Room Square Patterns

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Abstract

Suppose a Howell design $H(s,2n)$, H, contains as a subarray an $m \times m$ array M which contains a Room square of order m and possibly other pairs in the "empty" cells of the $RS(m)$. Then we say that H contains a Room square pattern of order m, a $RSP(m)$. (i.e. H contains the non-empty cells of a $RS(m)$.) If M contains only the pairs of a $RS(m)$, then H contains a $RS(m)$ as a sub-design. In this paper, we are interested in the existence of Howell designs with RSP sub-designs. This investigation is motivated by the fact that RSP sub-designs occur naturally in several constructions for Howell designs; in addition, it is possible to construct $H(n, n+\alpha)$ with $RSP(m)$ sub-designs for values of n and m where $H(n, n+\alpha)$ with $RS(m)$ sub-designs can not exist.

1 Introduction

A *Howell design* of side s and order $2n$, or more briefly an $H(s, 2n)$, is an $s \times s$ array in which each cell is either empty or contains an unordered pair of elements from some $(2n)$-set V such that

1. every element of V occurs in precisely one cell of each row and each column, and

2. every unordered pair of elements from V is in at most one cell of the array.

It follows immediately from the definition of an $H(s, 2n)$ that $n \le s \le 2n-1$.

An $H(2n-1, 2n)$ is also called a *Room square* of order $2n-1$ or a $RS(2n-1)$. The spectrum of Room squares was completed in [20]: there exists a $RS(2n-1)$ for all positive integers n, $n \ne 2$ or 3. There is an extensive literature available on Room squares, see [20] and a recent survey [10]. At the other boundary, the existence of a pair of mutually orthogonal Latin squares of order n implies the existence of an $H(n, 2n)$. Thus, there is an $H(n, 2n)$ for n a positive integer, $n \ne 2$ or 6, [4]. (An $H(6, 12)$ is displayed in [14].) The spectrum for $H(s, 2n)$ was completed in two papers, [3, 21].

Theorem 1.1 *[3, 21] There exists an $H(s, 2n)$ for all positive integers s and n except when $(s, 2n) \in \{(2,4),\ (3,4),\ (5,6),\ (5,8)\}$.*

An $H^*(s, 2n)$ is an $H(s, 2n)$ in which there is a subset W of V, $|W| = 2n-s$, such that no pair of elements from W appears in the design. *-designs are useful in recursive constructions. We note that there exist $H^*(s, 2n)$ for s even with two exceptions: there is no $H^*(2, 4)$ and there is no $H^*(6, 12)$, [3]. Information on *-designs for s odd can be found in [21]. Many of the recursive constructions also use Howell designs in standard form. Suppose H is an $H^*(s, 2n)$ defined on $Z_s \cup W$. H is said to be in *standard form* if there is an element of W, say ∞, so that $\{i, \infty\}$ occurs in cell (i, i) for $i = 0, 1, \ldots, s-1$.

Suppose that H is an $H(s, 2n)$ defined on the symbol set V. A $t \times t$ subarray G of H is said to be a *Howell sub-design* $H(t, 2m)$ if it is itself a Howell design of side t defined on a symbol set $U \subseteq V$ of size $2m$. In view of Theorem 1.1, no Howell design can contain a Howell sub-design $H(t, 2m)$

when $(t, 2m) \in \{(2,4),\ (3,4),\ (5,6),\ (5,8)\}$. However, we can construct Howell designs which are *missing* sub-designs of these orders. In this case we can still speak of the pairs of elements which would occur in such sub-designs (if they existed).

An $H(s, 2n) - H(t, 2m)$ incomplete Howell design is an $H(s, 2n)$ which contains an $H(t, 2m)$ sub-design or an $H(s, 2n)$ which is missing an $H(t, 2m)$ sub-design if the $H(t, 2m)$ sub-design does not exist. A formal definition and results on incomplete Howell designs can be found in [6]. The case where the sub-design is a Room square, $t = 2m - 1$, has been of particular interest. In this paper, we are interested in the existence of Howell designs with Room square patterns.

Suppose an $H(s, 2n)$, H, contains as a subarray an $m \times m$ array M which contains a $RS(m)$ and possibly other pairs in the "empty" cells of the $RS(m)$. Then we say that H contains a *Room square pattern* of order m, a $RSP(m)$. (i.e. H contains the non-empty cells of a $RS(m)$.) If M contains only the pairs of a $RS(m)$, then H contains a $RS(m)$ as a sub-design. In Figure 1 a Howell design $H(20, 22)$ which contains a Room square pattern $RSP(7)$ is displayed. The Room square pattern defined on the symbol set $\{1, \ldots, 8\}$ can be found in the upper left 7×7 array. We note that the smallest $H(n, n+2)$ which contains a $RS(7)$ as as sub-design is an $H(22, 24)$.

