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ROOM SQUARES WITH HOLES OF SIDES 3, 5, AND 7

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We investigate Room squares with small holes: missing subsquares of sides 3, 5 or 7. We refer to a Room square of side s missing a subsquare of side t as an (s, t)-incomplete Room square. For any odd t, it has been shown that there is an integer $S(t)$ such that an (s, t)-incomplete Room square exists for all odd $s > S(t)$. In this paper we prove that $S(3) \leq 39$, $S(5) \leq 67$, and $S(7) \leq 53$.

1. Introduction

We investigate Room squares and subsquares. All terminology is as in [6] and [7], and we will assume the reader is familiar with the definitions contained therein. (These include group-divisible designs, transverial designs, and frames.)

We will refer to an incomplete Room square of side s missing a subsquare of side t as an *(s, t)-incomplete Room square.*

It is known that for any odd t, there is a constant $S(t)$ such that there exists an (s, t)-incomplete Room square for all odd $s > S(t)$.

The following was proved in [7].

Theorem 1.1. (1) $S(1) = 5$, $S(3) \leq 77$, $S(5) \leq 79$.

(2) *For all odd* $t \geq 7$, $S(t) \leq 6t + 39$.

(3) *For all odd* $t \geq 129$, $S(t) \leq 4t + 27$.

In this paper we improve the bounds on $S(3)$, $S(5)$, and $S(7)$. We prove that $S(3) \leq 39$, $S(5) \leq 67$, and $S(7) \leq 53$.

At this point we record the existence of two small arrays that were constructed by computer.

Lemma 1.2. *There exist* $(11, 3)$- *and* $(13, 3)$-*incomplete Room squares.*

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Proof. The (11, 3)-incomplete Room square is presented in [6]; we exhibit a (13, 3)-incomplete Room square in Fig. 1. □

			8, 10		11, 12				5, 7	6, 13		4, 9
			12, 13	9, 11					6, 8	4, 7		5, 10
					9, 10	5, 13			4, 12		6, 11	7, 8
	6, 7		0, 4		8, 13	10, 11	1, 5	3, 12		2, 9		
7, 12	11, 13			0, 5	1, 4	2, 6				3, 10	8, 9	
9, 13			2, 5		0, 6	1, 8	10, 12	4, 11			3, 7	
4, 8		5, 12		10, 13		0, 7	2, 11		1, 9			3, 6
	5, 9			6, 12		3, 4	0, 8	7, 13			2, 10	1, 11
6, 10		4, 13		1, 7				0, 9	3, 11	5, 8		2, 12
		6, 9	7, 11				3, 13	2, 8	0, 10	1, 12	4, 5	
	8, 12	7, 10	3, 9	2, 4				5, 6		0, 11	1, 13	
	4, 10	8, 11	1, 6		3, 5		7, 9		2, 13		0, 12	
5, 11				3, 8	2, 7	9, 12	4, 6	1, 10				0, 13

Fig. 1. A (13, 3)-incomplete Room square.

2. Constructions

In this section we present several constructions for incomplete Room squares and Room squares with subsquares.

The following two results are from [7].

Lemma 2.1. *Suppose there is a* TD(k, n), *where* $k \geq 6$. *For* $6 \leq i < k$, *let* $3 \leq d_i \leq n$. *Also, let* $0 \leq d_k \leq n$. *Then there is a*

$$\left(10n + 2\sum_{i=6}^{k} d_i + 1,\ 2d_k + 1\right)\text{-incomplete Room square.}$$

If $d_k \geq 3$, *then there exists a*

$$\left(10n + 2\sum_{i=1}^{k} d_i + 1,\ 2d_i + 1\right)\text{-incomplete Room square}$$

for all i, $6 \leq i \leq k$.

Lemma 2.2. *Suppose* $n \neq 2, 3, 6, 10$ *or* 14, *and* $0 \leq t \leq 3n$. *Then there is a* $(16n + 2t + 1, 2t + 1)$-*incomplete Room square. If* $t \neq 1$ *or* 2, *then there is a* $(16n + 2t + 1, 4n + 1)$-*incomplete Room square.*

The following is a variation of Lemma 2.1, useful in a few special situations.

