus, suppose $\Pi(4) = 3$ and $\Pi(9) = 4$. Then $\{3, \Pi(7)\} = \{\Pi(4), \Pi(7)\} = F(\Pi(9), \Pi(5)) = F(4, \Pi(5))$. Since $F(4, 2) = \{3, 6\}$, we have $\Pi(5) = 2$ and $\Pi(7) = 6$. Then $F(\Pi(4), \Pi(7)) = F(3, 6)$ must be empty. It is not, so we have a contradiction in this case as well.

The other cases are handled similarly.

LEMMA 3.2. *There are at least* $8!(3!)^3$ *distinct frames of type* $8^1 3^1$, *on a fixed symbol set.*

PROOF. We start with the $\{\{1, 2, 3\}, \{i\}: 4 \leqslant i \leqslant 11\}$-frame F_{11} presented in Figure 2. Let α and β be permutations of $\{1, 2, 3\}$ and let Π be a permutation of $\{1, 2, \ldots, 11\}$ such that $\{\Pi(1), \Pi(2), \Pi(3)\} = \{1, 2, 3\}$. Define the frame $F_{11}^{\alpha,\beta,\Pi}$ as follows:

for $4 \leqslant i, j \leqslant 11$, $F_{11}^{\alpha,\beta,\Pi}(\Pi(i), \Pi(j)) = \{\Pi(s), \Pi(t)\}$;
for $1 \leqslant i \leqslant 3$, $4 \leqslant j \leqslant 11$, $F_{11}^{\alpha,\beta,\Pi}(\alpha(i), \Pi(j)) = \{\Pi(s), \Pi(t)\}$;

and for $4 < i < 11$, $1 < j < 3$, $F_{11}^{\alpha,\beta\Pi}(\Pi(i), \beta(j)) = \{\pi(s), \Pi(t)\}$, where, in each case $F_{11}(i, j) = \{s, t\}$. It is easily shown that each $F_{11}^{\alpha,\beta,\Pi}$ thus constructed is a $\{\{1,2,3\},\{i\}: 4 \leqslant i \leqslant 11\}$-frame. We assert that the $(3!)^3 8!$ frames thus produced are distinct.

Again, it suffices to show that $F_{11}^{\alpha,\beta,\Pi} \neq F_{11}$ for any α, β, Π. Consider $F_{11}(7,5) = \{8,11\}$. *Then* $F_{11}^{\alpha,\beta,\Pi}(\Pi(7), \Pi(5)) = \{(8), \Pi(11)\}$. *Now* $\Pi(5)$, $\Pi(7)$, $\Pi(8)$, $\Pi(11) \in \{4,5,\ldots,11\}$ *so we see from* F_{11} *that* $\{\Pi(8), \Pi(11)\} = \{8,11\}$, $\{9,10\}$, $\{4,7\}$ or $\{5,6\}$. These four cases are easily disposed of, as in Lemma 3.1, proving the result.

THEOREM 3.3. *Suppose there exists a* TD(9, n), *with n odd and* $0 \leqslant t \leqslant n$. *Then there are at least*

$$(9!)^{(n-t)n} \cdot \left(8!\,(3!)^3\right)^{nt} \cdot \left(2(n!)^2\right)^8 \cdot 2((n+2t)!)^2$$

distinct Room squares of side $9n + 2t$, *on a fixed symbol set.*

PROOF. We apply Lemma 2.2. In the construction of the Room square of side $9n + 2t$, there are $(n - t)n$ blocks which miss W, each of which correspond to a frame of type 1^9. The remaining tn blocks correspond to a frame of type $1^8 3^1$. We thus obtain $(9!)^{(n-t)n}(8!(3!)^3)^{tn}$ distinct frames of type $n^8(n + 2t)^1$.

Now we fill in the holes of this frame. Let R be any Room squares of side n, by permuting rows and columns of R, and transposing (i.e. interchanging the function of rows and columns), we can obtain $2(n!)^2$ distinct Room squares of side n. We fill in eight Room squares of side n and one of side $n + 2t$; so each frame can be completed in at least $(2(n!)^2)^8 2((n + 2t)!)^2$ ways.

It is easily seen that all the Room squares obtained by this construction are distinct.