Our investigation of Howell designs with Room square pattern sub-designs is motivated by the fact that RSP sub-designs occur naturally in several constructions for Howell designs. In addition, it is possible to construct $H(n, n+\alpha)$ with $RSP(m)$ sub-designs for values of n and m where $H(n, n+\alpha)$ with $RS(m)$ sub-designs can not exist; an example is given in Figure 1.

In the next section, we consider the existence of Room squares with RSP sub-designs. Constructions and results for $H(n, n+\alpha)$ (and $\alpha > 1$) with RSP sub-designs are in Section 3.

2 Room squares with RSP sub-designs

Let n and s be odd positive integers. An $H(n, n+1) - H(s, s+1)$ is also known as an (n, s)-incomplete Room square (IRS). If there exists an $(n, s) - IRS$ and an $RS(s)$, then there exists an $RS(n)$ with a $RS(s)$ sub-design. The existence of $(n, s) - IRS$ was an open problem for over 20 years. The spectrum has now been established with only one possible exception.

Figure 1: An $H(20,22)$ which contains a $RSP(7)$

7 8	15 20	16 18	1 5		4 6	2 3					9 12			17 21		10 22	14 19	11 13	
3 4	1 8	9 22	12 19	2 6		5 7				16 21			14 15		10 17		13 20		11 18
6 1	4 5	2 8	13 17	12 15	3 7	10 16	9 11								18 21			14 22	19 20
13 14	7 2	5 6	3 8		10 20	4 1			15 16	9 18	11 22		17 19			12 21			
5 2		1 3	6 7	4 8	9 19			13 18	10 14			20 21		11 15			17 22		12 16
11 17	6 3		2 4	1 7	5 8		12 18	14 20	9 21	10 15		16 22			13 19				
21 22	10 18	7 4		3 5	1 2	6 8	14 17						12 13	16 19		11 20		9 15	
	12 22		10 21	9 14			2 19	4 11	7 18	6 13					1 15		5 16	3 20	8 17
		12 14			13 21		3 10		1 22	7 19	16 17		6 11	8 20			4 15	5 18	2 9
15 18	11 19				14 16		7 22	8 9	4 20		3 13				5 12	2 17		6 21	1 10
12 20						11 14	1 16			4 17	2 15	9 10	3 18		6 22	5 13	7 21	8 19	
10 19				13 22		12 17	6 20				4 21	1 18	7 9	5 14		8 15	3 11	2 16	
			14 18		11 12	19 22	8 21	6 16	5 17			3 15	2 20	1 13	4 9			7 10	
	9 17	15 19	11 16					5 21	8 13	3 22	1 20	6 14		7 12		4 18	2 10		
		11 21	15 22			18 20		3 17			5 19	8 12	4 16	6 10	2 14	1 9			7 13
		10 13		16 20	18 22		5 15		3 19	2 12		7 17	1 21		8 11		6 9		4 14
9 16				19 21			4 13	10 12				5 11	8 22	2 18	7 20	3 14		1 17	6 15
	14 21			17 18		9 13		7 15	2 11	5 20	8 10			4 22	3 16	6 19	1 12		
		17 20		10 11		15 21		1 19		8 14	6 18	2 13		3 9		7 16		4 12	5 22
	13 16		9 20		15 17			2 22	6 12	1 11	7 14	4 19	5 10				8 18		3 21

Theorem 2.1 *[11] Suppose n and s are odd positive integers, $n \geq 3s+2$, and $(n,s) \neq (5,1)$. Then there exists an $(n,s)-IRS$ except possibly for $(n,s)=(67,21)$.*

If there exists a $RS(n+m)$ which contains a $RSP(m)$ as a sub-design, then n must necessarily be greater than some lower bound which is a function of m.

Lemma 2.2 *If there exists an $H(n+m, n+m+1)$ (a $RS(n+m)$) with a $RSP(m)$ sub-design, then $n+m \geq \lceil 2m+(1/2)(1+\sqrt{8m+1})\rceil$.*

Proof: Let M be a set of cardinality $m+1$ and let N be a set of cardinality n where n is even (and m is odd). Suppose H is an $H(n+m, n+m+1)$ defined on $M \cup N$ which contains as a subarray a $RSP(m)$ defined on M. We write H in the following form where: R is an $m \times m$ array; B is an $n \times n$ array; A is an $m \times n$ array; and C is an $n \times m$ array.

$$H = \begin{array}{|c|c|} \hline R & A \\ \hline C & B \\ \hline \end{array}$$

Let n_{AC} denote the number of pairs in $A \cup C$ with both elements in N. Then $n_{AC} \geq mn - m(m-1)/2$. Since each pair in N must occur once in H, $\binom{n}{2} - n_{AC} \geq 0$. This gives $n^2-(1+2m)n+m^2-m \geq 0$. Solving this for n, we have $n \geq m+(1/2)(1+\sqrt{8m+1})$. The result follows since n is an integer. □

In view of Theorem 2.1, we are interested in constructing $RS(n)$ with $RSP(s)$ sub-designs for n an odd integer with $\lceil 2s+(1/2)(1+\sqrt{8s+1})\rceil \leq n \leq 3s$. We first note that the tripling construction due to Wallis can be used to construct $RS(3m)$ with $RSP(m)$ sub-designs.

