Fault tolerant routings with minimum optical index

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Abstract

We construct sets of routings in the complete directed graph $\vec{K}_n$ that tolerate up to f failures of nodes or arcs. These routings are optimal with respect to several desirable criteria. In addition, our routings have minimum (or close to minimum) possible optical index, which means that wavelengths can be assigned to the directed paths in the routings in an efficient manner. This property is useful in the context of optical networks.

Keywords: optical network, routing, fault-tolerance

1 Introduction

This paper studies the design of routings in optical networks. This has been a topic of considerable recent interest; see, for example, [1, 3, 4, 5, 10]. For background material on optical networks, we refer to Aggarwal *et al* [1]. The optical networks relevant to our work use wavelength division multiplexing; these are called *WDM optical networks*. In this model, routing nodes are joined by fiber-optic links, and each link can support some fixed number of wavelengths (see, for example, Beauquier *et al* [4] and Beauquier [3]). Each

routing path uses a particular wavelength, and two paths can use the same wavelength if and only if they have no nodes in common. In general, we want to minimize the total number of wavelengths used in the entwork.

We are particularly interested in constructing fault-tolerant routings, the study of which was initiated by Maňuch and Stacho [10]. We will want to provide a routing between any two nodes in the network, even if some number of nodes and/or links fail (up to a specified threshold, say f). Thus, given a source node x and a destination node y, we will require $f+1$ routings from x to y that have no internal nodes or links in common. Given that at most f components fail, at least one of these $f+1$ routings will contain no failed components.

Beauquier *et al* [4] suggest that it is most appropriate to model an optical network by a directed graph, say G. A routing therefore specifies a directed path in G from any vertex (i.e., node) x to any other vertex y. As mentioned above, we want to find a set of $f+1$ directed paths from x to y that are vertex-disjoint (except for x and y). Moreover, we want to find such a set of paths for all possible choices of x and y ($x \neq y$). Routings of this type are known as "f-tolerant routings" [10].

In addition to fault tolerance, other properties of routings are also useful, such as balancing and minimizing the use of each arc (i.e., directed edge), and finding an f-tolerant routing that can be split into a sequence of $f+1$ "levels". We already pointed out that an important issue in the context of optical networks is to assign wavelengths to the directed paths in a routing in such a way that no arc is assigned a particular wavelength in more than one directed path. Furthermore, we want to minimize the total number of wavelengths used, subject to this condition; i.e., we want to design routings that have minimum possible "optical index".

Some interesting results on these kinds of routings were presented by Gupta, Maňuch and Stacho [9] when the underlying network graph is a complete directed graph or a complete bipartite directed graph. In this paper, we concentrate on constructing routings for the complete directed graph that improve upon the routings presented in [9]. Some of the routings we construct are simultaneously optimal with respect to all of the criteria discussed above, including (levelled) optical index.

1.1 Our Contributions

This paper is organized as follows. In Section 2, we give the formal definitions of the routings we study. Section 3 describes a backtracking technique that we used to find several optimal small routings by computer. Section 4

presents a recursive construction which allows infinite classes of new optimal routings to be constructed. In Section 5, we analyze the optical indices for a class of routings derived from a simple algebraic formula. These routings exist for orders n that are odd prime powers, and their (levelled) optical indices are at most 25% higher than the theoretical optimal value. This improves the general results in [9], where the upper bound proven on the optical indices is roughly 50% higher than the theoretical optimal value. In Section 6, we show that optical indices can be further decreased if we do not require them to be levelled. In this case, we provide routings of the complete directed graph $\vec{K}_q$, having optical indices that are at most 9 away from the theoretical optimal value, whenever $q = 2p + 1$ and q and p are odd primes. Our results use mathematical techniques from design theory, graph theory and finite fields.

2 Optimal routings

In this section, we give precise mathematical definitions of the concepts discussed in the introduction.

Let $n \geq 3$ be a positive integer and let $\vec{K}_n$ denote the complete directed graph on a set of n vertices, say X. A P_2-routing of $\vec{K}_n$ is a set $\mathcal{L}$ of directed paths of length two in $\vec{K}_n$, such that the following properties are satisfied:

1. for all $x, y \in X$, $x \neq y$, there is a unique directed path in $\mathcal{L}$ having origin x and terminus y, and

2. every arc in $\vec{K}_n$ occurs in exactly two directed paths in $\mathcal{L}$.

For $0 \leq f \leq n - 2$, an *optimal, balanced, levelled, f-tolerant routing of* $\vec{K}_n$ is an $(f + 1)$-tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_f)$ that satisfies the following properties:

1. $\mathcal{L}_0$ consists of the $n(n - 1)$ arcs in $\vec{K}_n$ (i.e., all the possible directed paths of length 1),

2. for $1 \leq i \leq f$, $\mathcal{L}_i$ is a P_2-routing of $\vec{K}_n$, and

3. no directed path of length two occurs in more than one of the routings $\mathcal{L}_i$, $1 \leq i \leq f$.

