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Spanning Sets and Scattering Sets in Steiner Triple Systems

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A spanning set in a Steiner triple system is a set of elements for which each element not in the spanning set appears in at least one triple with a pair of elements from the spanning set. A scattering set is a set of elements that is independent, and for which each element not in the scattering set is in at most one triple with a pair of elements from the scattering set. For each $v \equiv 1, 3 \pmod 6$, we exhibit a Steiner triple system with a spanning set of minimum cardinality, and a Steiner triple system with a scattering set of maximum cardinality. In the process, we establish the existence of Steiner triple systems with complete arcs of the minimum possible cardinality.

1. BACKGROUND

A *Steiner triple system* of *order* v, or STS(v), is a v-element set V, and a set $\mathscr{B}$ of 3-element subsets of V called *triples*, for which each pair of elements from V appears in precisely one triple of $\mathscr{B}$. For $X \subseteq V$, the *spanned set* $\mathscr{C}(X)$ is the set $\{y \in V \backslash X : \{a, b, y\} \in \mathscr{B}, a, b \in X\}$. A set $X \subseteq V$

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is a *spanning set* if for every $v \in V \backslash X$, $v \in \mathscr{C}(X)$. An *independent set* or *arc* is a set $X \subseteq V$ containing no three elements which form a triple in $\mathscr{B}$. A *scattering set* $X \subseteq V$ is an independent set for which every $y \in V \backslash X$ has the property that y appears in a triple with at most one pair of elements from X.

Consider an STS $\mathscr{S} = (V, \mathscr{B})$ of order v. Let the *scattering number* $\text{scat}(\mathscr{S})$ be the size of a largest scattering set in $\mathscr{S}$, and the *spanning number* $\text{span}(\mathscr{S})$ be the size of a smallest spanning set in $\mathscr{S}$. Let $\text{scat}(v)$ be the maximum scattering number of an STS(v), and let $\text{span}(v)$ be the minimum spanning number. We first determine easy bounds on $\text{scat}(v)$ and $\text{span}(v)$. A set X of cardinality x spans at most $\binom{x}{2}$ distinct elements, with equality only if X is a scattering set. Hence for X to be a scattering set, $x + \binom{x}{2} = \binom{x+1}{2} \leqslant v$; similarly for X to be a spanning set, $\binom{x+1}{2} \geqslant v$. Rewriting this as an inequality on x, we have

$$\text{scat}(v) \leqslant \tfrac{1}{2}(\sqrt{8v+1} - 1) \leqslant \text{span}(v).$$

Naturally, $\text{scat}(v)$ and $\text{span}(v)$ are both integer, and hence more precisely we have the following inequalities, the second of which appears in [4]:

$$\text{scat}(v) \leqslant L(v) = \lfloor \tfrac{1}{2}(\sqrt{8v+1} - 1) \rfloor$$

and

$$\text{span}(v) \geqslant U(v) = \lceil \tfrac{1}{2}(\sqrt{8v+1} - 1) \rceil.$$

A Steiner triple system $\mathscr{S}$ of order v is *scattered* if $\text{scat}(\mathscr{S}) = L(v)$, and it is *spanned* if $\text{span}(\mathscr{S}) = U(v)$.

In this paper, our primary result is the construction, for every $v \equiv 1, 3 \pmod 6$ of a scattered Steiner triple system, and a spanned Steiner triple system.

At first glance, the definitions and resulting problems seem somewhat artificial. Hence we examine some motivation. Our original motivation arose in problems of *covering radius* (see [1 and Refs. therein]). Brualdi, Pless, and Wilson employed spanning sets in projective geometries to construct codes with small covering radius. The generalization to designs is natural in this context.

A second motivation arises from the large body of research on independent sets in Steiner triple systems [2, 6]. Independent sets have been studied geometrically as arcs. Of particular interest are maximal independent sets, or (in the geometric terminology) *complete arcs* (see, for example, [3]). De Resmini [4] has examined complete arcs of minimum cardinality in STS, and exhibited such an arc in an STS(15). She also determined the necessary conditions for $L(v) = U(v)$, the "tight" cases, namely that

$x \equiv 1, 2 \pmod 4$, $v = \binom{x+1}{2}$. It is straightforward to verify that every complete arc is a spanning set; the converse need not hold. Nevertheless, all of the spanning sets we produce are independent as well. Hence a consequence of our results is a determination of the size of minimum cardinality complete arcs in Steiner triple systems for every order.