Lemma 2.3. *Suppose there is a* TD(k, n), *where* $k \geq 6$. *For* $6 \leq i \leq k$ *suppose* $3 \leq d_i \leq n$. *Also, suppose* $\sum_{i=6}^{k} d_i \leq n$. *Then there exists a* $(10n + 2\sum_{i=6}^{n} d_i - 1, 2d_i + 1)$-*incomplete Room square, for* $6 \leq i \leq k$.

Sketch of Proof. Let $(X, \mathcal{G}, \mathcal{A})$ be a TD(k, n), $\mathcal{G} = \{G_1, \ldots, G_k\}$. Pick a point $x \in G_5$ and let B be a block containing x. For $6 \leq i \leq k$, delete $n - d_i$ points from G_i, in such a way that no block containing x contains more than one of the $\sum_{i=6}^{k} d_i$ points remaining in these groups. Finally delete x. Form a group-divisible design by taking as groups all blocks which contained x. This new GDD has the following properties:

(1) Every group has size at least 4.
(2) Except (possibly) for G_i, $6 \leq i \leq k$, every block has size at least 5.
(3) A group contains at most one pont of $\bigcup_{i=6}^{k} G_i$.
(4) Each group which meets a G_i has size 4 or 5.

Now, take two copies of each point and let ∞ be a new point. Replace each block G_i $(6 \leq i \leq k)$ by a Room square of side $2|G_i| + 1$. Replace every other block B by a frame of type $2^{|B|}$. Replace every group H not meeting any G_i $(6 \leq i \leq k)$ by a Room square of side $2|H| + 1$. Finally, replace a group meeting some G_i $(6 \leq i \leq k)$ by an $(11, 3)$- or $(13, 3)$-Room square (which exist by Lemma 1.2).

The result is a Room square, which contains subsquares of sides $2|G_i| + 1$, $6 \leq i \leq k$. □

There are several constructions in the literature known as product constructions. We will use the following.

Lemma 2.4 (Singular direct product [2]). *Suppose there exists a Room square of side* u, *and a* (v, w)-*incomplete Room square, where* $v - w \neq 6$. *Then there exists an* (s, w)-*incomplete Room square and an* (s, v)-*incomplete Room square, where* $s = u(v - w) + w$. *If* $w \neq 3$ *or* 5, *then there exists an* (s, u)-*incomplete Room square.*

A variation on the preceding construction is to start with a frame of type t^u rather than a Room square of side u (see [6]). The following can be proved.

Lemma 2.5 (Frame singular direct product). *Suppose there is a frame of type* t^u, *and there exists a* (v, w)-*incomplete Room square, where* $t \mid (v - w)$ *and* $(v - w)/t \neq 2$ *or* 6. *Then there exists an* (s, w)-*incomplete Room square and an* (s, v)-*incomplete Room square, where* $s = u(v - w) + w$.

Remark. A frame of type t^u is known to exist in the following cases (see [1]):

(1) $u = 4$ and $4 \mid t$,
(2) $u = 5$ and $\gcd(t, 210) > 1$,
(3) $u \geq 6$ and $t(u - 1)$ is even.

Still another variation was described in [3].

Lemma 2.6 (Singular indirect product). *Suppose there exists a Room square of side* u, *and a* (v, w)*-incomplete Room square. Let* $0<a<w$ *and suppose there exist a pair of orthogonal Latin squares of order* $v-a$ *containing (or missing) a pair of orthogonal Latin subsquares of order* $w-a$. *Then there exists an* $(s, u(w-a)+a)$*-incomplete Room square and an* (s, u)*-incomplete Room square, where* $s=u(v-a)+a$.

The next three constructions are due to Wallis.

Lemma 2.7 ([10]). *If* $u \geqslant 7$ *is odd, then there is a* $(4u+1, u)$*-incomplete Room square.*

Lemma 2.8 ([9]). *If* $u \geqslant 7$ *is odd, then there is a* $(5u, u)$*-incomplete Room square.*

Lemma 2.9 ([12]). *For all odd* $u \geqslant 3$ *there is a* $(3u+2, u)$*-incomplete Room square.*

The construction of Lemma 2.9 is generalized in [8].