Theorem 2.3 *[25] If there exists a $RS(m)$, then there is a $RS(3m)$ which contains as a subarray a $RSP(m)$.*

Combining Theorems 2.1 and 2.3 we have the following.

Corollary 2.4 *Let n and m be odd positive integers with $m \geq 7$ and $n \geq 3m$. Then there exists an $RS(n)$ with a $RSP(m)$ sub-design except possibly for $n = 67$ and $m = 21$.*

The frame direct product [13, 23] for Room squares can be used to construct Room squares with RSP sub-designs. We briefly indicate the proof and show how the RSP sub-design is constructed. (Frames are defined in Section 3.)

Theorem 2.5 *Suppose there exists a $RS(A)$ with a $RSP(B)$ sub-design, a pair of mutually orthogonal Latin squares of order $u-v$, and an $(u,v)-IRS$, then there is a $RS(A(u-v)+v)$ with a $RSP(B(u-v)+v)$ sub-design.*

Proof: Let R be a $RS(A)$ in standard form with a $RSP(B)$ sub-design. We write R so that the $B \times B$ array in the lower right hand corner of R contains a $RSP(B)$ (in standard form). Delete the main diagonal of R and (using a pair of mutually orthogonal Latin squares of order $(u-v)$) expand by $(u-v)$ to construct a Room frame of type $(u-v)^A$. We fill in the first $A-1$ holes of this frame using $(u,v)-IRS$s and pulling out a subarray of side v, see [13, 23]. In the last hole, we place a $RS(u)$ or (if $v \neq 3, 5$) a $(u,v)-IRS$ and a $RS(v)$. The resulting array R' is a $RS(A(u-v)+v)$. It is straightforward to verify that the square array of side $(B(u-v)+v)$ in the lower right hand corner of R' contains a $RSP(B(u-v)+v)$. □

Using Corollary 2.4 and Theorem 2.5 we now have a general construction for $RS(n)$ containing $RSP(m)$ where $n < 3m$.

Theorem 2.6 *Suppose $B \geq 7$ is odd, $u - v \neq 6$, and that there exists an $(u,v) - IRS$, then there exists a $RS(3x - 2v)$ containing an $RSP(x)$ sub-design for $x = (B(u-v)+v)$.*

Proof: In Theorem 2.5 use the $RS(3B)$ with a $RSP(B)$ sub-design that exists by Corollary 2.4. □

Using Theorem 2.6, it is now possible to construct infinite classes of $RS(n)$ containing $RSP(m)$ where $n < 3m$. The next corollary describes two examples where $u - v = 8$; the first one uses Theorem 2.6 with $(u,v) = (9,1)$, and the second one uses $(u,v) = (11,3)$.

Corollary 2.7 *(i) For every $m \equiv 9 \bmod 16$ with $m \geq 57$ there exists an $RS(3m-2)$ containing a $RSP(m)$ sub-design.*

(ii) For every $m \equiv 11 \bmod 16$ with $m \geq 59$ there exists an $RS(3m-6)$ containing a $RSP(m)$ sub-design.

3 Howell designs with RSP sub-designs

Our main constructions for Howell designs with RSP sub-designs are frame constructions. In order to describe these constructions, we need some definitions.

Let V be a set of v elements. Let $V_1, V_2, \ldots, V_n$ be a partition of V. A $\{V_1, V_2, \ldots, V_n\}$-*Room frame* F is a square array of side v which satisfies the properties listed below. We index the rows and columns of F by the elements of V.

1. Each cell is either empty or contains an unordered pair of symbols of V.

2. The subarrays indexed by $V_i \times V_i$ are empty for $i = 1, 2, \ldots, n$. (These subsquares often referred to as the *holes* of F.)

3. Row (or column) x contains each element of $V - V_i$ for $x \in V_i$.

4. The pairs occurring in F are precisely those $\{u, v\}$, where $\{u, v\} \in (V \times V) - \cup_{1 \leq i \leq n}(V_i \times V_i)$.

A $\{V_1, V_2, \ldots, V_n\}$-Room frame F is said to be *skew* if at most one of the cells (i, j) and (j, i) $(i \neq j)$ is nonempty.