A third motivation is algebraic in nature. To interpret a spanning set of a STS in the corresponding Steiner quasigroup, observe that every element of the quasigroup is either in the set, or is the product of a pair of elements in the set. Hence a spanning set can be viewed as a *2-generating* set. In general, a k-generating set is a subset for which each element is expressible as the product of at most k elements in the set. The spanning sets produced here are the smallest possible 2-generating sets in commutative idempotent (and semisymmetric) quasigroups (for orders 1, 3 (mod 6)). It is known that every quasigroup of order n has a generating set (n-generating set) of size at most log n, but no previous work seems to have determined k-generating sets for k small.

A fourth motivation is graph-theoretic. Every STS(v) corresponds to a so-called Steiner 1-factorization of the complete graph K_{v+1}. A spanning set in the STS is a set of vertices in the 1-factorization so that every 1-factor meets this set in at least one edge, while a scattering set requires that each 1-factor meet the set in at most one edge. Hence spanning (or scattering) sets can be viewed as subgraphs of a coloured complete graph, where factors are colours, in which every colour is represented at least once (or at most once).

The geometric, algebraic, and graph-theoretic viewpoints can all be useful for examining these problems and their applications. We should remark that spanning sets are apparently more interesting than scattering sets. We include the latter for two reasons. First, spanning is a "covering" problem and scattering is the dual "packing" problem. Second, in our recursive techniques, we employ scattering sets to form spanning sets.

2. Scattered Latin Squares

A latin square of order s on symbols $\{1, ..., s\}$ is (a, b)*-scattered* if

(1) the $a \times a$ subsquare determined by rows $1, ..., a$ and columns $1, ..., a$ contains a^2 distinct symbols, none of which are in the set $\{1, ..., b\}$;

(2) in rows $1, ..., a$, the symbols $1, ..., b$ appear in ab distinct columns; and

(3) in columns $1, ..., a$, the symbols $1, ..., b$ appear in ab distinct rows.

We first give an example of a (2, 2)-scattered latin square of order 7:

3	4	1	5	2	6	7
5	6	7	1	3	2	4
1	3	2	6	4	7	5
4	1	6	7	5	3	2
2	7	4	3	1	5	6
6	2	5	4	7	1	3
7	5	3	2	6	4	1

We require certain scattered latin squares in the construction of triple systems; hence we provide a general construction for them here.

LEMMA 2.1. *If* $b \leqslant a$ *and* $s \geqslant \max(a^2+b, ab+a+b)$, *then an* (a, b)-*scattered latin square of order* s *exists.*

Proof. Write $B = \lfloor (s-b)/a \rfloor$, $r = s-b-aB$ and $n = s-aB+1$. Note that $B \geqslant b+1$, $r \geqslant 0$ and $n \geqslant b+1$. Define first an $a \times a$ matrix $M = (m_{ij})$ in which $m_{ij} = (i-1)a + (j-1)$. M_i denotes the matrix obstained from M by adding $a \cdot i$ to each symbol mod aB. M_{ij} is obtained from M_i by taking column $k+j$ (mod a) of M_{ij} to be column k of M_i (that is, we cyclically rotate the columns). L_{ij} denotes the matrix obtained from M_{ij} by adding n to each entry of M_{ij}; not that each L_{ij} does not involve the symbols $1, \ldots, b+r$.

Next we form an $ab \times aB$ rectangle $\mathscr{L}$, which consists of $b \times B$ disjoint submatrices of size $a \times a$, which we call *principal* submatrices. The $b \times B$ matrix whose entries are the principal submatrices is called the *skeleton* of $\mathscr{L}$. The principal submatrix in the (i, j) position of the skeleton of $\mathscr{L}$ is chosen as follows. Let $k = j - i$ (mod B), $0 \leqslant k < B$. Place $L_{k,j-1}$ in position (i, j) of the skeleton. We depict the skeleton of $\mathscr{L}$ next, in order to simplify the verification of its properties:

$$\mathscr{L} = \begin{matrix} L_{00} & L_{10} & L_{20} & \cdots & L_{B-1,0} \\ L_{B-1,1} & L_{01} & L_{11} & \cdots & L_{B-2,1} \\ L_{B-2,2} & L_{B-1,2} & L_{02} & \cdots & L_{B-3,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{B-b+1,b-1} & L_{B-b+2,b-1} & L_{B-b+3,b-1} & \cdots & L_{B-b,b-1} \end{matrix}$$

We first verify that $\mathscr{L}$ is row latin. Any two principal submatrices in the same row of the skeleton are L_{ij} and L_{kj} with $i \neq k$. Hence the rows are disjoint unless $i \equiv k \pmod{B}$, which happens here only when $i = k$.