In the rest of this paper, we will refer to an optimal, balanced, levelled, f-tolerant routing of $\vec{K}_n$ simply as an *f-tolerant routing of $\vec{K}_n$*. For $0 \leq i \leq f$, $\mathcal{L}_i$ is called the ith *level* of the routing.

Observe that an $(n-2)$-tolerant routing of $\vec{K}_n$ has the property that every directed path of length two occurs exactly one of the $n-1$ levels of the routing.

Example 2.1. The (unique) 1-tolerant routing of $\vec{K}_3$ is as follows:

$$\mathcal{L}_0 : \quad 01, 02, 10, 12, 20, 21$$
$$\mathcal{L}_1 : \quad 021, 012, 120, 102, 210, 201$$

Note that we are using the notation xy to denote the arc $x \to y$, and xyz denotes the directed path $x \to y \to z$. $\qquad \square$

Let $\mathcal{R} = (\mathcal{L}_0, \ldots, \mathcal{L}_f)$ be an f-tolerant routing of $\vec{K}_n$. We want to assign a wavelength to each directed path in $\mathcal{R}$ in such a way that no arc is assigned a particular wavelength in more than one directed path. Furthermore, we want to minimize the total number of wavelengths used, subject to this condition. This motivates the following definition.

For $0 \le i \le f$, construct a graph whose vertices are the directed paths in $\mathcal{L}_0, \ldots, \mathcal{L}_i$. Two vertices in this graph are adjacent if they have a common arc. This graph is called the *ith path graph of $\mathcal{R}$* and the *ith optical index of $\mathcal{R}$*, denoted $w_i(\mathcal{R})$, is defined to be the chromatic number of the ith path graph. It should be clear that $w_i(\mathcal{R})$ is in fact the minimum number of wavelengths required for the first $i+1$ levels of the routing $\mathcal{R}$.

The following is proven in [9].

Theorem 2.1. *Let $\mathcal{R}$ be any f-tolerant routing of $\vec{K}_n$. Then $w_0(\mathcal{R}) = 1$. Further, for $1 \le i \le f$, it holds that $2f + 1 \le w_i(\mathcal{R}) \le 3f + 1$.*

An f-tolerant routing of $\vec{K}_n$, say $\mathcal{R}$, is said to have *minimum optical index* if $w_i(\mathcal{R}) = 2i + 1$ for all i such that $1 \le i \le f$. $\mathcal{R}$ is said to have *levelled minimum optical index* if the path graph of each level i has chromatic number equal to two, for $1 \le i \le f$. (Observe that the path graph of level i is a union of directed cycles, so it has chromatic number equal to two if and only if all these cycles have even length.)

It should be clear that a routing having levelled minimum optical index has minimum optical index.

Theorem 2.2. *The unique 1-tolerant routing of $\vec{K}_3$ has minimum optical index, but it does not have levelled minimum optical index.*

Proof. In the following routing $\mathcal{R}$, the superscripts denote wavelengths:

$$\mathcal{L}_0 : \quad 01^3, 02^2, 10^1, 12^2, 20^1, 21^3$$
$$\mathcal{L}_1 : \quad 021^1, 012^1, 120^3, 102^3, 210^2, 201^2$$

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The above assignment of wavelengths shows that $w_1(\mathcal{R}) = 3$, which is optimal.

It is also easy to see that the path graph of $\mathcal{L}_1$ consists of two cycles of order 3. Therefore, the routing $\mathcal{R}$ does not have levelled minimum optical index. $\qquad\square$

3   Routings for small orders

We construct routings for small orders by using a backtracking algorithm. We assume a certain automorphism group in order to reduce the search time. Here is a description of the strategy we use.

We take $X = \{0, \ldots, n-1\}$ and let π denote the permutation

$$(0\ 1\ \cdots\ n-3)(n-2)(n-1).$$

That is, π consists of a cycle of length $n-2$ together with two fixed points. We will search for a "base" P_2-routing of $\vec{K}_n$ defined on X, and then develop this base routing through the subgroup $\langle \pi \rangle$ of the symmetric group S_X in order to obtain an f-tolerant routing.

Under the action of $\langle \pi \rangle$, the $n(n-1)(n-2)$ directed paths of length two having vertices in X are partitioned into equivalence classes called "orbits". It is easy to see that there are a total of $n(n-1)$ orbits, each consisting of exactly $n-2$ directed paths.

Let P denote the set of $n(n-1)$ ordered pairs of distinct elements of X. Our base routing $\mathcal{L}$ will be derived from a function $\mathsf{m} : P \to X$ that satisfies the following properties:

1. for all $(x, y) \in P$, $\mathsf{m}(x, y) \neq x, y$,

2. the collection of all $2n(n-1)$ arcs of the form $(x, \mathsf{m}(x, y))$ and $(\mathsf{m}(x, y), y)$ contains every directed edge in P exactly twice, and

3. the $n(n-1)$ directed paths $(x, \mathsf{m}(x, y), y)$ contain exactly one directed path from each orbit under the action of $\langle \pi \rangle$.

