There are 23 nonisomorphic perfect one-factorizations of K_{14}

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Abstract: Using an orderly algorithm we established that there are exactly 23 nonisomorphic perfect one-factorizations of K_{14}. Seah and Stinson [13] had previously found 21 perfect one-factorizations of K_{14} with nontrivial automorphism group.

1 Introduction

A *one-factor* in a graph G is a set of edges in which every vertex appears precisely once. A *one-factorization* of G is a partition of the edge-set of G into one-factors. (We will sometimes refer to a one-factorization as an *OF*). If the complete graph on n vertices K_n has a one-factorization, then necessarily n is even and any such one-factorization contains $n-1$ one-factors each of which contains $n/2$ edges. Several excellent surveys on one-factorizations of the complete graph are [9], [12], and [15].

Two one-factorizations F and H of G, say $F = \{f_1, f_2, \ldots, f_k\}$, $H = \{h_1, h_2, \ldots, h_k\}$, are called *isomorphic* if there exists a map ϕ from the vertex-set of G onto itself such that $\{f_1\phi, f_2\phi, \ldots, f_k\phi\} = \{h_1, h_2, \ldots, h_k\}$. Here $f_i\phi$ is the set of all the edges $\{x\phi, y\phi\}$ where $\{x, y\}$ is an edge in F. The exact number of nonisomorphic one-factorizations of K_{2n} is known only for $2n \leq 12$. It is easy to see that there is a unique one-factorization of K_2, K_4, and K_6. There are exactly six for K_8 [2], 396 for K_{10} [6], and there are 526,915,620 nonisomorphic *OFs* of K12 [4]. In these searches, the orders of the automorphism groups of the factorizations were also found; this information can be used to calculate the exact number of distinct factorizations.

A *perfect* one-factorization (*P1F*) of graph G is a one-factorization in which the union of any pair of distinct one-factors forms a Hamiltonian cycle of G. Mendelsohn and Rosa conjectured that K_{2n} has a *P1F* for all $n \geq 2$ [9]. Theorems concerning the structure of automorphism group of a *P1F* were proven by Ihrig [7, 8]. The surveys cited above also review what is known about perfect one-factorizations of the complete graph.

It is known that up to isomorphism there is exactly one perfect one-factorization of K_n, for $n = 4, 6, 8, 10$, and there are exactly five for K_{12} [10]. Seah and Stinson [13, 14] determined that there are 21 nonisomorphic perfect one-factorizations of K_{14} that have nontrivial automorphism groups. We performed a complete enumeration of the nonisomorphic *P1Fs* of K_{14}, and determined that there are exactly 23. In the process we confirmed the result of Seah and Stinson.

2 Results

We generated the perfect one-factorizations of K_{14} with an *orderly* algorithm; it generates the nonisomorphic *OFs* of K_{14} in lexicographic order. The algorithm builds up each one-factorization by adding one one-factor at a time and rejects a partial one-factorization if it is not the lexicographically lowest representative of all the partial one-factorizations in its isomorphism class. In this way, the algorithm generates only the lowest representative of any isomorphism class of one-factorizations and as such never generates any *OFs* which are isomorphic to each other. This approach saves both time and space over algorithms which first generate distinct (but possibly isomorphic) one-factorizations and then use methods to winnow isomorphs.

Orderly algorithms have been used in other combinatorial searches including [1, 3, 11, 13, 14]. A systematic treatment of this method appears in [5]. Our algorithm is essentially the one described in [4].

The complete enumeration of the *P1Fs* of K_{14} required about 12 years of cpu time at a rate of 20 mips. However, since the algorithm performs a depth-first search through the tree of partial factorizations, we were able to complete the work in several months by distributing parts of the search to independent processors; a search from a partial solution can proceed independently from, and in parallel with, searches from other partial solutions.

We found that there are exactly 23 perfect one-factorizations of K_{14}. The 21 with nontrivial automorphism groups are those listed in [13] and [14]. The two *P1Fs*, F_{22} and F_{23}, with trivial automorphism groups are listed in Figure 1 with one one-factor per line, and each successive pair of vertices indicates an edge. Thus, the first line of F_{22} specifies the one-factor $\{(0,1),(2,3),(4,5),(6,7),(8,9),(10,11),(12,13)\}$.