The type of a $\{V_1, V_2, \ldots, V_n\}$-Room frame is the multiset $\{|V_1|, |V_2|, \ldots, |V_n|\}$. We use "exponential notation" to describe the type of a Room frame; a Room frame has type $t_1^{u_1} t_2^{u_2} \ldots t_n^{u_n}$ if there are u_i V_j's of cardinality t_i, $1 \leq i \leq k$. For convenience, in this paper we will use the terms frame and Room frame interchangeably.

The basic frame construction for Howell designs is stated below. It is often referred to as the "filling in the holes" construction.

Theorem 3.1 *[22] If there exists a $\{G_1, G_2, \ldots, G_m\}$-frame and $H^*(|G_i|, |G_i|+\alpha)$ for $i = 1, 2, \ldots, m$, then there exists an $H^*(\sum_{i=1}^m |G_i|, (\sum_{i=1}^m |G_i|) + \alpha)$ which contains an $H^*(|G_i|, |G_i| + \alpha)$ sub-design for all $i = 1, 2, \ldots, m$.*

The following corollaries of the basic frame construction are useful for constructing Howell designs with sub-designs.

Corollary 3.2 *If there exists a $\{G_1, G_2, \ldots, G_m\}$-frame and $H^*(|G_i|+w, |G_i|+w+\alpha) - H^*(w, w+\alpha)$ for $i = 1, 2, \ldots, m$, then there exists an $H^*((\sum_{i=1}^m |G_i|)+w, (\sum_{i=1}^m |G_i|) + w + \alpha) - H^*(w, w + \alpha)$.*

Corollary 3.3 *If there exists a $\{G_1, G_2, \ldots, G_m\}$-frame and $H^*(|G_i|+w, |G_i|+w+\alpha) - H^*(w, w+\alpha)$ for $i = 1, 2, \ldots, m-1$ and an $H^*(|G_m|+w, |G_m|+w+\alpha)$, then there exists an $H^*((\sum_{i=1}^m |G_i|) + w, (\sum_{i=1}^m |G_i|) + w + \alpha)$.*

It is clear that the basic frame construction can be used to produce $H(n, n + \alpha)$ which contain as sub-arrays $H(m, m + \alpha)$ for some m and n. There are two ways to use the basic frame construction to produce $H(n, n+\alpha)$ which contain as sub-arrays $H(m, m + 1)$ (Room squares) or Room square patterns. The first method is to fill in one of the holes of the Room frame with an $H^*(|G_i|, |G_i| + \alpha) - H(m, m + 1)$ or with an $H^*(|G_i|, |G_i| + \alpha)$ with an RSP sub-design. The second method is to construct $\{G_1, G_2, \ldots, G_m\}$-frames which contain as a sub-design a Room square pattern in standard form with the main diagonal deleted.

In [6], we used the first method to establish the following existence results for $H(n, n + 2)$ with Room square sub-designs. In each case, $H(2t, 2t + 2) - H(m, m + 1)$ are constructed for several small values of t and then these designs are used to fill in the holes of Room frames.

Theorem 3.4 *[6]*

(i) There exists an $H(n, n + 2) - H(3, 4)$ if and only if $n \equiv 0 \pmod 2$, $n \geq 10$.

(ii) There exists an $H(n, n + 2) - H(5, 6)$ if and only if $n \equiv 0 \pmod 2$, $n \geq 16$.

(iii) There exists an $H(n, n + 2) - H(7, 8)$ if and only if $n \equiv 0 \pmod 2$, $n \geq 22$.

In this section, we use the second method to construct $H(n, n+\alpha)$ with RSP sub-designs. Let $RSP^*(m)$ denote a $RSP(m)$ in standard form with the main diagonal deleted. We construct frames with RSP^* sub-designs and then fill in the holes with the appropriate Howell designs in standard form. We first construct frames with RSP^* sub-designs for frames of type 2^n.

Theorem 3.5 *Let $m \equiv 1 \pmod 2$, $m \geq 7$. There exists a frame of type 2^m with a $RSP^*(m)$ as a sub-design.*

Proof: The skew Room square construction for frames of type 2^m always produces frames with $RSP^*(m)$ sub-designs, (see [9, Theorem 2.6]). □

Frames with partitionable transversals are useful in constructing Howell designs with sub-designs. Let F be a Room frame of type $(2t)^n$ defined on $V \cup W$ where $|W| = 2t$. F has a set S of ℓ *holey ordered partitionable transversals with respect to the hole* W if the transversals in S satisfy the following properties. Let $S = \{S^1, S^2, \ldots, S^\ell\}$.

1. S^j contains $2t(n-1)$ pairs, one from each row and column indexed by V, which can be partitioned into two sets S_1^j and S_2^j, $|S_1^j| = |S_2^j| = t(n-1)$, where every element of V occurs precisely once in S_i^j, $i = 1, 2$.

2. The pairs in F can be ordered so that every element in V occurs precisely once as a first coordinate and precisely once as a second coordinate in the pairs of S^j.