We next verify that $\mathscr{L}$ is column latin. In column k of L_{ij}, all symbols are congruent to $n+k-j \pmod{a}$. Two principal submatrices in the same column of the skeleton, say L_{ij} and L_{kl}, appear in rows $j+1$ and $l+1$ of the skeleton. If their columns intersect, we must have $j \equiv l \pmod{a}$. However, since $b \leqslant a$, this implies that $j = l$.

At this point, we have constructed a matrix of size $ab \times aB$ (which is, in fact, a latin rectangle on symbols $n, ..., s$). We want to extend this rectangle $\mathscr{L}$ to an $ab \times s$ latin rectangle. To do this, we use standard methods of projecting transversals, but at the same time we introduce properties (2) and (3) of the scattered latin square. Transversals $T_1, ..., T_b$ are chosen as follows. To form T_i, select the entries of $\mathscr{L}$ which are on the principal diagonals of the principal submatrices $L_{B-i,0}, ..., L_{B-i,b-1}$. Next r transversals $S_1, ..., S_r$ are found as follows. There are r further disjoint transversals of the submatrix $L_{B-1,0}$. Each can be extended to a transversal of $\mathscr{L}$ by taking the transversal of the same positions in each submatrix $L_{B-1,0}, ..., L_{B-1,b-1}$. The $b+r$ transversals produced in this way are pairwise disjoint.

For $i = 1, ..., b$, the symbols in T_i are moved to column $s-r-b+i$ (keeping the entries in the same rows), and the cells of the transversal are filled with symbol i. For $i = 1, ..., r$, a similar operation moves elements of S_i to column $s-r+i$, and fills each cell of the transversal with symbol $b+i$. The result is an $ab \times s$ latin rectangle which meets conditions (1) and (2) of a scattered latin square. It nearly meets condition (3), in that each symbol $1, ..., b-1$ appears in the first a columns in different rows. However, symbol b does not yet appear in the first a columns. This guarantees that any completion of this rectangle to a latin square is (a, b)-scattered. Every latin rectangle has a completion to a latin square. ∎

To illustrate the construction of scattered latin squares, we give an example. Take $a=2$, $b=1$ and $s=5$. We compute $B=2$, $r=0$ and $n=2$. Then $M = \left[\begin{smallmatrix}01\\23\end{smallmatrix}\right]$. The matrix $\mathscr{L}$ is a 2×4 matrix,

$$\begin{bmatrix}2345\\4523\end{bmatrix}.$$

In this case, we project only one transversal, to obtain

$$\begin{bmatrix}23154\\45213\end{bmatrix}.$$

Any completion of this 2×5 rectangle to a 5×5 latin square yields a (2, 1)-scattered latin square as required.

In applying these scattered latin squares, we observe that one may choose a set of $a_1 \leqslant a$ rows, $a_2 \leqslant a$ columns, and $a_3 \leqslant b$ symbols in an (a, b)-scattered latin square, so that each pair of chosen rows and columns contains a different symbol, and each chosen symbol appears in different columns (rows) in each chosen row (column, respectively). This allows us to choose different numbers of rows and columns.

3. The Recursive Construction

In this section, we adapt standard tripling constructions to produce spanned and scattered STS. We describe the general construction for spanned STS first.

Construction 3.1. For $i = 1, 2, 3$, let $\mathscr{S}_i$ be an STS$(s+w)$ containing a sub-STS(w). Suppose that $\mathscr{S}_i$ has a scattering set X_i of size $a_i + e$ which intersects the subsystem in e elements. Denote by d_i the number of elements in the subsystem which are spanned by X_i. Suppose that $a_1 \geqslant a_2 \geqslant a_3$, and there exists an (a_1, a_3)-scattered latin square of order s. Finally suppose that the following numerical conditions hold:

(1) $d_1 + d_2 + d_3 + e - 2\binom{e}{2} \geqslant w$, and

(2) if $\{i, j, k\} = \{1, 2, 3\}$, then $a_i a_j + \binom{a_k + e}{2} + a_k \geqslant s + d_k$.