$\mathcal{L}$ just consists of all the paths $(x, \mathsf{m}(x, y)), y)$, $x \neq y$.

Theorem 3.1. *Suppose there is a function* $\mathsf{m} : P \to X$ *that satisfies properties 1–3 defined above. Then there is an f-tolerant routing of $\vec{K}_n$. Furthermore, if the path graph of the resulting base routing $\mathcal{L}$ has chromatic number equal to 2, then the f-tolerant routing has levelled minimum optical index.*

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Proof. As usual, $\mathcal{L}_0$ just consists of the arcs in $\vec{K}_n$ (the set P, in this case). We define $\mathcal{L}_1 = \mathcal{L}$, and for $2 \leq i \leq n - 2$, we construct $\mathcal{L}_i$ by applying the permutation π to all the directed paths in $\mathcal{L}_{i-1}$. It is not hard to see that properties 1–3 guarantee that we obtain an f-tolerant routing of $\vec{K}_n$ by carrying out this process.

It is also straightforward to see that the path graphs of the $n - 2$ levels $\mathcal{L}_1, \ldots, \mathcal{L}_{n-2}$ are all identical. Therefore, if the path graph of $\mathcal{L}$ has chromatic number equal to 2, it follows immediately that the routing $(\mathcal{L}_0, \ldots, \mathcal{L}_{n-2})$ has levelled minimum optical index. $\qquad\square$

Theorem 3.2. *For $4 \leq n \leq 8$, there is an f-tolerant routing of $\vec{K}_n$ that has levelled minimum optical index.*

Proof. Base routings satisfying the relevant hypotheses of Theorem 3.1 were found by computer and are presented in the Appendix. $\qquad\square$

4   Recursive constructions

It seems natural to try and develop recursive constructions for $(n - 2)$-tolerant routings of $\vec{K}_n$. There are in fact many possible constructions that can be described, for example, involving designs such as large sets of Steiner triple systems. However, if we also desire routings that have levelled minimum optical index, then the problem seems to be considerably more difficult. Because there is no 1-tolerant routing of $\vec{K}_3$ that has levelled minimum optical index, it seems difficult to preserve the property of levelled minimum optical index using recursive constructions based on designs having block size three. However, in this section, we describe one recursive construction, based on designs having block size four, that can be used to produce infinite classes of $(n - 2)$-tolerant routings of $\vec{K}_n$ having levelled minimum optical index.

A t-(v, k, λ)-*design* is a pair $(X, \mathcal{A})$, where X is a v-set of elements called *points* and $\mathcal{A}$ is a collection of k-subsets of X called *blocks*, such that every t-subset of points is contained in exactly λ blocks. A t-(v, k, λ)-design $(X, \mathcal{A})$ is said to be *partitionable* if it is possible to partition $\mathcal{A}$ into

$$\ell = \frac{v - t + 1}{k - t + 1}$$

subcollections $\mathcal{A}_i$ $(1 \leq i \leq \ell)$ such that $(X, \mathcal{A}_i)$ is a $(t - 1)$-(v, k, λ)-design for all i, $1 \leq i \leq \ell$.

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Theorem 4.1. *Suppose there exists a partitionable* 3-$(n, 4, 1)$-*design. Then there exists an* $(n - 2)$-*tolerant routing of* $\vec{K}_n$ *that has levelled minimum optical index.*

Proof. Let $(X, \mathcal{A})$ be a 3-$(n, 4, 1)$-design that can be partitioned into $\ell = (n - 2)/2$ 2-$(n, 4, 1)$-designs, $(X, \mathcal{A}_i)$, $1 \le i \le \ell$. For each block $A \in \mathcal{A}$, let $(\mathcal{L}_0^A, \mathcal{L}_1^A, \mathcal{L}_2^A)$ be a 2-tolerant routing of $\vec{K}_4$ on the set A, that has levelled minimum optical index. Further, we will stipulate that $\mathcal{L}_0^A$ consists of the 12 ordered pairs of elements of A.

Now, for $1 \le i \le \ell$ and $j = 1, 2$, define

$$\mathcal{L}_{i,j} = \bigcup_{A \in \mathcal{A}_i} \mathcal{L}_j^A.$$

Also, define $\mathcal{L}_0$ to be the set of all $n(n - 1)$ ordered pairs of elements of X.

It is not difficult to prove that

$$\mathcal{R} = (\mathcal{L}_0, \mathcal{L}_{1,1}, \mathcal{L}_{1,2}, \ldots, \mathcal{L}_{\ell,1}\mathcal{L}_{\ell,2})$$

is an $(n - 2)$-tolerant routing of $\vec{K}_n$.

