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The Existence of Room 5-Cubes

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1. INTRODUCTION

A Room t-cube of side n is a t-dimensional cube satisfying the property that each of its 2-dimensional projections is a Room square. More precisely, a *Room t-cube of side n* is a t-dimensional array of side n on a set S of $n+1$ objects called symbols (usually $\{0, 1, 2, ..., n-1, \infty\}$) which satisfy the following conditions:

(i) Each cell is either empty or contains an unordered pair of distinct symbols from S.

(ii) Each symbol occurs in every $(t-1)$-dimensional flat in the array exactly once.

(iii) Every unordered pair of symbols occurs precisely once in the array.

A Room t-cube is standardized if it also satisfies (iv) the ith diagonal cell contains the pair of symbols $\{\infty, i\}$. There are several other equivalent forms that are taken by Room t-cubes, one in terms of graphs and one in terms of Latin squares. The first of these forms is as a set of orthogonal one-factorizations of K_n. Let G be a graph with an even number of vertices. A *one-factor* in G is a set of (pairwise disjoint) edges which between them contain each vertex exactly once. A *one-factorization* is a partition of all the edges of G into pairwise disjoint one-factors. Two one-factorizations F_1 and F_2 are *orthogonal* if any one-factor in F_1 and any one-factor in F_2 have at most one edge in common. The following theorem from [12] shows the connection between orthogonal one-factorizations of K_n and Room t-cubes.

THEOREM 1.1 (Horton). *The existence of t pairwise orthogonal one-factorizations of K_{n+1} is equivalent to the existence of a Room t-cube of side n.*

Another formulation of Room t-cubes is in terms of orthogonal symmetric Latin squares. Two Latin squares R and C are said to be *orthogonal*

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symmetric Latin squares if they satisfy the following three properties: (i) R and C are both symmetric; (ii) R and C both have ith diagonal entry i; and (iii) if R and C have (i, j) entries α and β, respectively, where $i<j$, then there are not numbers k and l for which $k<l$ and R and C have (k, l) entries α and β respectively, unless $k=i$ and $l=j$. We see that two symmetric Latin squares with property (iii) are as close to orthogonal as is possible without sacrificing symmetry, thus the term orthogonal symmetric Latin squares. The following theorem relating Room t-cubes and pairwise orthogonal symmetric Latin squares can be found in [10].

THEOREM 1.2. *The existence of t pairwise orthogonal symmetric Latin squares of side n is equivalent to the existence of a Room t-cube of side n.*

In each of the above definitions it is easily seen that n is necessarily odd. Let $v(n)$ denote the size of the largest possible set of pairwise orthogonal symmetric Latin squares of side n or equivalently the largest t such that there exists a Room t-cube of side n. Much work has been done on finding values for $v(n)$. Some results on $v(n)$ are given in the next theorem. Other results can be found in [2, 5, 10].

THEOREM 1.3. (a) $v(1)=\infty$, $v(3)=1$, $v(5)=1$, $v(7)=3$ [14].

(b) $v(9)=4$ [9].

(c) *$v(n) \rightarrow \infty$ as $n \rightarrow \infty$* [10].

(d) *If $n=2^k t+1$ is a prime power with t odd, then $v(n) \geqslant t$* [3].

(e) *For all odd $n \geqslant 7$, $v(n) \geqslant 3$* [8].

In this paper we improve upon Theorem 1.3(e) above. We will prove

THEOREM 1.4. *If n is odd and $n \geqslant 11$ (except possibly $n=15$), then there is a Room 5-cube of side n* (i.e., $v(n) \geqslant 5$).

In Section 2 we give the main constructions needed for the proof. In Section 3 we establish the theorem for "small" values of n, those $\leqslant 4575$, then in Section 4 we complete the proof.

2. MAIN CONTRUCTIONS

The main recursive construction for Room 5-cubes uses 5-dimensional frames. These were first defined by Dinitz and Stinson [7]. The definition is as follows: Let T and U be sets with $|T|=t$ and $|U|=u$. A $t \times u$ by $t \times u$

array S will be called a *t-frame of order u* if it satisfies the following properties:

(1) Each cell is either empty or contains an unordered pair of elements of $U \times T$.

(2) There exist u empty t by t subsquares of S, no two of them containing any cell in the same row or column. These subsquares will be denoted Su_i and are called holes (it will usually be convenient to place these empty arrays on the diagonal of S).

(3) A row or column of S which meets Su_i contains each element of $(U\backslash\{u_i\}) \times T$ exactly once, and contains no element of $\{u_i\} \times T$.

(4) Each unordered pair of elements $\{(u_1, t_1), (u_2, t_2)\}$ with $u_1 \neq u_2$, occurs in a unique cell of S.