See [16, 17, 24, 6, 8] for further information on partitionable transversals in frames.

The next construction uses a patterned starter and a skew strong starter which can be Latin square ordered [15]. A skew strong starter S of order q is said to be *Latin square ordered* if the array of ordered pairs generated by $S \cup -S$ (in the usual Room square construction) is the array of pairs formed by the superposition of a pair of $OPILS$ of type 1^q. (See [19, 17, 22, 10] for definitions and results on starters and [23, 15] for definitions and results on orthogonal partitioned incomplete Latin squares, $OPILS$.)

Example 3.6 $S = \{\{2,3\}, \{4,6\}, \{1,5\}\}$ is a Latin square ordered skew strong starter of order 7. $S \cup -S$ generates the following square, L.

	6,2	5,4	4,6	3,1	2,3	1,5
2,6		0,3	6,5	5,0	4,2	3,4
4,5	3,0		1,4	0,6	6,1	5,3
6,4	5,6	4,1		2,5	1,0	0,2
1,3	0,5	6,0	5,2		3,6	2,1
3,2	2,4	1,6	0,1	6,3		4,0
5,1	4,3	3,5	2,0	1,2	0,4	

L can be decomposed into the following two $OPILs$ of type 1^7.

	6	5	4	3	2	1
2		0	6	5	4	3
4	3		1	0	6	5
6	5	4		2	1	0
1	0	6	5		3	2
3	2	1	0	6		4
5	4	3	2	1	0	

	2	4	6	1	3	5
6		3	5	0	2	4
5	0		4	6	1	3
4	6	1		5	0	2
3	5	0	2		6	1
2	4	6	1	3		0
1	3	5	0	2	4	

Theorem 3.7 *If there exists a skew strong starter of order q which can be Latin square ordered, then there is a frame of type 2^q with a holey ordered partitionable transversal which contains a $RSP^*(q)$.*

Proof: Let G an additive abelian group on an odd number q of elements, $G = \{0, x_1, x_2, \ldots, x_{q-1}\}$. Let $\overline{G}$ be a copy of G, $\overline{G} = \{\overline{0}, \overline{x_1}, \overline{x_2}, \ldots, \overline{x_{q-1}}\}$.

Let S be a skew strong starter of order q defined on G, and let $-\overline{S}$ be the skew complement of S defined on $\overline{G}$. The pairs in S can be Latin square ordered, $S = \{(s_i, t_i) \mid i = 1, 2, \ldots, (q-1)/2\}$ where $\cup_{i=1}^{(q-1)/2}\{s_i, t_i\} = G - \{0\}$. Similarly, the pairs in $-\overline{S} = \{(\overline{u_i}, \overline{v_i}) \mid i = 1, 2, \ldots, (q-1)/2\}$ where $\cup_{i=1}^{(q-1)/2}\{\overline{u_i}, \overline{v_i} \mid i = 1, 2, \ldots, (q-1)/2\} = \overline{G} - \{\overline{0}\}$. B_1 will denote the set $\{(s_i, \overline{t_i}) \mid i = 1, 2, \ldots, (q-1)/2\}$ where $s_i \in G$ and $\overline{t_i} \in \overline{G}$, and B_2 the set $\{(u_i, \overline{v_i}) \mid i = 1, 2, \ldots, (q-1)/2\}$ where $u_i \in G$ and $\overline{v_i} \in \overline{G}$. Let L_1 denote the $q \times q$ array generated by B_1 and its corresponding adder, and let L_2 denote the $q \times q$ array generated by B_2 and its adder. Then the array $L = L_1 \circ L_2$ is the array formed by the superposition of a pairs of $OPILS$ of type 1^q. Let R_1 be the $q \times q$ array of pairs generated by S and let R_2 be the $q \times q$

array of pairs generated by $-\overline{S}$. R will denote the array of pairs formed by the superposition of R_1 and R_2. We construct a frame F of type 2^q with an ordered partitionable transversal as follows.

$$F = \begin{array}{|c|c|}\hline R & \\ \hline & L \\ \hline \end{array}$$

We use the patterned starter P to find an ordered partitionable transversal for F. Let P be the patterned starter defined on G. The Latin square ordering of the pairs in S induces an ordering of the pairs in P: $P = \{(x_i, y_i) \mid i = 1, 2, \ldots, (q-1)/2\}$. Let $\overline{P}$ denote the Latin square ordering of the pairs in $-\overline{S}$: $\overline{P} = \{(\overline{x_i}, \overline{y_i}) \mid i = 1, 2, \ldots, (q-1)/2\}$. The pairs constructed from the patterned starter in L are $P' = \{(y_i, \overline{x_i}), (\overline{y_i}, x_i) \mid i = 1, 2, \ldots, (q-1)/2\}$. The pairs in $P \cup \overline{P} \cup P'$ form an ordered partitionable transversal of F; the partitioning is $P_1 = P \cup \overline{P}$ and $P_2 = P'$.