F_{22}	F_{23}
0 1 2 3 4 5 6 7 8 9 10 11 12 13	0 1 2 3 4 5 6 7 8 9 10 11 12 13
0 2 1 4 3 6 5 8 7 10 9 12 11 13	0 2 1 4 3 6 5 8 7 10 9 12 11 13
0 3 1 5 2 7 4 9 6 12 8 11 10 13	0 3 1 5 2 7 4 10 6 9 8 13 11 12
0 4 1 7 2 11 3 12 5 9 6 10 8 13	0 4 1 8 2 9 3 13 5 6 7 11 10 12
0 5 1 12 2 13 3 10 4 6 7 8 9 11	0 5 1 10 2 11 3 9 4 12 6 13 7 8
0 6 1 11 2 9 3 4 5 12 7 13 8 10	0 6 1 3 2 12 4 13 5 10 7 9 8 11
0 7 1 10 2 5 3 11 4 13 6 9 8 12	0 7 1 11 2 8 3 4 5 13 6 12 9 10
0 8 1 2 3 13 4 10 5 6 7 9 11 12	0 8 1 12 2 13 3 11 4 7 5 9 6 10
0 9 1 13 2 8 3 5 4 7 6 11 10 12	0 9 1 2 3 7 4 6 5 11 8 12 10 13
0 10 1 6 2 4 3 8 5 11 7 12 9 13	0 10 1 9 2 5 3 12 4 8 6 11 7 13
0 11 1 3 2 12 4 8 5 7 6 13 9 10	0 11 1 13 2 6 3 5 4 9 7 12 8 10
0 12 1 9 2 10 3 7 4 11 5 13 6 8	0 12 1 6 2 10 3 8 4 11 5 7 9 13
0 13 1 8 2 6 3 9 4 12 5 10 7 11	0 13 1 7 2 4 3 10 5 12 6 8 9 11

Figure 1: The two perfect one-factorizations of K_{14} with trivial automorphism groups

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References

[1] J.W. Brown, Enumeration of Latin squares with application to order 8, *J. Combin. Theory B* **5** (1968), 177–184.

[2] L.E. Dickson and F.H. Safford, Solution to problem 8 (group theory). *Amer. Math. Monthly* **13** (1906), 150–151.

[3] J.H. Dinitz and D.K. Garnick, Holey factorizations, *Ars Combinatoria*, to appear.

[4] J.H. Dinitz, D.K. Garnick and B.D. McKay, There are 526,915,620 nonisomorphic one-factorizations of K_{12}, *Journal Combin. Design* **2** (1994), 273–285.

[5] I.A. Faradzhev, Constructive enumeration of combinatorial objects, *Problemes Combinatoires et Theorie des Graphes Colloque Internat.* CNRS 260. CNRS Paris (1978), 131–135.

[6] E.N. Gelling and R.E. Odeh, On 1-factorizations of the complete graph and the relationship to round-robin schedules. *Congressus Num.* **9** (1974), 213–221.

[7] E.C. Ihrig, Symmetry groups related to the construction of perfect one-factorizations of K_{2n}, *J. Combin. Theory B* **40** (1986), 121–151.

[8] E.C. Ihrig, The structure of symmetry groups of perfect one-factorizations of K_{2n}, *J. Combin. Theory B* **47** (1989), 307–329.

[9] E. Mendelsohn and A. Rosa, One-factorizations of the complete graph – a survey, *J. Graph Theory* **9** (1985), 43–65.

[10] L. Petrenyuk and A. Petrenyuk, Intersection of perfect one-factorizations of complete graphs, *Cybernetics* **16** (1980), 6–9.

[11] R.C. Read, Every one a winner, *Annals of Discrete Math.* **2** (1978), 107–120.

[12] E. Seah, Perfect one-factorizations of the complete graph – a survey, *Bull. ICA* **1** (1991), 59–70.

[13] E. Seah and D.R. Stinson, Some perfect one-factorizations for K_{14}. *Ann. Discrete Math.* **34** (1987), 419–436.

[14] E. Seah and D.R. Stinson, On the enumeration of one-factorizations of the complete graph containing prescribed automorphism groups. *Math Comp.* **50** (1988), 607–618.

[15] W.D. Wallis, One-factorizations of the complete graph, in *Contemporary Design Theory: A Collection of Surveys* (eds. J.H. Dinitz and D.R. Stinson), Wiley, 1992, 593–639.