An *n-dimensional t-frame of order u* is an n-dimensional cube of side $t \times u$, which satisfies property(1) and such that each two-dimensional projection is a t-frame of order u. Informally an n-dimensional t-frame of order u is a Room n-cube of side $t \times u$ "missing" a spanning set of u disjoint Room n-cubes of side t. It is convenient to index the cells of S by the elements of $(U \times T)^n$, so that the cells of the rows meeting any Su_i are $(\{u_i\} \times T) \times (U \times T) \times (U \times T) \times \cdots \times (U \times T)$. For brevity we may rever to an n-dimensional t-frame of order u as an (n, t, u)-frame. Obviously, we are interested in $(5, t, u)$-frames.

Let a Room n-cube be described with symbols $U \cup \{\infty\}$ where $|U| = u$ and $\infty \notin U$. If the contents of all cells containing ∞ are removed, one obtains a 1-frame, specifically an $(n, 1, u)$-frame. Given a 1-frame, the holes can be filled in to obtain a Room n-cube. Thus we have that a $(n, 1, u)$-frame is equivalent to a Room n-cube of side u. We will make use of this fact in many of the constructions which follow.

We require several definitions concerning designs. A *group-divisible design* (GDD) is a triple $(X, \mathfrak{G}, \mathcal{A})$, where X is a finite set (of points), $\mathfrak{G}$ is a partition of X into subsets called groups, and $\mathcal{A}$ is a set of subsets of X into subsets called (blocks), such that (1) every unordered pair of points $\{x_1, x_2\}$ not contained in a group is contained in a unique block and (2) a group and a block contain at most one point in common.

A *Latin square* (of order s) based on a symbol set S, where $|S| = s$, is an s by s array L of the symbols of S, such that each symbol occurs precisely once in each row and each column. Two Latin squares, L and M of order s based on symbol sets S and T, respectively, are said to be *orthogonal* provided their superimposition yields every ordered pair in $S \times T$ exactly once. Several Latin square are mutually orthogonal if each pair is orthogonal. We refer to a set of mutually orthogonal Latin squares as a set of MOLS. Let $N(n)$ denote the maximum number of MOLS of size n. The following is a well-known result concerning MOLS (see [11]).

LEMMA 2.1. *Suppose* $n \geqslant 2$ *has prime power factorization* $n = \prod p^i$. *Then* $N(n) \geqslant \min\{p^i - 1\}$.

A special type of group divisible design associated with sets of MOLS is called a transversal design. A *transversal design* $\mathrm{TD}(m, n)$ is a $\mathrm{GDD}(X, \mathfrak{G}, \mathscr{A})$ in which $|X| = mn$, $\mathfrak{G}$ consists of m groups, each of cardinality n, and $\mathscr{A}$ consists of n^2 blocks each size m. The following is also well known (see [11]).

LEMMA 2.2. *The existence of a* $\mathrm{TD}(m, n)$ *is equivalent to the existence of* $m-2$ MOLS *of under* n.

For notation we will say that $n \in R_t$ if there is a Room t-cube of side n. We can now present the main recursive construction for Room 5-cubes.

THEOREM 2.3. *If* $N(s) \geqslant n-1$ *and there exist a* (d, t, n)*-frame and a* $(d, t, n+1)$*-frame, and if* $ts+1 \in R_d$ *and* $ta+1 \in R_d$ *with* $a \leqslant s$, *then* $tns + ta + 1 \in R_d$.

Proof. We will present the proof in the two-dimensional ($d=2$) case for ease of readability. The higher dimensional cases (in particular $d=5$) are proven similarly.

Since $N(s) \geqslant n-1$, there exists a $\mathrm{TD}(s, n+1)$ which we will denote by $(X, \mathfrak{G}, \mathscr{A})$. For each $x \in X$ and $y \in X$, let S_x be a set of size t with $S_x \cap S_y = \varnothing$ if $x \neq y$. For $G \in \mathfrak{G}$, let $S_G = \bigcup_{x \in G} S_x$. Now from some group delete $s-a$ elements and call this "short" group G_0, so $|G_0| = a$. Let $S_{G_0} = \bigcup_{x \in G_0} S_x$.

For every $A \in \mathscr{A}$, there is a frame F_A on the symbols $S_A = \bigcup_{x \in A} S_x$ since either $|A| = n+1$, or $|A| = n$ if $A \cap G_0 = \varnothing$.

