

PAIRWISE ORTHOGONAL SYMMETRIC LATIN SQUARES

Jeffrey H. Dinitz
University of Vermont

The purpose of this paper is to update all previous lists of known lower bounds for the number of pairwise orthogonal symmetric Latin squares (POSLS) of side n .

Gross, Mullin, and Wallis [7] define Latin squares R and C , of side t , to be orthogonal symmetric if they satisfy the following three properties: (i) R and C are both symmetric; (ii) R and C both have i th diagonal entry i ; and (iii) if R and C have (i,j) entries ρ and γ , respectively, where $i < j$, then there are not numbers k and ℓ for which $k < \ell$ and R and C have (k,ℓ) entries ρ and γ , respectively, unless $k = i$ and $\ell = j$. It can be seen that necessarily t must be odd.

A Room square of side t is a t by t array of cells, whose entries are chosen from a set S of $t + 1$ symbols which satisfies the following conditions: (i) every cell of the array is either empty or contains an unordered pair of distinct symbols from S ; (ii) each symbol occurs in every row and in every column of the array; and (iii) every unordered pair of symbols occurs precisely once in the array (see [9]). A Room n -cube of side t (Room n -design [7,8]) is an n dimensional array of side t which enjoys the property that each two-dimensional projection of the array is a Room square of side t . Thus a Room square is a Room 2-cube and a Room cube is a Room 3-cube. Note that again t is necessarily odd.

The connection between Room n -cubes and POSLS is given in the following theorem.

Theorem 1 (Bruck [2], Horton [8]): A Room n -cube of side t is equivalent to n pairwise orthogonal symmetric Latin squares of side t .

We can ask questions about POSLS similar to those which have been asked concerning orthogonal Latin squares. Given an odd positive integer t , write $\nu(t)$ for the size of the largest possible set of POSLS of side t , or equivalently, for the largest n for which a Room n -cube of side t exists. In 1973 Gross, Mullin, and Wallis [7] gave a list summarizing the known lower bounds for $\nu(t)$. In 1979 Beaman and Wallis [1] gave an updated version of the 1973 list. In the spirit of those

previous lists we give the current list of known lower bounds on $v(t)$, with t odd.

- A) $v(1) = 1$, $v(3) = 1$, $v(5) = 1$, and $v(7) = 3$.
- B) $v(t) \geq 3$ when $t > 5$. That is, Room cubes exist for all odd sides except 3 or 5 [6].
- C) If $q = 2^nk + 1$ is a prime power, with k odd, then $v(q) \geq k$ [3].
As a special case we have that if $q = 2k + 1$ is a prime power with k odd, then $v(q) = \frac{q-1}{2}$ [8].
- D) If there exists a pairwise balanced design with index unity, on v objects and with block size in the set K , and if there exist t POSLS of order k for every $k \in K$, then $v(v) \geq t$ [7].
- E) If there exists an (n, \mathcal{L}, u) -frame, a Room n -cube of side v containing a sub-Room n -cube of side w , and if $N(\frac{v-w}{2}) \geq n$, then $v(u(v-w) + w) \geq n$ [5]. Here $N(n)$ is the number of pairwise orthogonal Latin squares of side n . The following two cases are special cases of E.
- F) If $v(u) \geq n$, and if there is a Room n -cube of side v containing a sub-Room n -cube of side w , and if $N(v-w) \geq n$, then $v(u(v-w) + w) \geq n$ [8].
- F') As a special case of F, we have $v(uv) \geq \min\{v(u), v(v), N(v)\}$ and $v(u(v-1) + 1) \geq \min\{v(u), v(v), N(v-1)\}$.
- G) $v(13) \geq 5$, $v(15) \geq 4$, $v(17) \geq 4$, $v(21) \geq 4$, $v(25) \geq 7$, $v(29) \geq 13$, $v(37) \geq 15$, $v(41) \geq 9$, $v(53) \geq 17$, $v(61) \geq 21$, and $v(101) \geq 31$ [4].