Lemma 2.10. *Suppose* $t>3$. *Also, let* $u \geqslant 7$ *be odd,* $u \neq 11$. *Then there is a Room square of side* $tu+t-1$ *which contains subsquares of sides* u *and* $2t-1$.

One further construction is useful in certain circumstances. It is a slight extension of the tripling construction in [11].

Lemma 2.11. *Suppose there exists a* (v, w)*-incomplete Room square. Then there exists a* $(3v, w)$*-incomplete Room square.*

Our last lemma is a trivial, though useful, observation.

Lemma 2.12. *If there exists a* (u, v)*-incomplete Room square and* (v, w)*-incomplete Room square, then there exists a* (u, w)*-incomplete Room square.*

3. Room squares with small holes

The following result is proved in [7, Theorem 3.6].

Lemma 3.1. *If* $t=3$, 5, *or* 7 *and* $s \geqslant 76+t$ *is odd, then there exists an* (s, t)*-incomplete Room square.*

In this section we construct numerous (s, t)*-incomplete Room squares* ($t=3, 5$, and 7) not covered by the above result.

Lemma 3.2. *There exists a* (57, 3)-*incomplete Room square.*

Proof. This incomplete Room square is constructed by applying Lemma 2.1.

Write $57 = 10 \cdot 5 + 2 \cdot 3 + 1$. We construct a (57, 11)-incomplete Room square; fill in an (11, 3)-incomplete Room square (Lemmata 2.12 and 1.2). □

We present the remaining constructions for (*s*, 3)-incomplete Room squares in tabular form (Table 1). In many cases, the cited construction will guarantee only a subsquare of side 11 or 13; but Lemma 2.12 will then apply.

Table 1. Construction of (*s*, 3)-incomplete Room squares

s	Equation	Authority	Subsquare
77	$7(11-0)+0$	Lemma 2.4	11
75	$9(11-3)+3$	Lemma 2.4	
73	$7(13-3)+3$	Lemma 2.4	
71	$7(11-1)+1$	Lemma 2.4	11
69	$5 \cdot 13+4$	Lemma 2.10	13
67	$16 \cdot 4+2 \cdot 1+1$	Lemma 2.2	
65	$5 \cdot 13$	Lemma 2.8	13
63	$6(13-3)+3$	Lemma 2.5 ($t=2$)	
61	$5(13-1)+1$	Lemma 25 ($t=3$)	13
59	$5 \cdot 11+4$	Lemma 2.10	11
57		Lemma 3.2	
55	$4 \cdot 13+3$	Lemma 2.10	13
53	$5(13-3)+3$	Lemma 2.5 ($t=2$)	
51	$5(11-1)+1$	Lemma 2.5 ($t=2$)	11
49	$4(13-1)+1$	Lemma 2.5 ($t=3$)	13
47	$4 \cdot 11+3$	Lemma 2.10	11
45	$4 \cdot 11+1$	Lemma 2.7	
43	$5(11-3)+3$	Lemma 2.5 ($t=2$)	
41	$3 \cdot 13+2$	Lemma 2.9	13
35	$3 \cdot 11+2$	Lemma 2.9	11
33	$3 \cdot 11$	Lemma 2.11	
13		Lemma 1.2	
11		Lemma 1.2	

Thus we have

Theorem 3.3. *There exist* (*s*, 3)-*incomplete Room squares for* $s = 11, 13, 33, 35$, *and for all odd* $s \geq 41$.

Next, we consider (*s*, 5)-incomplete Room squares. Note that a (17, 5)-incomplete Room square exists, by Lemma 2.9.