We first construct a t-frame whose rows and columns are indexed by $S = \bigcup_{x \in X} S_x$. We let $A(x, y)$ denote the block in the TD containing $\{x, y\}$. Let F be defined by

$$F(s, t) = \begin{cases} \varnothing & \text{if } \{s, t\} \subseteq S \text{ for some } G \in \mathfrak{G} \text{ or } G = G_0, \\ F_{A(x,y)}(s, t) & \text{otherwise, where } s \in S_x \text{ and } t \in S_y. \end{cases}$$

Note that F is $tns + ta$ by $tns + ta$ in size.

We now construct a Room square from this frame F. We will basically just add a border and fill in the "holes" with Room squares.

Let Ω, ∞ be such that $\{\Omega, \infty\} \cap S = \varnothing$. Add a new row and column to F indexed by ∞. If $G \in \mathfrak{G}$, then let R_G be a Room square of order $ts+1$ indexed by $S'_G = S_G \cup \{\infty\}$ on the symbols $S_G \cup \{\infty, \Omega\}$ with

$R_G(\infty, \infty) = \{\infty, \Omega\}$. We define the Room square R of side $tns + ta + 1$ indexed by $S \cup \{\infty\}$ as

$$R(s, t) = \begin{cases} R_G(s, t) & \text{if } \{s, t\} \subseteq S'_G \quad \text{for some } G \in \mathfrak{G}, \\ F(s, t) & \text{otherwise.} \end{cases}$$

We first show that each pair of symbols occur in precisely one cell. Pick two symbols $s \in S_x$ and $t \in S_y$, if $\{x, y\} \subseteq G$ for some group G, then $\{x, y\}$ occurs in a unique cell of R_G. If x and y are in different groups, then $\{x, y\}$ occurs in a unique cell of $F_{A(x,y)}$. Now if $s \in S_x$, then $\{\infty, s\}$ and $\{\Omega, s\}$ occur in R_G where $x \in G$. Finally $\{\infty, \Omega\}$ occur in cell $R(\infty, \infty)$. Thus each pair of elements occurs together exactly once.

Now pick a row $r \in S_x$ and a symbol $s \in S_y$, if $\{x, y\} \subseteq G$ for some group G, then s occurs in a unique cell of row r in R_G, and in no other cell in row r. If x and y are in different groups, then s occurs in a unique cell in row r in $F_{A(x,y)}$ and in no other cell in row r. If $r \in S_x$ then ∞ and Ω occur in unique cells in row r in R_G where $x \in G$. Finally if $r = \infty$, and if $s \in S_y$, then s occurs in row ∞ in R_G where $y \in G$. Also $\{\Omega, \infty\}$ is in cell (∞, ∞), completing the proof.

Again we note that the proof of Theorem 2.3 is easily extended from the 2-dimensional to the d-dimensional case. We of course are interested in the case where $d = 5$. We will state this case as

COROLLARY 2.4. *If $N(s) \geq n - 1$ and there exists a $(5, t, n)$-frame, a $(5, t, n+1)$-frame, and if $ts + 1 \in R_5$ and $ta + 1 \in R_5$ with $a \leq s$, then $tns + ta + 1 \in R_5$.*

In order to use Corollary 2.4 we need to be able to construct 5-frames and Room 5-cubes for small order s. To do so we will use what are termed frame starters. Frame starters were first defined in [7] and are just a generalization of the starters used to construct Room squares.

Let G be an additive abelian group, and H a subgroup. Denote $|G| = g$, $|H| = h$, and suppose $g - h$ is even. An *$(h, g/h)$-frame starter* in $G \backslash H$ is a set of unordered pairs $S = \{\{s_i, t_i\}, 1 \leq i \leq (g-h)/2\}$ satisfying

(1) $\{s_i\} \cup \{t_i\} = G \backslash H$, and

(2) $\{\pm(s_i - t_i)\} = G \backslash H$.

If $H = \{0\}$, then we get a $(1, g)$-frame starter. A $(1, g)$-frame starter is called a *starter* of order g (note g must be odd) and is equivalent to the well-known starters used to construct Room squares.

Let $A = \{\{s_i, t_i\}\}$ and $B = \{\{u_i, v_i\}\}$ be two frame starters. We may assume that $t_i - s_i = v_i - u_i$, for $1 \leq i \leq (g-h)/2$. A and B are *orthogonal frame starters* provided $u_j - s_j = u_i - s_i$ implies $i = j$ and $u_i - s_i \notin H$ for all i.

Several starters are *pairwise orthogonal* if each pair of starters is orthogonal. A frame starter $A = \{\{s_i, t_i\}\}$ is *strong* if $s_i + t_i = s_j + t_j$ implies $i = j$ and if $s_i + t_i \notin H$ for all i. The special frame starter $P = \{\{s_i, t_i\}\}$ where $s_i = -t_i$ for all i is called the *patterned frame starter*. It is obvious that this is only a starter in $G \backslash H$ if $|G|$ is odd. The following two theorems are proven in [7].