Further, observe that the path graph of any level $\mathcal{L}_{i,j}$ is a disjoint union of the path graphs of the constituent levels $\mathcal{L}_j^A$ $(A \in \mathcal{A}_i)$. This implies that all the path graphs of the levels of $\mathcal{R}$ have chromatic number equal to 2, as claimed. $\square$

Determining the existence of partitionable 3-$(n, 4, 1)$-designs is a difficult unsolved problem. Nevertheless, there are some known infinite classes of partitionable 3-$(n, 4, 1)$-designs which are recorded in the following theorem.

Theorem 4.2. *For all* $n = 4^k$, $n = 2(7^k + 1)$ *and* $n = 2(31^k + 1)$, *there exists a partitionable* 3-$(n, 4, 1)$-*design.*

Proof. For $n = 4^k$, the result is due to Baker [2], while for $n = 2(7^k + 1)$ and $n = 2(31^k + 1)$, the result is found in Teirlinck [11]. $\square$

Corollary 4.3. *For all* $n = 4^k$, $n = 2(7^k + 1)$ *and* $n = 2(31^k + 1)$, *there exists an* $(n - 2)$-*tolerant routing of* $\vec{K}_n$ *that has levelled minimum optical index.*

5 Analysis of an algebraic construction

In this section, we analyze an algebraic construction based on certain classes of latin squares. First, we require some definitions. A *latin square of order* n based on symbol set X is an $n \times n$ array L whose entries are elements of X, such that every row and every column of L contains every symbol in exactly one cell. If the rows and columns of L are indexed by X, then L is *idempotent* if $L(x, x) = x$ for all $x \in X$. Two idempotent latin squares of order n, say L_1 and L_2, defined on the same symbol set X, are *disjoint* if $L_1(x, y) \neq L_2(x, y)$ for all $x \neq y$. $L_1, \ldots, L_f$ are disjoint idempotent latin squares of order n if every pair of squares L_i, L_j ($i \neq j$) are disjoint idempotent latin squares of order n. It is easy to see that, if $L_1, \ldots, L_f$ are disjoint idempotent latin squares of order n, then $f \leq n - 2$.

The following construction is implicit in [9].

Theorem 5.1. *Suppose that* $L_1, \ldots, L_f$ *are* f *disjoint idempotent latin squares of order* n *defined on symbol set* X. *For* $1 \leq i \leq f$, *define*

$$\mathcal{L}_i = \{xzy : z = L_i(x, y), x \neq y\}.$$

Also, define

$$\mathcal{L}_0 = \{xy : x \neq y\}.$$

Then $(\mathcal{L}_0, \ldots, \mathcal{L}_f)$ *is an* f-*tolerant routing of* $\vec{K}_n$.

Remark 5.1. Since it is known that $n - 2$ disjoint idempotent latin squares of order n exist for all $n \neq 6$ (see [12, 6, 7]), it follows immediately that there is an $(n - 2)$-tolerant routing of $\vec{K}_n$ for all $n \neq 6$.

We now analyze a specific construction for disjoint idempotent latin squares and determine the optical indices of the resulting routings. This construction is based on a simple formula, which could be advantageous in implementing the routing strategy in an efficient way. Suppose that q is an odd prime power and let $X = \mathbb{F}_q$. For all $a \in \mathbb{F}_q$, $a \neq 0, 1$. Define the latin square $L_a(x, y) = ax + (1 - a)y$. The following result is elementary.

Lemma 5.2. *Suppose that* q *is an odd prime power. For all* $a \in \mathbb{F}_q$, $a \neq 0, 1$, L_a *(as defined above) is an idempotent latin square of order* q. *Furthermore,* L_a *and* L_b *are disjoint if* $a \neq b$.

From the set of $q - 2$ latin squares L_a ($a \in \mathbb{F}_q$, $a \neq 0, 1$) we obtain a $(q - 2)$-tolerant routing of $\vec{K}_q$. We now determine the chromatic number of the path graph of each level of this routing. Consider L_a and the level

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$\mathcal{L}_a$ that it generates. The directed paths in $\mathcal{L}_a$ are xzy, $z = ax + (1 - a)y$, $x \neq y$. We can express y as a function of x and z as follows:

$$y = \frac{z - ax}{1 - a}.$$

Thus we have

$$\mathcal{L}_a = \{P_{x,z} = xzy : y = (z - ax)(1 - a)^{-1}, x \neq z\}.$$

For all ordered pairs (x, z) where $x \neq z$, the second edge of $P_{x,z}$ is the first edge of $P_{z,y}$, where $y = (z - ax)(1 - a)^{-1}$. Hence there is an edge joining $P_{x,z}$ and $P_{z,y}$ in the path graph, where y is as defined above. Since every arc occurs once as a first edge and once as a second edge in a path in $\mathcal{L}_a$, we have accounted for all the edges in the path graph of $\mathcal{L}_a$.