Since R contains a $RSP^*(q)$, namely R_1 (or R_2), then F contains a $RSP^*(q)$.□

Corollary 3.8 *Let q be an odd prime power, $q \geq 7$, $q \neq 9$. There exists a frame of type 2^q which contains both a $RSP^*(q)$ and an ordered partitionable transversal.*

Proof: For every odd prime power $q \geq 7$, $q \neq 9$, there exist skew strong starters of order q which can be Latin square ordered [15].□

We use frames of type 2^n with RSP^* sub-designs and the existence of frames of type 1^n ($RS(n)$) together with some standard frame constructions to construct frames with RSP^* sub-designs and larger hole sizes. The first two constructions, the Direct Product and the Fundamental Construction, are well known.

Theorem 3.9 *(Direct Product) [23] If there exists a frame of type $t_1^{u_1} t_2^{u_2} \ldots t_n^{u_n}$ and a pair of mutually orthogonal Latin squares of side m, then there is a frame of type $(mt_1)^{u_1}(mt_2)^{u_2} \ldots (mt_n)^{u_n}$.*

Note that if the frame of type $t_1^{u_1} t_2^{u_2} \dots t_n^{u_n}$ has a $RSP^*(s)$ sub-design, then the resulting frame also has a $RSP^*(s)$ sub-design.

The Fundamental Construction is stated in terms of group divisible designs. A *group divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$ where (i) X is a set (called *points*), (ii) $\mathcal{G}$ is a partition of X into subsets (called *groups*), (iii) $\mathcal{A}$ is a family of subsets of X (called *blocks*) such that a group and a block contain at most one common point, and (iv) every pair of points from distinct groups occurs in exactly one block. The *group-type* (or *type*) of a GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. Usually an "exponential notation" is used to describe the type of a GDD: a GDD of type $t_1^{u_1} t_2^{u_2} \dots t_n^{u_n}$ is a GDD where there are u_i groups of size t_i, for $1 \leq i \leq k$. A K-GDD is a GDD with blocks of size k with $k \in K$. A *transversal design* $TD(k, n)$ is a $\{k\}$-GDD of type n^k. (We note that the pairs in a Room frame form a GDD with block size 2 and groups $V_1, V_2, \dots, V_n$.)

Theorem 3.10 *(Fundamental Construction) [26, 23] Let $(X, \mathcal{G}, \mathcal{A})$ be a GDD, and let $w : X \to Z^+ \cup \{0\}$ (w is called a weighting). For every $A \in \mathcal{A}$, suppose there is a Room frame of type $\{w(x) : x \in A\}$, then there is a Room frame of type $\{\sum_{x \in G} w(x) : \; G \in \mathcal{G}\}$.*

If one of the Room frames of type $\{w(x) : x \in A\}$ used in this construction contains as a sub-design a $RSP^*(s)$, then the resulting frame also contains a sub-design a $RSP^*(s)$.

Theorem 3.6 of [16] can be also used to construct frames with RSP sub-designs. This construction uses incomplete orthogonal arrays. Let V be a finite set of size n and let K be a subset of V of size n. An *incomplete orthogonal array* $IA(n, k, s)$ is an $(n^2 - k^2) \times s$ array written on the symbol set V such that every ordered pair of $(V \times V) - (K \times K)$ occurs in every pair of columns of the array. An $IA(n, k, s)$ is equivalent to a set of $s - 2$ mutually orthogonal Latin squares of order n which are missing a subsquare of order k. (We need not be able to fill in the $k \times k$ missing subsquares with Latin squares of order k.) The existence of an $IA(m + u, u, 4)$ is equivalent to the existence of a pair of incomplete mutually orthogonal Latin squares, $IMOLS(m + u; u)$. Recent results on IAs or $IMOLS$ can be found in [2].

Theorem 3.11 *[16] Suppose there exist*

1. *a frame of type $(2t)^n$ with a $RSP(n)$ sub-design and with a set S of ℓ holey ordered partitionable transversals with respect to the same hole,*

2. *a pair of orthogonal Latin squares of side m, and*

3. *$IA(m + k_j, k_j, 4)$ where $\sum_{j=1}^{\ell} k_j = k$.*

Then there exists a frame of type $(2tm)^{n-1}(2tm + 2k)$ with a $RSP(n)$ sub-design.

Similarly, Construction 3.4 from [24] can be used to construct frames with RSP^* sub-designs (see [24] for the additional definitions). This construction is the analogue of Theorem 3.11 for frames of type t^n with t odd.