LEMMA 2.5. *If $A = \{\{s_i, t_i\}\}$ is a strong frame starter, then A and $-A = \{\{-s_i, -t_i\}\}$ are orthogonal frame starters.*

LEMMA 2.6. *If there is a strong frame starter in $G \backslash H$ with $|G|$ odd, then there are 3 pairwise orthogonal frame starters in $G \backslash H$. (These are A, $-A$ and P).*

If A and B are strong frame starters, then we say that A and B are *orthogonal strong frame starters* provided A is orthogonal to B and $-B$. We have

LEMMA 2.7. *If there exist two orthogonal* strong *frame starters in $G \backslash H$, then*

(a) *if $|G|$ is even, there are 4 orthogonal frame starters in $G \backslash H$,*

(b) *if $|G|$ is odd, there are 5 orthogonal frame starters in $G \backslash H$.*

Proof. Let A and B be orthogonal strong frame starters in $G \backslash H$, then A, $-A$, B, $-B$ are 4 pairwise orthogonal frame starters in $G \backslash H$. If $|G|$ is odd, then A, $-A$, B, $-B$, and P are all pairwise orthogonal starters where P is the patterned frame starter.

The connection between orthogonal frame starters and frames is given by the following theorem (also proven in [7]).

THEOREM 2.8. *If there exist n pairwise orthogonal frame starters in $G \backslash H$ with $|G| = g$ and $|H| = h$, then there exists an (n, h, u)-frame where $u = g/h$.*

We will wish to construct Room 5-cubes directly from strong starters. Remembering that a Room 5-cube of side u is equivalent $(5, 1, u)$ frame with $|G| = u$ being odd and using Lemma 2.7(b) and Theorem 2.8 we have

THEOREM 2.9. *If there exist two orthogonal strong starters of order u, then $u \in R_5$.*

Proof. $|G| = u$ and $|H| = 1$, so by Lemma 2.7(b), there are 5 pairwise orthogonal frame starters in $G \backslash H = G \backslash \{0\}$. Thus by Theorem 2.8 there is a $(5, 1, u)$-frame and thus a Room 5-cube of side u.

We will also need to construct 2-frames ($h=2$) directly from strong frame starters. Since this implies G is even, then by Theorem 2.7(a) we only get 4 orthogonal starters. Thus, in order to get 5 orthogonal starters we must add another starter orthogonal to the original 4. Analogous to the previous theorem, we have

LEMMA 2.10. *If there exist two orthogonal strong frame starters A and B in $G\backslash H$ (with $|G|=g$ even and $|H|=h$) and if there exists a frame starter C in $G\backslash H$ with C orthogonal to A, $-A$, B, $-B$, then there exists a $(5, h, u)$-frame, with $u=g/h$.*

Proof. A, $-A$, B, $-B$, and C form a set of 5 pairwise orthogonal frame starters in $G\backslash H$ so by Theorem 2.8 there exists a $(5, h, u)$-frame with $u=g/h$.

THEOREM 2.11. *There exist $(5, 2, u)$-frames for $u=12$, 13, 16, 17, 20, and 21.*

Proof. In the Appendix frame starters A, B, and C are given for $u=16$, 17, 20, and 21 which satisfy the conditions of Lemma 2.10. For $u=12$ and $u=13$ we give 5 orthogonal 2-frame starters. For each value of u we have $G=Z_{2u}$, $H=\{0, u\}$.

Some comments are in order concerning the sets of orthogonal frame starters given in the Appendix. The starters for $u=12$ were found by a purely backtracking program. For $u=13$, the starters given in the Appendix are derived from the ones found by Dinitz and Stinson in [7]. In that paper, three orthogonal 2-frame starters of order 13 are found without the aid of computer by use of cyclotomic methods in Galois fields. It turns out that these three starters and their negative starters are all orthogonal and so in fact there are 6 pairwise orthogonal 2-frame starters of order 13. In the Appendix these are called A, $-A$, B, $-B$, C, and $-C$. The starters were originally found in $Z_{13}\times Z_2$, we have written them in Z_{26} in the Appendix.