In order to further analyze the structure of this path graph, we consider the recurrence relation

$$x_{i+2} = \frac{x_{i+1} - a\,x_i}{1 - a}, \tag{1}$$

$i = 0, 1, \dots$. The characteristic polynomial of (1) is

$$f(z) = z^2 - \frac{z - a}{1 - a}.$$

The polynomial $f(z)$ has roots 1 and $a/(1 - a)$. Note that $a/(1 - a) = 1$ when $a = 2^{-1}$; in this situation, $f(z)$ has a double root.

Using standard methods, the recurrence relation (1) can be shown to have the following solution:

Lemma 5.3. *Suppose that $a \in \mathbb{F}_q$, $a \neq 0, 1, 2^{-1}$. Then the recurrence relation (1) has the solution*

$$x_i = c_0 + c_1\gamma^i,$$

where $\gamma = a(1 - a)^{-1}$, $c_0 = (x_1 - \gamma x_0)(1 - \gamma)^{-1}$ and $c_1 = (x_0 - x_1)(1 - \gamma)^{-1}$. If $a = 2^{-1}$, then the recurrence relation (1) has the solution

$$x_i = x_0 + i(x_1 - x_0).$$

Using Lemma 5.3, it is not difficult to determine the structure of the path graph of $\mathcal{L}_a$.

Theorem 5.4. *Suppose that $a \in \mathbb{F}_q$, $a \neq 0, 1, 2^{-1}$. Then the path graph of $\mathcal{L}_a$ consists of disjoint cycles of order d, where d is the (multiplicative) order of $a/(1 - a)$ in $\mathbb{F}_q{}^*$. If $a = 2^{-1}$, then the path graph of $\mathcal{L}_a$ consists of disjoint cycles of order q.*

Proof. Pick any arc (x_0, x_1) (where $x_0 \neq x_1$), and construct a sequence $S_{x_0,x_1} = (x_0, x_1, x_2, \ldots)$ by applying the recurrence relation (1). When $a \neq 2^{-1}$, it is easy to see that $x_j = x_i$ if and only if d divides $j - i$, where d is the order of γ in $\mathbb{F}_q{}^*$. On the other hand, when $a = 2^{-1}$, it follows that $x_j = x_i$ if and only if q divides $j - i$. In any case, the sequence S_{x_0,x_1} is periodic.

Now, each possible sequence S_{x_0,x_1} gives rise to a cycle in the path graph of $\mathcal{L}_a$. To be specific, the sequence S_{x_0,x_1} corresponds to the cycle

$$P_{x_0,x_1} - P_{x_1,x_2} - \cdots - P_{x_0,x_1},$$

where the order of the cycle equals the period of the sequence S_{x_0,x_1}. The entire path graph of $\mathcal{L}_a$ consists of a disjoint union of cycles of the same length. $\square$

Example 5.1. Suppose that $q = 7$. Then $2^{-1} = 4$, and the path graph of $\mathcal{L}_4$ consists of six disjoint cycles of order 7. When $a = 2, 3, 5, 6$, it can be verified that $a/(1 - a) = 5, 2, 4, 3$ (respectively). The values 2 and 4 have order 3, and 3 and 5 have order 6 in $\mathbb{F}_7{}^*$. Therefore $\mathcal{L}_2$ and $\mathcal{L}_6$ each consist of seven disjoint cycles of order 6, while $\mathcal{L}_3$ and $\mathcal{L}_5$ each consist of fourteen disjoint cycles of order 3.

To illustrate, let's look at $\mathcal{L}_2$ in more detail. If we take $x_0 = 0$, $x_1 = 1$, then we obtain $x_2 = 6$, $x_3 = 3$, $x_4 = 2$, $x_5 = 4$, and $x_{i+6} = x_i$ for all $i \geq 0$. The following six paths form a cycle in the path graph of $\mathcal{L}_2$: $016 - 163 - 632 - 324 - 240 - 401 - 016$. The seven 6-cycles in this path graph are as follows:

$$016 - 163 - 632 - 324 - 240 - 401 - 016$$
$$120 - 204 - 043 - 435 - 351 - 512 - 120$$
$$231 - 315 - 154 - 546 - 462 - 623 - 231$$
$$342 - 426 - 265 - 650 - 503 - 034 - 342$$
$$453 - 530 - 306 - 061 - 614 - 145 - 453$$
$$564 - 641 - 410 - 102 - 025 - 256 - 564$$
$$605 - 052 - 521 - 213 - 136 - 360 - 605.$$

$\square$

We are nearly in a position to compute a bound on the optical indices of our routings. The bound depends on the number of elements in $\mathbb{F}_q{}^*$ having odd order. We will use the following easy lemma.