Theorem 3.12 *[24] Suppose there is a frame of type t^g with a $RSP^*(g)$ sub-design and ℓ disjoint complete ordered transversals. For $1 \leq i \leq \ell$, let $u_i \geq 0$ be and integer, and let m be an even positive integer. Suppose there exists a $SOLSSOM(m)$ such that the SOM is unipotent. Suppose also that there exist a pair of partitionable $IMOLS(m + u_i; u_i)$ for $1 \leq i \leq \ell$. Then there is a frame of type $(mt)^g(2u)^1$ with a $RSP^*(g)$ sub-design where $u = \sum_{i=1}^{\ell} u_i$.*

Note that if we use frames of type 1^g with g odd and $g \geq 7$, then the resulting frames contain as sub-designs $RS(g)$ with the main diagonal deleted.

Frames of type 2^q with $RSP^*(q)$ sub-designs can be used in the constructions described in Theorems 3.9 and 3.10 to produce Howell designs with $RSP(q)$ sub-designs. An example is the following.

Lemma 3.13 *Let $q \equiv 1 \pmod 2$, $q \geq 7$. Suppose there exists a set of $q-2$ mutually orthogonal Latin squares of order m (a $TD(q, m)$) and suppose $n = (q-2)m + i + j$ where $2 \leq i, j \leq m$. Then there is an $H(2n, 2n + 2)$ which contains a $RSP(q)$ sub-design.*

Proof: The $q - 2$ mutually orthogonal Latin squares of order m are used to construct a $\{q, q-1, q-2\}$-GDD of type $m^{q-2}(i)^1(j)^1$. Since there exists a frame of type 2^q with a $RSP^*(q)$ sub-design by Theorem 3.5 and frames of type 2^k for $k \geq 5$ [9, 22], we apply the Fundamental Construction, Theorem 3.10, to produce frames of type $(2m)^{q-2}(2i)^1(2j)^1$ with $RSP^*(q)$ sub-designs. Fill in the holes with Howell designs in standard position to construct $H(2n, 2n + 2)$. □

This construction illustrates that it is quite easy to establish the existence of $H(n, n + \alpha)$ $(n \geq n_0)$ which contain $RSP(q)$ sub-designs *without* constructing any small designs. Since we have already established the existence of $H(n, n+2) - RS(7)$ in Theorem 3.4, we will consider the next case for $RSP(q)$s, $q = 9$. It is important to contrast the techniques that we use for this case to the techniques we used for $H(n, n+2) - RS(q)$ for $q = 3, 5$ and 7 in [6]. Although it is well within our computing power to construct $H(n, n+2) - RS(9)$ as we did for $RS(7)$, for larger orders of Room squares this becomes more difficult.

We first note the following bound for $RSP(q)$ sub-designs.

Lemma 3.14 *If there exists an $H(n+m, n+m+2)$ with a $RSP(m)$ sub-design, then $n + m \geq \lceil 2m + \sqrt{2m} \rceil$.*

Proof: The proof is similar to that of Lemma 2.2. Let M be a set of cardinality $m+1$ and let N be a set of cardinality $n+1$ where n is odd. Suppose H is an $H(n+m, n+m+1)$ defined on $M \cup N$ which contains as a subarray a $RSP(m)$ defined on M. We write H in the following form where: R is an $m \times m$ array; B is an $n \times n$ array; A is an $m \times n$ array; and C is an $n \times m$ array.

$$H = \begin{array}{|c|c|} \hline R & A \\ \hline C & B \\ \hline \end{array}$$

H is missing $(n+m+2)/2$ pairs and these pairs partition the set $N \cup M$. Since each pair in M must occur once in the design, the number of pairs with both elements in N which do not occur in H is $(n-m)/2$. Let n_{AC} denote the number of pairs in $A \cup C$ with both elements in N. Then $n_{AC} \geq (n+1)m - (m(m-1)/2)$. Since there are $\binom{n+1}{2} - (n-m)/2$ pairs with both elements in N contained in H, we have the inequality

$$\binom{n+1}{2} - (n-m)/2 \geq (n+1)m - (m(m-1)/2).$$

This gives us $n^2 - 2mn + m^2 \geq 2m$ or $(n-m)^2 \geq 2m$. $\square$

This gives us a bound of $n \geq 18$ for an $H(n, n+2)$ which contains a $RSP(7)$. Since a $RS(7)$ is a $RSP(7)$, there are only 2 values needed to complete the spectrum of these designs. Figure 1 contains an $H(20, 22)$ with a $RSP(7)$ sub-design. This combined with the results of Theorem 3.4 gives the following:

Theorem 3.15 *There exists an $H(n, n+2)$ which contains a $RSP(7)$ as a sub-design for $n \geq 18$ except possibly for $n = 18$.*

The lower bound for n for an $H(n, n+2)$ which contains a $RSP(9)$ is 24.