The other starters given in the Appendix were all found by using a version of the hill climbing algorithm for strong starters originally described by Dinitz and Stinson in [6]. The original program only found one strong starter. Here we first found a strong frame starter A, then by hill climbing attempted to find a strong starter B such that A and B were orthogonal strong frame starters. On the average, using this method an orthogonal mate was found for about 6 out of every 100 frame starters. When an A and B were found we again used hill climbing, or in the case $u=16$ used an exhaustive search, to find a frame starter C where C is orthogonal to A, $-A$, B, and $-B$. (Note that since in all these cases since $|G|$ is even we cannot use P, the patterned strter.) For $u=16$, we found 15 sets of orthogonal strong frame starters which did not have a mate before finding the set given in the Appendix which did work. For $u = 17$, 20, and 21 we could not perform an exhaustive search so we found the orthogonal starter C by hill climbing. It took many attempts but eventually all were found. We are now ready to apply Corollary 2.4.

THEOREM 2.12. (a) *If* $N(s) \geq 11$, $2s+1 \in R_5$ *and* $2t+1 \in R_5$ *with* $t \leq s$, *then* $24s+2t+1 \in R_5$.

(b) *If* $N(s) \geq 15$, $2s+1 \in R_5$ *and* $2t+1 \in R_5$ *with* $t \leq s$, *then* $32s+2t+1 \in R_5$.

(c) *If* $N(s) \geq 19$, $2s+1 \in R_5$ *and* $2t+1 \in R_5$ *with* $t \leq s$, *then* $40s+2t+1 \in R_5$.

Proof. Use Corollary 2.4 with $t = 2$ and use the frames found in Theorem 2.11.

In order to effectively use Theorem 2.12 we need to find a large set of consecutive "small" numbers all of which are in R_5. In the next section we show that if $17 \leq n \leq 4575$ and n is odd, then $n \in R_5$.

3. SMALL VALUES

We begin this section with a useful and well-known construction. The proof can be found in [7].

THEOREM 3.1. *Suppose the following exist*:

(i) *a* $(5, t, u)$-*frame*,

(ii) *a* $(5, 1, v)$-*frame with a* sub$(5, 1, u)$-*frame*,

(iii) 5 MOLS *of order* $(v - w)/t$,

then $u(v - w) + \in R_5$.

The reader should be reminded that a $(5, 1, u)$-frame is equivalent to a Room 5-cube of side u. Also, note that every $(5, 1, u)$-frame has sub-frames of sides 0 and 1.

THEOREM 3.2. *If* $n = 11, 13, 17, 19, \ldots, 355$, *then* $n \in R_5$.

Proof. If $n = 13, 17, 21, 25, 33, 35, 39$, then $n \in R_5$ was shown in [5].

All $n \in \{11, 19, 23, 27, 29, 31, 37, 41, 43, 45, 47, 55, 59, 61, 67, 71, 73, 79, 83, 89, 101, 103, 107, 109, 113, 121, 125, 127, 131, 139, 149, 151, 157, 163, 167, 169, 173, 179, 181, 191, 197, 199, 211, 223, 227, 229, 233, 239, 241, 243, 251, 263, 269, 271, 277, 281, 283, 289, 293, 307, 311, 313, 317, 331, 337, 343, 347, 349, 353} are prime powers of the form $n = 2^k t + 1$ with t odd and $t \geqslant 5$. Thus by Theorem 1.3(d) $n \in R_5$.

The following table lists values of n which are in R_5 by use of Theorem 3.1. For all n we use $t = 1$ except for $n = 235$ and 289, where $t = 2$. The existence of the required sets of Latin squares can be checked in [1].

$133 = 11(13-1)+1$	$253 = 11 \times 23$
$143 = 11 \times 13$	$273 = 17(17-1)+1$
$177 = 11(17-1)+1$	$275 = 11 \times 25$
$187 = 11 \times 17$	$289 = 17(19-1)+1$
$205 = 17(13-1)+1$	$297 = 11 \times 27$
$209 = 11 \times 19$	$299 = 13 \times 23$
$221 = 13 \times 17$	$301 = 25(13-1)+1$
$231 = 21 \times 11$	$305 = 19(17-1)+1$
$235 = 13(19-1)+1$	$319 = 11 \times 29$
$247 = 19 \times 13$	$323 = 17 \times 19$

By Theorem 2.12(a), with $s = 13$, we have that $24 \times 13 + 2t + 1 \in R_5$ for $t = 5$, 6, 8, 9, 10, 11, 12, and 13. Thus {323, 325, 329, 331, 333, 335, 337, 339} $\subseteq R_5$.