Lemma 5.5. *Suppose q is an odd prime power and write $q - 1 = 2^s t$, where t is odd. Then $\mathbb{F}_q{}^*$ has exactly t elements of odd order (including 1, which has order 1).*

Proof. Let $\omega \in \mathbb{F}_q{}^*$ be a primitive element. For any i, ω^i has order equal to $(q-1)/\gcd(q-1,i)$. Hence, ω^i has odd order if and only if 2^s divides i, i.e., if $i = j\,2^s$ for some j where $0 \le j \le t-1$. $\qquad\square$

Consider the mapping θ defined as $\theta(a) = a/(1-a)$, $a \ne 0, 1$. It is easy to see that $\theta : \mathbb{F}_q\backslash\{0,1\} \to \mathbb{F}_q\backslash\{0,-1\}$ is a bijection. Using Lemma 5.5, information about the chromatic numbers of the path graphs of $\mathcal{L}_a$ can be summarized as follows:

- if $a = 2^{-1}$, then $a/(1-a) = 1$ and the path graph of $\mathcal{L}_a$ has chromatic number equal to 3,

- there are $t-1$ elements $a \ne 2^{-1}$ such that $a/(1-a)$ has odd order (the path graphs of these $\mathcal{L}_a$'s have chromatic number equal to 3), and

- for the remaining $q - 2 - t$ values of a, the path graphs of the $\mathcal{L}_a$'s have chromatic number equal to 2.

Suppose we name the levels of the routing as $\mathcal{L}_0, \ldots, \mathcal{L}_{q-2}$ so that the following conditions are satisfied:

- the path graph of $\mathcal{L}_0$ has chromatic number equal to 1 ($\mathcal{L}_0$ just consists of the arcs of $\vec{K}_q$, as usual),

- for $1 \le i \le q - 2 - t$, the path graph of $\mathcal{L}_i$ has chromatic number equal to 2, and

- for $q - 1 - t \le i \le q - 2$, the path graph of $\mathcal{L}_i$ has chromatic number equal to 3.

Then we obtain the following result.

Theorem 5.6. *Suppose q is an odd prime power and write $q - 1 = 2^s t$, where t is odd. Then there is a $(q-2)$-tolerant routing $\mathcal{R}$ of $\vec{K}_q$ such that $w_i(\mathcal{R}) = 2i + 1$ for $0 \le i \le q - 2 - t$, and $w_i(\mathcal{R}) = 3i + 1 - (q - 2 - t)$ for $q - 2 - t < i \le q - 2$.*

Observing that the value of s defined in Theorem 5.6 always satisfies the inequality $s \ge 1$, we obtain $t \le (q-1)/2$. The following result is an immediate corollary.

Corollary 5.7. *Suppose q is an odd prime power. Then there is a $(q-2)$-tolerant routing $\mathcal{R}$ of $\vec{K}_q$ such that $w_i(\mathcal{R}) \le (5i + 3)/2$ for all i. (In particular, $w_{q-2}(\mathcal{R}) \le (5q - 7)/2$.)*

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Remark 5.2. In [9, Theorem 2], it is shown for all prime powers q that there exists a $(q-2)$-tolerant routing $\mathcal{R}$ of $\vec{K}_q$ such that $w_i(\mathcal{R}) \leq 3i+1$ for all i. For the odd prime powers $q \equiv 3 \pmod 4$, Corollary 5.7 improves this bound by about 17%. The optical indices of our routings are about 25% greater than the theoretical minimum value.

For odd prime powers $q \equiv 1 \pmod 4$, our results are even better. Here, we have $s \geq 2$ in Theorem 5.6, and it can be verified that this improves the bound from [9] by about 25% (or more). The optical indices of our routings are no more than 12.5% greater than the theoretical minimum value (asymptotically).

6   On the $(q-2)$-th optical index

In this subsection, we continue our study of the routings described in Section 5. We look specifically at the optical index $\omega_{q-2}(\mathcal{R})$ of the $(q-2)$-tolerant routing $\mathcal{R}$ of $\vec{K}_q$. That is, we are no longer considering the levelled optical indices in this section.

In order to keep the analysis from becoming too messy, we will confine our attention to integers q which have the form $q = 2p+1$, where p is an odd prime. For such q, Theorem 5.6 says that $\omega_{q-2}(\mathcal{R}) \leq (5q-7)/2$. We show now that this can be considerably improved, to $2q+6$. This very close to the theoretical minimum value, $2q-3$.

Let $q = 2p+1$ where p is an odd prime and define

$$A = \{a \in \mathbb{F}_q^* : a/(1-a) \text{ has order } p\}.$$

Then $|A| = p-1$.

For any $a \in A$, let $\mathcal{M}_\gamma = \mathcal{L}_a$, where $\gamma = a/(1-a)$. It is easy to see that

$$\mathcal{M}_\gamma = \mathcal{M}_\gamma^+ \cup \mathcal{M}_\gamma^-,$$

where

$$\mathcal{M}_\gamma^+ = \{(j+\gamma^h, j+\gamma^{h+1}, j+\gamma^{h+2}) : j \in \mathbb{F}_q, 0 \leq h \leq p-1\}$$

and

$$\mathcal{M}_\gamma^- = \{(j-\gamma^h, j-\gamma^{h+1}, j-\gamma^{h+2}) : j \in \mathbb{F}_q, 0 \leq h \leq p-1\}.$$

Each of $\mathcal{M}_\gamma^+$ and $\mathcal{M}_\gamma^-$ consists of pq paths.