Lemma 3.4. *For odd s*, $69 \leq s \leq 89$, *there exists an* (*s*, 5)-*incomplete Room square.*

Proof. Apply Lemma 2.2 with $n=4$, $2\leq t\leq 12$. For $3\leq t\leq 12$, we construct a $(16n+2t+1, 17)$-incomplete square, which gives rise to a $(16n+2t+1, 5)$-incomplete Room square, by filling in a $(17, 5)$-Room square. For $t=2$, a $(69, 5)$-incomplete Room square is produced directly. □

Lemma 3.5. *There exists a* $(65, 5)$-*incomplete Room square.*

Proof. $65=5(17-5)+5$. Apply Lemma 2.5 with $t=3$. □

Lemma 3.6. *There exists a* $(55, 5)$-*incomplete Room square.*

Proof. Write $55=10\cdot 5+2\cdot 2+1$, and apply Lemma 2.1. □

Lemma 3.7. *There exists a* $(53, 5)$-*incomplete Room square.*

Proof. $53=3\cdot 17+2$. Applying Lemma 2.9, we construct a $(53, 17)$-incomplete Room square. Then fill in a $(17, 5)$-incomplete Room square (Lemma 2.12). □

Lemma 3.8. *There exists a* $(51, 5)$-*incomplete Room square.*

Proof. $51=3\cdot 17$. Apply Lemma 2.11. □

Summarizing Lemmata 2.9, 3.1 and 3.4–3.8 we have

Theorem 3.9. *There exists an* $(s, 5)$-*incomplete Room square for* $s=17, 51, 53, 55, 65$ *and all odd* $s\geq 69$.

We now construct $(s, 7)$-incomplete Room squares.

Lemma 3.10. *There exists a* $(75, 7)$-*incomplete Room square.*

Proof. Apply Lemma 2.3 with $k=6$, $n=7$, $d_6=3$. □

Lemma 3.11. *There exists an* $(81, 7)$-*incomplete Room square.*

Proof. Apply Lemma 2.3 with $k=7$, $n=7$ and $d_6=d_7=3$. □

Lemma 3.12. *There exists an* $(83, 7)$-*incomplete Room square.*

Proof. $83=10\cdot 7+2(3+3)+1$. Apply Lemma 2.1 with $n=k=7$, $d_6=d_7=3$. □

We present the remaining constructions in tabular form (Table 2).

As a consequence, we have

Theorem 3.13. *There exists an* $(s, 7)$*-incomplete Room square for* $s = 23$, 29, 31, 35, 39, 43, 47, 49, *and all odd* $s \geq 55$.

Proof. Lemmata 3.1, 3.10–3.12, and Table 2. □

Table 2. Construction of $(s, 7)$-incomplete Room squares

Order	Equation	Authority	Remark
23	$3 \cdot 7+2$	Lemma 2.9	
29	$4 \cdot 7+1$	Lemma 2.7	
31	$4 \cdot 7+3$	Lemma 2.10	
35	$5 \cdot 7$	Lemma 2.8	
39	$5 \cdot 7+4$	Lemma 2.10	
43	$7(7-1)+1$	Lemma 2.5	$t=2$
47	$6 \cdot 7+5$	Lemma 2.10	
49	$7 \cdot 7$	Lemma 2.4	
55	$9(7-1)+1$	Lemma 2.5	$t=2$
57	$7(9-1)+1$	Lemma 2.4	
59	$7(11-3)+3$	Lemma 2.4	
61	$10(7-1)+1$	Lemma 2.5	$t=2$
63	$7 \cdot 9$	Lemma 2.4	
65	$7(11-2)+2$	Lemma 2.6	$w=3$
67	$11(7-1)+1$	Lemma 2.5	$t=2$
69	$3 \cdot 23$	Lemma 2.4	(23, 7)-incomplete Room square
71	$7(11-1)+1$	Lemma 2.4	
73	$7(13-3)+3$	Lemma 2.4	
75		Lemma 3.10	
77	$7 \cdot 11$	Lemma 2.4	
79	$13(7-1)+1$	Lemma 2.5	$t=2$
81		Lemma 3.11	
83		Lemma 3.12	
85	$7(13-1)+1$	Lemma 2.4	
87	$3 \cdot 29$	Lemma 2.11	(29, 7)-incomplete Room square
89	$3 \cdot 29+2$	Lemma 2.9	(29, 7)-incomplete Room square

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