The remaining cases are all solved by use of Theorem 2.9. Pairs of orthogonal strong starters of order n are given in [4] for $n = 45$, 49, 51, 55, 57, 63, 65, 69, 75, 77, 85, 87, 91, 93, 95, 97, 99, 103, 105, 111, 115, 117, 119, 123, 129, 135, 141, 145, 147, 153, 155, 159, 161, 165, 171, 175, 183, 185, 189, 193, 195, 201, 203, 207, 213, 215, 217, 219, 225, 237, 245, 249, 255, 257, 259, 261, 265, 267, 279, 285, 287, 291, 293, 295, 303, 309, 315, 321, 327, 345, and 355. These sets of orthogonal strong starters were constructed by use of the hill-climbing algorithm for strong starters in the manner described in the comments following Theorem 2.11. The listing of these strong starters is also available from the author on an IBM compatable floppy disk.

Now that we have a large set of consecutive small orders for which Room 5-cubes exist we can use Theorem 2.12 and Theorem 3.2 to get the following theorem.

THEOREM 3.3. *Let* $m = \min(2s+1, 353)$ *and assume* $2s+1 \in R_5$.

(a) *If* $N(s) \geqslant 11$, *then* $\{24s+11, 24s+13, 24s+17, \ldots, 24s+m\} \subseteq R_5$,

(b) *If* $N(s) \geqslant 15$, *then* $\{32s+11, 32s+13, 32s+17, \ldots, 32s+m\} \subseteq R_5$,

(c) *If* $N(s) \geqslant 19$, *then* $\{40s+11, 40s+13, 40s+17, \ldots, 40s+m\} \subseteq R_5$.

We can now construct Room 5-cubes for many more small values.

THEOREM 3.4. *If* $357 \leqslant n \leqslant 4575$ *and n is odd, then* $n \in R_5$.

Proof. If $n \in \{359, 361, 367, 373, 379, 383, 389, 449, 457, 461, 463, 499, 503, 509, 521, 547, 601, 607, 653, 853, 857, 859, 863, 991\}$, then n is a prime power and Theorem 1.3(d) applies to prove $n \in R_5$.

If $n \in \{365, 371, 381, 387, 393, 423, 445, 447, 453, 471, 501, 505, 511, 515, 519, 549, 553, 603, 615, 655, 711, 873, 879, 1143\}$, then we have again used the computer to construct 2 orthogonal strong starters of order n. By use of Theorem 2.9 we have that $n \in R_5$. In order to save space, these starters are given in [4]. They are also available from the author on an IBM compatible floppy disk.

All of the remaining values of n are constructed in the following table, where we give the values for n and the authority used to imply the existence of a Room 5-cube side n. Again all necessary results concerning MOLS can be found in [1].

n	Construction	Authority
357	21×17	Theorem 3.1
363	33×11	Theorem 3.1
369	$23(17-1)+1$	Theorem 3.1
375	$17(23-1)+1$	Theorem 3.1, $t=2$
377	13×29	Theorem 3.1
385	35×11	Theorem 3.1
387		Theorem 2.9
391	17×23	Theorem 3.1
395–397		Theorem 3.3(a), $s=16$
399	21×19	Theorem 3.1
401–417		Theorem 3.3(a), $s=16$
419–421		Theorem 3.3(a), $s=17$
425–443		Theorem 3.3(a), $s=17$
451		Theorem 3.1
455	35×13	Theorem 3.1
459	27×17	Theorem 3.1
465	$29(17-1)+1$	Theorem 3.1
467–469		Theorem 3.3(a), $s=19$
473–495		Theorem 3.3(a), $s=19$
497	$31(17-1)+1$	Theorem 3.1
507	39×13	Theorem 3.1
513	27×19	Theorem 3.1
517	11×47	Theorem 3.1
523–525		Theorem 3.3(b), $s=16$
527	17×31	Theorem 3.1
529–545		Theorem 3.3(b), $s=16$
551	19×29	Theorem 3.1
555–557		Theorem 3.3(b), $s=17$
559	43×13	Theorem 3.1
561–567		Theorem 3.3(b), $s=17$
569–599		Theorem 3.3(a), $s=23$