To illustrate this notation, we present a small example.

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Example 6.1. Suppose that $q = 7$; then $p = 3$. From Example 5.1, it can be seen that $A = \{3, 5\}$. Since $3/(-2) = 2$ and $5/(-4) = 4$ in $\mathbb{F}_7$, it follows that $\mathcal{M}_2 = \mathcal{L}_3$ and $\mathcal{M}_4 = \mathcal{L}_5$.

Consider $\gamma = 2$. As noted in Example 5.1, $\mathcal{M}_2$ consists of 14 cycles of length 3. $\mathcal{M}_2^+$ consists of the 21 paths $241, 124$ and 412 developed modulo 7, and $\mathcal{M}_2^-$ consists of the 21 paths $536, 653$ and 365 developed modulo 7. $\qquad\blacksquare$

Let $\beta \in \mathbb{F}_q^*$ be any fixed element of order p. Then $\{\beta^i : 1 \leq i \leq p - 1\}$ consists of all the elements in $\mathbb{F}_q$ of order p and it is immediate that

$$\{\mathcal{L}_a : a \in A\} = \{\mathcal{M}_{\beta^i} : 1 \leq i \leq p - 1\}.$$

We will consider all the paths in $\{\mathcal{M}_{\beta^i} : 1 \leq i \leq p - 1\}$. For every cycle in the associated path graph corresponding to these paths, we will remove one path. This changes every such cycle to a path on p vertices, which can be coloured using two colours. Then we will recolour the removed paths in such a way that the overall optical index is reduced.

For $1 \leq i \leq p - 1$, define

$$\mathcal{N}_{\beta^i}^+ = \{(j + 1, j + \beta^i, j + \beta^{2i}) : j \in \mathbb{F}_q\}$$

and

$$\mathcal{N}_{\beta^i}^- = \{(j - 1, j - \beta^i, j - \beta^{2i}) : j \in \mathbb{F}_q\}.$$

Then $|\mathcal{N}_{\beta^i}^+| = |\mathcal{N}_{\beta^i}^-| = q$ for every i. Further, it can be verified that $\mathcal{N}_{\beta^i}^+$ consists of one path from every cycle in the path graph of $\mathcal{M}_{\beta^i}^+$, and $\mathcal{N}_{\beta^i}^-$ consists of one path from every cycle in the path graph of $\mathcal{M}_{\beta^i}^-$.

Finally, define

$$\mathcal{N}^+ = \bigcup_{i=1}^{p-1} \mathcal{N}_{\beta^i}^+$$

and

$$\mathcal{N}^- = \bigcup_{i=1}^{p-1} \mathcal{N}_{\beta^i}^-.$$

Observe that $\mathcal{N}^+ \cup \mathcal{N}^+$ consists all the removed paths.

Lemma 6.1. *The path graph of each of $\mathcal{N}^+$ and $\mathcal{N}^-$ has chromatic number not exceeding 4.*

Proof. First we consider the path graph of $\mathcal{N}^+$. Each set $\mathcal{N}^+_{\beta^i}$ consists of one orbit of q paths under the action of $\mathbb{F}_q$. Take the orbit representative $P_i = (1, \beta^i, \beta^{2i})$, and denote the differences generated by the path P_i as $x_i = \beta^i - 1$ (the *first difference*) and $y_i = \beta^{2i} - \beta^i$ (the *second diference*). Construct the graph G with vertex set $\{P_i : 1 \leq i \leq p - 1\}$, in which P_i is adjacent to P_j if and only if $\{x_i, y_i\} \cap \{x_j, y_j\} \neq \emptyset$. To show that the path graph of $\mathcal{N}^+$ has chromatic number not exceeding 4, it suffices to show that G has chromatic number not exceeding 4.

Consider a path P_i having differences x_i, y_i as defined above. It is not hard to see that x_i can occur at most twice as a second difference, y_i can occur at most once as a first difference, and at most one additional time as a second difference. Therefore the degree of P_i in G is at most 4. It is also straightforward to show that G contains no copy of K_4 as a subgraph. Therefore, applying Brooks' Theorem (see, for example, [13]), it follows that G has chromatic number not exceeding 4.