Theorem 3.16 *There exists an $H(n, n+2)$ which contains a $RSP(9)$ sub-design for all $n \geq 24$.*

Proof: The constructions for $n = 26, 28, 30, 32, 34$, and 38 are listed in [7], and constructions for $n = 24, 46, 48, 50, 52, 58, 60, 66$, and 70 can be found at the web site `http://www.emba.uvm.edu/~dinitz/rsp/rsp.html`. Constructions for the remaining cases for $n \leq 74$ are listed in Table 1.

Table 1

n	Construction	Comments
36	3.9	1^9-frame, $m = 4$
	3.1	$H(4,6)$
40	3.12	1^9-frame, $\ell = 4, m = 4, u = 2$
	3.1	$H(4,6)$
42	3.12	1^9-frame, $\ell = 4, m = 4, u = 3$
	3.1	$H(4,6)$, $H(6,8)$
44	3.12	1^9-frame, $\ell = 4, m = 4, u = 4$
	3.1	$H(4,6)$, $H(8,10)$
54	3.9	2^9-frame, $m = 3$
	3.1	$H(6,8)$
56	3.9	2^9-frame, $m = 3$
	3.3	$H(8,10) - H(2,4)[6]$, $H(8,10)$
62	3.10	$\{7,9\} - GDD$ of type $6^9 8$
	3.1	$H(6,8)$, $H(8,10)$
64	3.10	$\{7,9\} - GDD$ of type $6^9 8$
	3.3	$H(8,10) - H(2,4)$, $H(10,12)$
68	3.10	$\{7,9\} - GDD$ of type $8^8 2^1$ (see [18])
	3.3	$H(10,12) - H(2,4)$, $H(4,6)$
72	3.9	1^9-frame, $m = 8$
	3.1	$H(8,10)$
74	3.9	1^9-frame, $m = 8$
	3.3	$H(10,12) - H(2,4)[6]$, $H(4,6)$

For $76 \leq n \leq 98$, we use Theorem 3.12 with a frame of type 1^9, $m = 8$, $\ell = 4$ and $2 \leq u \leq 12$ to construct a frame of type $8^9(2u)$ with a $RSP^*(9)$. We fill in the holes using Theorem 3.1 and Corollary 3.3 (for $n = 98$).

Let $n = 2v$. For $50 \leq v \leq 59$, we construct GDDs with at least one block of size 9 and then use Theorems 3.10 ($w(x) = 2$ for all x) and 3.1 to construct the Howell designs, noting that the block of side 9 assures a sub RSP(9) by Theorem 3.5. Begin with a resolvable transversal design TD(7,9) and delete all but i points from the first group and all but j points from the second group, where $0 \leq i \leq j \leq 9$. Now use one of the resolution classes of blocks as the groups to obtain a $\{i, j, 5, 6, 7, 9\}$-GDD of type $5^{9-j} 6^{j-i} 7^i$ which has at least 5 blocks of size 9. Let $i = 0$ and $5 \leq j \leq 9$ for $50 \leq v \leq 54$ and let

$i = 5$ and $5 \le j \le 9$ for $55 \le v \le 59$.

For $v \ge 60$, we write $v = 7 \cdot m + i + j$ where $m \ge 8,\ 2 \le i, j \le m$ and there exists a set of 7 mutually orthogonal Latin squares of order m. Since there exist 7 mutually orthogonal Latin squares of order m for $m > 780$ (see [1]), we need only find a set $M = \{m_1, \ldots, m_k\}$ such that there exist 7 mutually orthogonal Latin squares of order m for $m \in M$ and $9m_i \ge 7m_{i+1} + 4$. Such a set is $M = \{8, 9, 11, 13, 16, 19, 23, 25, 31, 41, 47, 56, 71, 88, 109, 139, 176, 225, 288, 369, 473, 607, 621\}$. Since we can construct $GDD(v; \{7, 8, 9\}; \{m, i, j\}; 0, 1)$, we apply Theorem 3.10, (Lemma 3.13). □

The constructions using frames with $RSP(q)$ sub-designs can be used to easily construct $H^*(n, n + \alpha)$ for larger values of α. The following is an example for $RSP(7)$ sub-designs.

Lemma 3.17 *Let ℓ be an integer, $1 \le \ell \le 6$. There exists an $H^*(n, n + \ell)$ which contains a $RSP(7)$ sub-design for $n \ge 168$ and $n + \ell$ even.*

Proof: Since there exists a frame of type 2^7 with a partitionable transversal and a $RSP^*(7)$ (by Corollary 3.8), we apply Theorem 3.11 with $n = 7,\ m \ge 12$ and $0 \le k \le 6$ to construct a frame containing a $RSP(7)$ sub-design. Fill in this frame using Theorem 3.1 or Corollary 3.3 to construct the desired Howell design. □

Note again that it is easy to extend the constructions for $RSP(q)$ sub-designs to constructions for larger α. We would have to work much harder to get similar results for Room squares.

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