The Existence of Referee Squares

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Abstract

In this short note, we prove a conjecture of Anderson, Hamilton and Hilton [1] on the existence of referee squares.

1 Introduction

Let n be an odd integer. A referee square of order n is an $n \times n$ array R based on $S = \{1, 2, \ldots, n\}$ such that

1. each cell is either empty or contains an unordered pair of distinct symbols on S,

2. each $i \in S$ occurs precisely once in each row (except the ith) and in each column (except ith column), and does not occur in the ith row and ith column,

3. each unordered pair of distinct elements of S occurs in exactly one cell of R,

4. the main diagonal cells are non-empty.

Note the close relationship between referee squares and the more well-known object, the Room square (see [4] for a survey of Room squares).

Referee squares were first introduced in [1] where it was conjectured that they exist for all odd orders $n \geq 3$ with $n \neq 5$. In 1998 Y.S. Liaw [5] made progress towards this conjecture by proving the following:

Theorem 1.1 *(Liaw [5]) There exists a referee square of order n for any odd composite integer n and for all* $3 \leq n \leq 47$ *except that there is no referee square of order 5.*

In this short note, we solve the existence problem completely by proving the following result.

Theorem 1.2 *If* $n \geq 3$, $n \neq 5$ *and* n *odd, there exists a referee square of order* n.

2 Constructions

The main recursive construction uses frames. But in order to apply the recursion, a few small orders are needed first. In order to obtain these we use variant of a strong starter.

A *strong referee starter* S of order v in $\mathbb{Z}_v$ is a set of unordered pairs $S = \{\{s_i, t_i\} : 1 \leq i \leq (v-1)/2\}$ which satisfies the following properties:

1. $\{s_i : 1 \leq i \leq (v-1)/2\} \cup \{t_i : 1 \leq i \leq (v-1)/2\} = \{1, 2, \ldots, n-1\}$

2. $\{\pm(s_i - t_i) : 1 \leq i \leq (v-1)/2\} = \mathbb{Z}_v \backslash \{0\} = \{1, 2, \ldots, n-1\}$

3. if $s_i + t_i \equiv s_j + t_j \pmod{v}$, then $i = j$

4. for some i, $s_i + t_i \equiv 0 \pmod{v}$.

In [5] (Theorem 2.1) it is proven that a referee square of order v exists if there exists two starters with distinct distances and which contains a zero distance. It is easy to show that given a strong referee starter S, the two starters S and $-S$ satisfy this property. Hence the existence of a strong referee starter of order v, implies the existence of a referee square of order v.

Lemma 2.1 *There exist referee squares of orders* $n = 53, 59, 61, 79$.

Proof: For each of these orders we give a strong referee starter. In each case, the pair with difference 1 has sum congruent to zero modulo n.

$$n = 53$$

26,27 43,45 34,37 20,24 8,13 5,11 3,10 6,14 40,49 47,4 18,29 30,42 41,1 9,23 17,32 19,35 38,2 33,51 12,31 16,36 39,7 46,15 21,44 28,52 25,50 22,48

$$n = 59$$

29,30 12,14 1,4 27,31 45,50 49,55 13,20 40,48 34,43 56,7 36,47 41,53 22,35 18,32 46,2 9,25 11,28 33,51 57,17 6,26 3,24 52,15 19,42 58,23 39,5 54,21 10,37 16,44 38,8

$$n = 61$$

30,31 32,34 6,9 4,8 40,45 52,58 37,44 47,55 24,33 26,36 28,39 59,10 12,25 50,3 14,29 7,23 5,22 60,17 35,54 43,2 41,1 57,18 53,15 48,11 13,38 16,42 19,46 21,49 27,56 51,20

$$n = 79$$

39,40 74,76 58,61 37,41 44,49 23,29 68,75 6,14 22,31 9,19 59,70 36,48 7,20 66,1 2,17 26,42 60,77 51,69 63,3 5,25 11,32 12,34 27,50 38,62 28,53 45,71 30,57 55,4 43,72 73,24 15,46 35,67 64,18 78,33 65,21 56,13 52,10 16,54 8,47

Now for the frame constructions. Let S be a set, and let $\{S_1, S_2, \ldots, S_n\}$ be a partition of S. An $\{S_1, S_2, \ldots, S_n\}$-*Room frame* is an $|S| \times |S|$ array, F, indexed by S, that satisfies the following properties:

1. Every cell of F either is empty or contains an unordered pair of distinct symbols of S.

2. The subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are referred to as *holes*).

3. Each symbol $x \notin S_i$ occurs in row (or column) s, for any $s \in S_i$.

4. The pairs in F are those $\{s, t\}$, where $(s, t) \in (S \times S) \setminus \cup_{i=1}^{n} (S_i \times S_i)$.

As is usually done in the literature, we refer to a Room frame simply as a *frame*. The *type* of a frame F is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. We usually use an "exponential" notation to describe types: a type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ denotes u_i occurrences of holes of size t_i, $1 \leq i \leq k$.

