JOURNAL OF COMBINATORIAL THEORY, Series A 29, 395–398 (1980)

Note

Some New Row-Complete Latin Squares

D. S. ARCHDEACON, J. H. DINITZ, AND D. R. STINSON

Department of Mathematics, The Ohio State University, Columbus, Ohio 43210

AND

T. W. TILLSON

Hewlett–Packard Corporation, Fort Collins, Colorado 80525

Communicated by the Managing Editors

Received February 22, 1980

Row-complete Latin squares of orders 9, 15, 21 and 27 are given. The square of order 9 is the smallest possible odd order row-complete Latin square.

A *row-complete Latin square* (RCLS) of order n, L, is a Latin square in which for any two distinct symbols, x, y, there exist i, j, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n-1$ such that $L(i,j) = x$ and $L(i,j+1) = y$. *Column complete Latin squares* are defined analogously, and a square is *complete* if it is both row and column complete. RCLS are used in statistics for the design of sequential experiments.

A finite group G is *sequencible* if its elements can be arranged into a sequence $a_1, a_2, ..., a_n$ in such a way that the partial products $a_1, a_1 a_2, ..., a_1 a_2 \cdots a_n$ are all distinct. Gordon [2] has shown that the existence of a sequencible group of order n is a sufficient condition for the existence of a complete Latin square of order n. In the same paper Gordon proves the existence of a sequencible goup of order $2n$ for all n.

Very little is known about RCLS of odd order. Sequencible groups nave been found for orders 21, 27, 39, 55 and 57 (see [1, 3, 5]); however, it is known there are no sequencible groups of order 9 or 15. We offer the following construction.

Let $n = p \cdot q$ and consider the group $G = \mathbb{Z}_p \times \mathbb{Z}_q$. A $q \times n$ array, A, with entries from $\mathbb{Z}_p \times \mathbb{Z}_q$ will be called a *generating set* of order n if:

0097-3165/80/060395–04$02.00/0

(1) each symbol $(i,j) \in G$ occurs once in each row,

(2) for each $i \in \mathbb{Z}_q$ and each column c of A there exists a row r with $A(r,c)=(\alpha,i)$ for some $\alpha \in \mathbb{Z}_p$ (each $\mathbb{Z}_q$ element appears once in each column),

(3) if $A(r_1,c_1)=(i,j)$ and $A(r_1,c_1+1)=(i',j')$, and if $A(r_2,c_2)=(k,j)$ and $A(r_2,c_2+1)=(k',j')$, then $i-i' \equiv k-k' \pmod p$ implies $r_1=r_2$ and $c_1=c_2$.

By developing the elements in Z_p cyclically while leaving the Z_q elements constant the resulting square of side pq is a RCLS.

Using a backtracking algorithm we have found generating sets of orders 9, 15, 21 and 27, with $q=3$ (see Table 1). By an exhaustive search it has been

TABLE 1[a]

$\mathbb{Z}_q$		012012120	012210	012210	012210
	$q=3$	100220211	120021	120021	120021
		221101002	201102	201102	201102
$\mathbb{Z}_p$		000121122			
	$p=3$	010102212			
		012110202			
		000121132	434243		
	$p=5$	001402342	314321		
		020402424	331311		
		000121122	353434	646565	
	$p=7$	003024364	515242	151663	
		046304212	540253	165136	
		000121122	353434	647677	585868
	$p=9$	030021356	157647	382562	818474
		048615265	287051	841437	373206

[a] For notational purposes the $\mathbb{Z}_q$ and $\mathbb{Z}_p$ patterns are given separately. The $\mathbb{Z}_q$ pattern for a given p is found by taking the first $3 \cdot p$ columns of the given $\mathbb{Z}_q$ pattern.

determined that there do not exist RCLS of orders 3, 5 and 7. Thus, the resulting square of order 9 (Fig. 1) is the smallest odd order RCLS. No RCLS of order 9 or 15 were previously known (RCLS of order 15 in Fig. 2). These squares fail the quadrangle criterion (or a definition see [3]); hence they are not generated by sequencible groups. Moreover, the square of order 9 is not a row permutation of a complete square (other squares with these properties are given in [4]). Finally it should be noted for the given $\mathbb{Z}_q$

1	4	7	2	6	8	5	9	3
2	5	8	3	4	9	6	7	1
3	6	9	1	5	7	4	8	2
4	2	1	8	7	3	9	5	6
5	3	2	9	8	1	7	6	4
6	1	3	7	9	2	8	4	5
7	8	6	5	2	4	3	1	9
8	9	4	6	3	5	1	2	7
9	7	5	4	1	6	2	3	8

FIG. 1. RCSL of order 9.

pattern, $q = 3$ and $p = 5$, we have found many generating sets giving rise to non-isomorphic RCLS.

As an example, the first row of the generating set for $n = 9$ is

$$(0,0)\ (0,1)\ (0,2)\ (1,0)\ (2,1)\ (1,2)\ (1,1)\ (2,2)$$

Develop the first coordinate mod $p = 3$ to get the following three rows.

$$\begin{matrix} (0,0) & (0,1) & (0,2) & (1,0) & (2,1) & (1,2) & (1,1) & (2,2) & (2,0) \\ (1,0) & (1,1) & (1,2) & (2,0) & (0,1) & (2,2) & (2,1) & (0,2) & (0,0) \\ (2,0) & (2,1) & (2,2) & (0,0) & (1,1) & (0,2) & (0,1) & (1,2) & (1,0) \end{matrix}$$

Now use the mapping $\varphi: Z_3 \times Z_3 \rightarrow \{1, 2,..., 9\}$ defined by $\varphi(i,j) = i + 3j + 1$ to get the first three rows of the RCLS of order 9 in Fig. 1. The other 6 rows are constructed analogously from the remaining 2 rows of the generating set.

1	6	11	2	8	12	7	14	3	5	9	15	13	10	4
2	7	12	3	9	13	8	15	4	1	10	11	14	6	5
3	8	13	4	10	14	9	11	5	2	6	12	15	7	1
4	9	14	5	6	15	10	12	1	3	7	13	11	8	2
5	10	15	1	7	11	6	13	2	4	8	14	12	9	3
6	1	2	15	11	3	14	10	8	9	12	5	4	13	7
7	2	3	11	12	4	15	6	9	10	13	1	5	14	8
8	3	4	12	13	5	11	7	10	6	14	2	1	15	9
9	4	5	13	14	1	12	8	6	7	15	3	2	11	10
10	5	1	14	15	2	13	9	7	8	11	4	3	12	6
11	13	6	10	1	8	5	3	15	14	4	7	9	2	12
12	14	7	6	2	9	1	4	11	15	5	8	10	3	13
13	15	8	7	3	10	2	5	12	11	1	9	6	4	14
14	11	9	8	4	6	3	1	13	12	2	10	7	5	15
15	12	10	9	5	7	4	2	14	13	3	6	8	1	11

FIG. 2. RCLS order 15

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