COROLLARY 3.4. *Suppose there exists a* TD(9, n) *and* $9n \leqslant s \leqslant 11$, *where n and s are odd. Then*

$$\mathrm{NR}(s) \geqslant \frac{(9!)^{n^2} 2^9 (n!)^{18}}{((11n)!)^2 (11n+1)!}.$$

PROOF. First, write $s = 9n + 2t$, where $0 \leqslant t \leqslant n$. We divide the lower bound for the number of distinct Room squares of side s obtained in Theorem 3.3 by the number of possible distinct Room squares isomorphic to a given Room square of side s. This number can be at most $((9n + 2t)!)^2(9n + 2t + 1)!$ (allowing perumtations of rows, columns and symbols). Thus,

$$\mathrm{NR}(s) = \mathrm{NR}(9n+2t) \geqslant \frac{(9!)^{(n-t)n}\left(8!\,(3!)^3\right)^{tn}\left(2(n!)^2\right)^8 2((n+2t)!)^2}{((9n+2t)!)^2(9n+2t+1)!}$$
$$> \frac{(9!)^{n^2} 2^9 (n!)^{18}}{((11n)!)^2 (11n+1)!}.$$

We examine the behavior of the above quantity with the following version of Stirlings's formula (see **[9]**).

LEMMA 3.5. *$n! = (2\pi n)^{1/2}(n/e)^n e^{\alpha_n}$ where $1/(12n+1) < \alpha_n < 1/12n$.*

THEOREM 3.6. *Suppose there is a TD(9, n) and $9n \leqslant s \leqslant 11n$ where n and s are odd. Then*

$$\mathrm{NR}(s) > c \cdot \exp\left(n^2 \ln(9!) - 15n\ln(n) - (33\ln 11 - 15)n + \frac{13}{2}\ln(n)\right),$$

where $c = (2^{29}\pi^{15}/(9 \cdot 11^3))^{1/2}$.

PROOF. From Corollary 3.4, we have

$$\mathrm{NR}(s) > \frac{2^9 \cdot (9!)n^2 \cdot (n!)^{18}}{((11n)!)^3(11n+1)}$$

$$\geqslant \frac{2^9(9!)^{n^2}(2\pi n)^9(n/e)^{18n}e^{18/(12n+1)}}{(22\pi n)^{3/2}(11n/e)^{33n}e^{3/(12\cdot 11n)}(11n+1)} \qquad \text{(Lemma 3.5)}$$

$$> \frac{2^9(9!)^{n^2}(2\pi n)^9(n/e)^{18n}}{(22\pi n)^{3/2}(11n/e)^{33n} \cdot 12n}$$

$$= \left(\frac{2^{29}\cdot\pi^{15}}{9\cdot 11^3}\right)^{1/2} \exp\left(n^2\ln(9!) - 15n\ln(n) - (33\ln 11 - 15)n + \frac{13}{2}\ln(n)\right).$$

The above bound is dependent upon the existence of certain transversal designs. The following result is well known.

LEMMA 3.7. *Let n have prime power factorization $\pi p_i^{\alpha_i}$. If $k \leqslant 1 + \min\{p_i^{\alpha_i}\}$, then there exists a TD(k, n).*

One can then prove the following by use of the above lemma and a simple arithmetic argument (see [**13**]).

COROLLARY 3.8. *If $s \geqslant 153$, then there is a positive integer n, with $9n \leqslant s \leqslant 11n$, such that a TD(9, n) exists.*

Elementary calculus shows that the bound of Corollary 3.6 is an increasing function of n (for $n \geqslant 9$, for example). Thus we have

THEOREM 3.9. *For $s \geqslant 153$ odd,*

$$\mathrm{NR}(s) > c \cdot \exp\left(\frac{s^2}{121}\ln(9!) - \frac{15}{11}s\ln(s) + \left(\frac{15}{11} - \frac{18}{11}\ln(11)\right)s + \frac{13}{2}\ln(s)\right),$$

where $c = (2^{29}\pi^{15}/(9\cdot 11^{16}))^{1/2} > .19$.

COROLLARY 3.10. (1) *For $s \geqslant 153$, s odd,* $\mathrm{NR}(s) > .19e^{.04s^2} > 1$.
(2) *For $s \geqslant 1001$, s odd,* $\mathrm{NR}(s) > .19e^{.09s^2}$.
(3) $\mathrm{NR}(1001) > 10^{40850}$.

4. Summary and remarks. We have shown that the number of nonisomorphic (and inequivalent) Room squares grows extremely rapidly. The techniques in this paper are quite general. A similar argument involving group-divisible designs could be used, for example, to show that the number of nonisomorphic (v, k, 1) balanced incomplete block designs approaches infinity as v approaches infinity.

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