n	Construction	Authority
605	55×11	Theorem 3.1
609	21×29	Theorem 3.1
611–613		Theorem 3.3(a), $s = 25$
617–651		Theorem 3.3(a), $s = 25$
657	$41(17-1)+1$	Theorem 3.1
659–661		Theorem 3.3(a), $s = 27$
663	39×17	Theorem 3.1
665–703		Theorem 3.3(a), $s = 27$
705	55×11	Theorem 3.1
707–709		Theorem 3.3(a), $s = 29$
713–755		Theorem 3.3(a), $s = 29$
757		Theorem 3.3(a), $s = 31$
759		Theorem 3.3(b), $s = 23$
761–807		Theorem 3.3(a), $s = 31$
809–833		Theorem 3.3(a), $s = 32$
835–851		Theorem 3.3(b), $s = 25$
855	45×19	Theorem 3.1
861	21×41	Theorem 3.1
865	$27(33-1)+1$	Theorem 3.1
867	51×17	Theorem 3.1
869	11×79	Theorem 3.1
871	13×67	Theorem 3.1
875–877		Theorem 3.3(b), $s = 27$
881–903		Theorem 3.3(b), $s = 27$
905–963		Theorem 3.3(a), $s = 37$
965–987		Theorem 3.3(b), $s = 29$
989	23×43	Theorem 3.1
993	$31(33-1)+1$	Theorem 3.1
995–997		Theorem 3.3(a), $s = 41$
999	27×37	Theorem 3.1
1001–1067		Theorem 3.3(a), $s = 41$
1069–1119		Theorem 3.3(a), $s = 43$
1121–1135		Theorem 3.3(a), $s = 27$
1137	$71(17-1)+1$	Theorem 3.1
1139–1141		Theorem 3.3(a), $s = 47$
1145–1223		Theorem 3.3(a), $s = 43$
1225–1259		Theorem 3.3(b), $s = 37$
1261–1287		Theorem 3.3(c), $s = 31$
1289–1379		Theorem 3.3(a), $s = 53$
1381–1395		Theorem 3.3(b), $s = 41$
1397–1431		Theorem 3.3(b), $s = 43$
1433–1535		Theorem 3.3(a), $s = 59$
1537–1587		Theorem 3.3(a), $s = 61$
1589–1605		Theorem 3.3(a), $s = 64$
1607–1623		Theorem 3.3(b), $s = 49$
1625–1743		Theorem 3.3(a), $s = 67$
1745–1847		Theorem 3.3(a), $s = 71$
1849–1899		Theorem 3.3(a), $s = 73$

n	Construction	Authority
1901–1911		Theorem 3.3(c), $s=47$
1913–2055		Theorem 3.3(a), $s=79$
2057–2159		Theorem 3.3(a), $s=83$
2161–2315		Theorem 3.3(a), $s=89$
2317–2343		Theorem 3.3(b), $s=71$
2345–2523		Theorem 3.3(a), $s=97$
2525–2679		Theorem 3.3(a), $s=103$
2681–2835		Theorem 3.3(a), $s=109$
2837–2939		Theorem 3.3(a), $s=113$
2941–3147		Theorem 3.3(a), $s=121$
3149–3303		Theorem 3.3(a), $s=127$
3305–3563		Theorem 3.3(a), $s=137$
3565–3615		Theorem 3.3(a), $s=139$
3617–3875		Theorem 3.3(a), $s=149$
3877–4083		Theorem 3.3(a), $s=157$
4085–4343		Theorem 3.3(a), $s=167$
4345–4575		Theorem 3.3(a), $s=179$

4. The Spectrum

In order to complete the spectrum we need only show that if $n \geqslant 4577$, then $n \in R_5$. We need a preliminary lemma and then we can proceed with the theorem.

Lemma 4.1. *Let a and b be positive numbers.* If $b-a \geqslant 14$, *then there exists some integer* $c \in [a, b]$ *with* $N(c) \geqslant 11$.

Proof. Using MacNeish's Theorem (Lemma 2.1), $N(c) \geqslant 11$ if 2, 3, 5, 7, 11 all do not divide into c. It is easy to check that in Z_{2310} $(2310 = 2 \times 3 \times 5 \times 7 \times 11)$ the largest gap between numbers that are relatively prime to 2, 3, 5, 7, and 11 is 14. Thus the largest possible gap between numbers n where $N(n) \geqslant 11$ is 14.

Theorem 4.2. *If* $s \geqslant 4577$, *s odd, then* $s \in R_5$.

Proof. Let $s \geqslant 4577$ and by way of induction assume that $t \in R_5$ for all odd t, $17 \leqslant t \leqslant s-2$. Now let $s = 2m+1$ and pick r such that $(m-176)/12 \leqslant r \leqslant (m-8)/12$ and $N(r) \geqslant 11$. This can be done by Lemma 4.1 since $(m-8)/12 - (m-1776)/12 \geqslant 14$. Thus

$$12r + 8 \leqslant m \leqslant 12r + 176,$$

and

$$24r+17 \leqslant 2m+1 \leqslant 24r+353,$$

therefore

$$24r+17 \leqslant s \leqslant 24r+353.$$

Now by Theorem 3.3(a), we will have $s \in R_5$ if $2r+1 \in R_5$, $N(r) \geqslant 11$ and if $\min(2r+1, 353)=353$. We already have $N(r) \geqslant 11$. Since $24r+353 \geqslant s \geqslant 4577$, then $r \geqslant 176$ and so $\min(2r+1, 353)=353$. Also $2r+1<2m+1=s$ so by induction $2r+1 \in R_5$. Thus by Theorem 3.3(a), $s \in R_5$ completing the proof.