In [1], Beaman and Wallis give a set of 4 symmetric Latin squares of side 9 which were claimed to be pairwise orthogonal. Upon inspection this turns out not to be true, and indeed no subset of size 3 is pairwise orthogonal. Using a computer search, we have found a new set of 3 POSLS of side 9, which are listed in Table 1. This set cannot be extended to 4 POSLS of side 9. Thus $v(9) \geq 3$; however, the exact value of $v(9)$ is still unknown.

We update the list of known lower bounds on $v(t)$ for small values of t , which appeared in [1] and [7]; the up-to-date list is shown in Table 2. In [5] Dinitz and Stinson give a list of lower bounds for $v(t)$ for $t \leq 1000$.

1	3	2	5	4	7	6	9	8
3	2	1	6	7	8	9	4	5
2	1	3	7	8	9	5	6	4
5	6	7	4	9	3	8	1	2
4	7	8	9	5	2	1	3	6
7	8	9	3	2	6	4	5	1
6	9	5	8	1	4	7	2	3
9	4	6	1	3	5	2	8	7
8	5	4	2	6	1	3	7	9

1	4	8	6	7	5	9	3	2
4	2	7	1	3	9	5	6	8
8	7	3	9	6	4	1	2	5
6	1	9	4	8	2	3	5	7
7	3	6	8	5	1	2	9	4
5	9	4	2	1	6	8	7	3
9	5	1	3	2	8	7	4	6
3	6	2	5	9	7	4	8	1
2	8	5	7	4	3	6	1	9

1	5	7	3	8	9	2	4	6
5	2	4	9	6	3	8	7	1
7	4	3	8	9	1	6	5	2
3	9	8	4	2	7	1	6	5
8	6	9	2	5	4	3	1	7
9	3	1	7	4	6	5	2	8
2	8	6	1	3	5	7	9	4
4	7	5	6	1	2	9	8	3
6	1	2	5	7	8	4	3	9

Table 1. 3 Pairwise orthogonal symmetric Latin squares of order 9

- 1) I. R. Beaman and W. D. Wallis, Pairwise orthogonal symmetric Latin squares Proc. 9th S-E Conf. on Combinatorics, Graph Theory, and Computing, Boca Raton, Florida (1978), pp. 113-118.
- 2) R. H. Bruck, "What is a loop?", in Studies in Modern Algebra, Prentice-Hall, 1963.
- 3) J. H. Dinitz, New lower bounds for the number of pairwise orthogonal symmetric Latin squares, Proc. 10th S-E Conf. on Combinatorics, Graph Theory, and Computing, Boca Raton, Florida (1979), pp. 393-398.
- 4) J. H. Dinitz, Room n-cubes of low order, preprint.
- 5) J. H. Dinitz and D. R. Stinson, The construction and uses of frames, Ars Combinatoria, Vol. 10 (1980), pp. 31-53.
- 6) J. H. Dinitz and D. R. Stinson, The spectrum of Room cubes, submitted.
- 7) K. B. Gross, R. C. Mullin, and W. D. Wallis, The number of pairwise orthogonal symmetric Latin squares, Utilitas Math. 4 (1973), pp. 239-251.
- 8) J. D. Horton, Room designs and one-factorizations, Aeq. Math., to appear.
- 9) R. C. Mullin and W. D. Wallis, The existence of Room squares, Aeq. Math. 13 (1975), pp. 1-7.

t	v(t)	t	v(t)	t	v(t)
1	1	35	3	69	3
3	1	37	15	71	35
5	1	39	3	73	9
7	3	41	9	75	3
9	3	43	21	77	3
11	5	45	3	79	39
13	5	47	23	81	5
15	4	49	3	83	41
17	4	51	3	85	3
19	9	53	17	87	3
21	4	55	3	89	11
23	11	57	3	91	3
25	7	59	29	93	3
27	13	61	21	95	3
29	13	63	3	97	3
31	15	65	3	99	3
33	3	67	33	101	31

Table 2. Known lower bounds for $v(t)$, $1 \leq t \leq 101$