Theorem 2.2 *Suppose there exists a frame of type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$, if there exists a referee square for t_i, $1 \leq i \leq k$, then there exists a referee square of order $\sum_{i=1}^{k} t_i u_i$.*

Proof: Fill in each hole $S_i \times S_i$ of side $t \times t$ by putting in a referee square of order t containing the symbols of S_i. □

Let K be a set of positive integers. A *group divisible design* K-GDD is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{A})$ where

1. $\mathcal{X}$ is a finite set of *points*,

2. $\mathcal{G} = \{\mathcal{S}_\rangle : \infty \leq \rangle \leq \backslash\}$ is a set of subsets of $\mathcal{X}$, called *groups*, which partition $\mathcal{X}$,

3. $\mathcal{A}$ is a collection of subsets of $\mathcal{X}$ with sizes from K, called *blocks*, such that every pair of points from distinct groups occurs in exactly 1 block, and

4. no pair of points belonging to a group occurs in any block.

The *type* of a GDD is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. Again the "exponential" notation is used to describe types: a type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ denotes u_i occurrences of groups of size t_i, $1 \leq i \leq k$. The following is the Fundamental Frame Construction (see [4]).

Theorem 2.3 *Let $(\mathcal{X}, \mathcal{G}, \mathcal{A})$ be a GDD, and let $w : \mathcal{X} \rightarrow Z^+ \cup \{0\}$ be a weight function on $\mathcal{X}$. Suppose that for each block $A \in \mathcal{A}$, there exists a frame of type $\{w(x) : x \in A\}$. Then there is a frame of type $\{\sum_{x \in G_i} w(x) : G_i \in \mathcal{G}\}$.*

We are now in position to solve the existence problem for referee squares. Denote $[a, b]$ as the set of odd integer between a and b and let R denote the set of positive integers n such that there exists a referee square of order n.

Lemma 2.4 *If $7 \leq n \leq 187$, then there exists a referee square of order n.*

Proof: If $7 \leq n \leq 51$, the result is obtained by Liaw [5]. When $n = 53, 59, 61, 79$, the result is obtained from Lemma 2.1. For $n = 55, 57, 63, 65, 77$ and 145, the result is obtained by Liaw since n is composite.

From a 5-GDD of type $(2m + 1)^5$ (which exists for all $m \geq 2$ [2]), give weight 3 to every point in the first four groups and weight 1 or 3 to the points in the last group. Since both frames of type 3^5 and $3^4 1^1$ exist [3], by Theorem 2.3 there exists a frame of type $(6m + 3)^4(2k + 1)$ when $m \leq k \leq 3m + 1$. Clearly, there exists a referee square of order $6m + 3$ since it is composite. When $2 \leq m \leq 8$, a referee square of order $2k+1$ exists when $3 \leq k \leq 3m+1$. Therefore, a referee square of order n can be constructed for all odd n such that $13(2m + 1) \leq n \leq 15(2m + 1)$. Apply this to $m = 2, 3, 4, 5$ to obtain $[67, 75] \cup [91, 105] \cup [117, 135] \cup [143, 165] \subset R$.

In a similar manner from a 9-GDD of type $(2m + 1)^9$ give weight 1 to the first eight groups and 1 or 3 to the last group to obtain a frame of type $(2m + 1)^8(2k + 1)$ for all $m \leq k \leq 3m + 1$. This is possible since there exist frames of type 1^9 and $1^8 3^1$ (see [4]). When $3 \leq m \leq 10$, there exists a referee square for each of the possible hole sides in the frame. Hence, we can construct a referee square of order n when $18m + 9 \leq n \leq 22m + 11$. Take $m = 4, 5, 6, 8$ to obtain $[81, 99] \cup [99, 121] \cup [117, 143] \cup [153, 187] \subset R$. □

Corollary 2.5 *There exists a referee square for order n if and only if n odd and $n \geq 7$ or $n = 3$.*

Proof: Begin with a 5-GDD of type $(2m+1)^5$ (these exist for all $m > 1$ [2]). Give weight 3 to every point in the first four groups and weight 1 or 3 to the points in last group. Since both frame of type 3^5 and $3^4 1^1$ exist ([3]), there exists a frame of type $(6m+3)^4(2m+1+2i)^1$ for $0 \leq i \leq 2m+1$. Again, there exists a referee square of order $6m + 3$ since it is composite. Therefore, if there exists a referee square of order $2m+1+2i$ for $0 \leq i \leq 2m+1$, then there exists a referee square of order $26m+13+2i$ for $0 \leq i \leq 2m+1$. Translating the notation, if $[2m+1, 6m+3] \subset R$, then $[26m+13, 30m+15] \subset R$. Since $[7, 187] \subset R$, by choosing $m \geq 6$ the result follows by induction. □

References

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