Now by use of Theorems 3.2, 3.4, and 4.2 we have our result.

THEOREM 4.3. *If $n \geqslant 11$ is odd (except possibly $n=15$), then $n \in R_5$.*

A comment is in order concerning the case $n=15$. We have performed an exhaustive search and have found that there is no set of 5 pairwise-orthogonal starters of order 15. In [5] a set of 4 pairwise-orthogonal starters is given. We, however, do not hesitate to conjecture that $15 \in R_5$.

Since there are Room 4-cubes of orders 9 [9] and 15, then the following theorem holds.

THEOREM 4.4. *There exists a Room 4-cube of side n if and only if n is odd and $n \geqslant 9$.*

APPENDIX

$u=12$

$A=$ 1,2 3,5 4,7 9,13 14,19 15,21 16,23 10,18 8,17 20,6 11,22
$B=$ 2,3 5,7 17,20 15,19 11,16 8,14 23,6 1,9 13,22 18,4 10,21
$C=$ 3,4 7,9 10,13 18,22 15,20 5,11 14,21 17,1 23,8 16,2 19,6
$D=$ 7,8 2,4 18,21 11,15 1,6 16,22 10,17 19,3 5,14 13,23 9,20
$E=$ 17,18 9,11 23,2 10,14 22,3 1,7 13,20 8,16 21,6 19,5 4,15

$u=13$

$A=$ 11,12 15,17 7,10 18,22 3,8 19,25 20,1 23,5 21,4 6,16 24,9 2,14
$-A=$ 14,15 9,11 16,19 4,8 18,23 1,7 25,6 21,3 22,5 10,20 17,2 12,24
$B=$ 17,18 3,5 25,2 16,20 7,12 9,15 4,11 19,1 23,6 14,24 10,21 22,8
$-B=$ 8,9 21,23 24,1 6,10 14,19 11,17 15,22 25,7 20,3 2,12 5,16 18,4
$C=$ 15,16 7,9 19,22 8,12 23,2 21,1 18,25 3,11 5,14 20,4 6,17 24,10
$-C=$ 10,11 17,19 4,7 14,18 24,3 25,5 1,8 15,23 12,21 22,6 9,20 16,2

$u=16$

$A=$ 17,18 22,24 26,29 10,14 2,7 3,9 31,6 25,1 12,21 5,15 19,30 8,20 23,4 13,27 28,11
$B=$ 8,9 25,27 31,2 7,11 17,22 4,10 12,19 29,5 21,30 23,1 13,24 14,26 15,28 6,20 3,18
$C=$ 21,22 12,14 17,20 1,5 13,18 2,8 19,26 28,4 29,6 25,3 31,10 15,27 30,11 9,23 24,7

$u = 17$

A = 23,24 3,5 6,9 7,11 25,30 16,22 12,19 10,18 26,1 28,4 31,8 21,33 14,27 15,29 32,13 20,2
B = 3,4 8,10 12,15 31,1 22,27 7,13 25,32 20,28 14,23 26,2 18,29 33,11 6,19 16,30 9,24 5,21
C = 19,20 26,28 5,8 12,16 33,4 31,3 23,30 7,15 27,2 1,11 10,21 13,25 9,22 18,32 14,29 24,6

$u = 20$

A = 31,32 7,9 22,25 37,1 11,16 29,35 27,34 10,18 36,5 13,23 4,15 21,33 6,19 28,2 39,14 8,24 26,3 12,30 38,17
B = 5,6 1,3 32,35 21,25 33,38 36,2 24,31 11,19 8,17 16,26 23,34 10,22 14,27 4,18 37,12 39,15 13,30 29,7 9,28
C = 36,37 30,32 24,27 6,10 29,34 8,14 21,28 17,25 9,18 35,5 1,12 31,3 2,15 39,13 11,26 7,23 16,33 4,22 19,38

$u = 21$

A = 27,28 24,26 29,32 8,12 2,7 9,15 16,23 37,3 34,1 10,20 36,5 18,30 33,4 41,13 25,40 6,22 14,31 35,11 19,38 39,17
B = 25,26 38,40 17,20 24,28 10,15 5,11 23,30 41,7 9,18 4,14 33,2 27,39 22,35 34,6 1,16 29,3 19,36 37,13 31,8 12,32
C = 18,19 10,12 2,5 25,29 1,6 14,20 38,3 27,35 31,40 23,33 39,8 34,4 11,24 22,36 13,28 16,32 9,26 41,17 30,7 37,15

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