By similar arguments, it can be shown that the path graph of $\mathcal{N}^-$ has chromatic number not exceeding 4. $\qquad\square$

Example 6.2. Let $q = 23$, so $p = 11$. The element $\beta = 2$ has order 11 in $\mathbb{F}_{23}{}^*$. We tabulate the first and second differences arising from the P_i's as follows:

i	β^i	$\beta^i - 1$	$\beta^{2i} - \beta^i$
1	2	1	2
2	4	3	12
3	8	7	10
4	16	15	10
5	9	8	3
6	18	17	7
7	13	12	18
8	3	2	6
9	6	5	7
10	12	11	17

The resulting graph G consists of three connected components: $\{P_1 P_8\}$, $\{P_5 P_2, P_2 P_7\}$, and $\{P_3 P_4, P_3 P_9, P_3 P_6, P_6 P_9, P_6 P_{10}\}$. It is easy to see that G has chromatic number equal to 3. Therefore $\mathcal{N}^+$ and $\mathcal{N}^-$ both have chromatic number equal to 3. $\qquad\square$

Theorem 6.2. *Suppose that $q = 2p + 1$ is prime, where p is an odd prime. Then there exists a $(q - 2)$-tolerant routing $\mathcal{R}$ of $\vec{K}_q$ such that $\omega_{q-2}(\mathcal{R}) \leq 2q + 6$.*

Proof. We proved in Corollary 5.7 that $\omega_{q-2}(\mathcal{R}) \leq (5q - 7)/2$. Using the technique described above, on each of $p - 1$ levels, we have reduced the number of wavelengths from 3 to 2 by removing a path from every cycle of length p in the relevant path graphs. Therefore, we have decreased the total number of wavelengths by $p - 1$ by removing the paths in $\mathcal{N}^+$ and $\mathcal{N}^-$. Then, applying Lemma 6.1, we use at most 8 new wavelengths for these paths. The total number of wavelengths is thus bounded above by

$$\frac{5q - 7}{2} - (p - 1) + 8 = \frac{5q - 7}{2} - \frac{q - 3}{2} + 8 = 2q + 6.$$

$\square$

Remark 6.1. The bound of $2q + 6$ in the theorem above is only 9 away from the theoretical optimal value, $2q - 3$.

Example 6.3. We continue Example 6.2. Here, the total number of wavelengths is

$$\frac{5q - 7}{2} - (p - 1) + 6 = 2q + 4 = 50.$$

Therefore, we have decreased the number of wavelengths by 4 in this particular case. $\square$

7 Summary

We have introduced three techniques for constructing $(n - 2)$-tolerant routings of $\vec{K}_n$ which have optimal or near-optimal optical index. The first of these techniques is a backtracking algorithm that we successfully used to find optimal routings for $n \leq 8$. These routings are constructed from certain base routings found by computer. Even though we have found only a few small examples, we conjecture that routings of this type exist for all integers $n \geq 4$.

Our second technique is a recursive construction based on so-called partitionable designs. This method yields infinite classes of routings that have minimum levelled optical index. We are hopeful that additional infinite classes of routings having minimum levelled optical index can be found using design-theoretic techniques.

Our third technique involves analyzing a simple algebraic formula that can be used to construct $(q - 2)$-tolerant routings of $\vec{K}_q$ when q is an odd prime power. A careful analysis of the structure of the routings allows us to show that these routings have levelled optical indices that are roughly 25%

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above the theoretical optimal values (in the worst case). This improves the general results in [9], where the upper bound proven on the (levelled) optical index is about 50% higher than the theoretical optimal value. Furthermore, we are able to find routings whose (non-levelled) optical indices are at most 9 away from the theoretical optimal value, whenever $q = 2p + 1$ and q and p are odd primes.

Acknowledgements

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A   Small Solutions

Base routing for $n = 4$:

$$031, 032, 013, 120, 102, 123, 230, 201, 213, 310, 321, 302$$

The path graph is a 12-cycle.

Base routing for $n = 5$:

$$021, 012, 013, 024, 140, 142, 103, 134, 230, 231, 243, 204$$
$$320, 321, 342, 304, 410, 431, 412, 403$$

The path graph consists of two 10-cycles.

Base routing for $n = 6$:

$$021, 012, 013, 034, 035, 140, 142, 153, 154, 135, 210, 251$$
$$253, 204, 245, 320, 341, 312, 304, 305, 450, 431, 432, 423$$
$$405, 510, 541, 502, 523, 524$$

The path graph is a 30-cycle.

Base routing for $n = 7$

$$021, 012, 013, 024, 035, 036, 140, 152, 163, 154, 105, 106$$
$$240, 261, 263, 254, 235, 256, 320, 361, 342, 314, 305, 346$$
$$430, 421, 432, 453, 465, 416, 560, 531, 512, 513, 504, 506$$
$$620, 641, 652, 623, 604, 645$$

The path graph consists of a 38-cycle and a 4-cycle.

Base routing for $n = 8$

$$021, 012, 013, 024, 035, 036, 047, 120, 142, 153, 134, 165$$
$$156, 107, 250, 271, 263, 274, 205, 216, 267, 360, 371, 352$$
$$314, 325, 376, 307, 410, 431, 452, 463, 475, 406, 457, 540$$
$$561, 572, 543, 534, 506, 517, 670, 641, 642, 623, 624, 605$$
$$617, 730, 751, 732, 723, 704, 765, 746$$

The path graph consists of a 40-cycle and a 16-cycle.