Then there is an STS$(3s+w)$ containing a spanning set of size $a_1 + a_2 + a_3 + e$. Moreover, this spanning set is independent.

Proof. Let S be a set of size s, and W a disjoint set of size w. We construct an STS$(3s+w)$ on the set of elements $(S \times \{1, 2, 3\}) \cup W$. $\mathscr{S}_i$ is placed on elements $(S \times \{i\}) \cup W$, with the sub-STS(w) on W, in such a way that the elements of $X_i \backslash W$ are $\{(1, i), (2, i), \ldots, (a_i, i)\}$. We also fix $X \subseteq W$, $|X| = e$, and require that $X_i \cap W = X$. We are otherwise free to permute the elements of $\mathscr{S}_i$. Hence we can ensure that any set of $\binom{a_i + e}{2} - d_i$ elements from $(S \times \{i\}) \backslash X_i$ is spanned by triples in $\mathscr{S}_i$. Similarly, we can ensure that any set of $d_i - \binom{e}{2}$ elements of $W \backslash (X \cup \mathscr{C}(X))$ can be spanned by $\mathscr{S}_i$; note that $|\mathscr{C}(X)| = \binom{e}{2}$ elements of W are spanned by triples in the sub-STS(w).

Next we form triples from the (a_1, a_3)-scattered latin square, by taking $\{(i, 1), (j, 2), (k, 3)\}$ to be a triple exactly when symbol k appears in the (i, j) entry of the latin square. Since we are free to place the $\mathscr{S}_i$ so as to span elements which are not spanned by the "latin square triples," we need only ensure that within $(S \times \{i\})$ and within W, all elements can be spanned.

Numerical conditions (1) and (2) in the construction ensure this. Finally, the fact that the chosen STS are scattered ensures that the spanning set produced is also independent. ▮

In general, we apply Construction 3.1 with $w = 0, 1, 3, 7$. When $w = 0$, we must have $e = d_1 = d_2 = d_3 = 0$; every scattered STS(s) yields the required triple systems. When $w = 1$, we have either (i) $e = 1$ and $d_1 = d_2 = d_3 = 0$, or (ii) $e = 0$, $d_1 + d_2 + d_3 \geqslant 1$, and $d_1, d_2, d_3 \leqslant 1$. Every scattered STS($s + 1$) provides a solution of both types, depending on whether we choose the sub-STS(1) (i.e., element) to be in the scattering set or not. Similarly for $w = 3$, we may have (i) $e = 0$, $d_1 + d_2 + d_3 \geqslant 3$, $d_1, d_2, d_3 \leqslant 3$, (ii) $e = 1$, $d_1 + d_2 + d_3 \geqslant 2$, $d_1, d_2, d_3 \leqslant 2$, or (iii) $e = 2$, $d_1 = d_2 = d_3 = 1$. Every scattered STS($s + 3$) on at least nine elements provides an STS of all three types, by choosing the sub-STS(3) to be a triple with zero, one or two elements of the scattering set. Hence for $w \in \{0, 1, 3\}$, we can employ and desired values for d_1, d_2, d_3. The only difficulty arises when we apply the construction with $w = 7$. We return to this point later.

A very similar construction works for scattered STS:

CONSTRUCTION 3.2. For $i = 1, 2, 3$, let $\mathscr{S}_i$ be an STS($s + w$) containing a sub-STS(w). Suppose that $\mathscr{S}_i$ has a scattering set X_i of size $a_i + e$ which intersects the subsystem in e elements. Denote by d_i the number of elements in the subsystem which are spanned by X_i. Suppose that $a_1 \geqslant a_2 \geqslant a_3$, and there exists an (a_1, a_3)-scattered latin square of order s; suppose further that $a_1 + a_2 + a_3 + e = L(3s + w)$. Finally suppose that the following numerical conditions hold:

(1) $d_1 + d_2 + d_3 + e - 2\binom{e}{2} \leqslant w$, and

(2) if $\{i, j, k\} = \{1, 2, 3\}$, then $a_i a_j + \binom{a_k + e}{2} + a_k \leqslant s + d_k$.

Then there is an STS($3s + w$) containing a scattering set of size $a_1 + a_2 + a_3 + e$.

Proof. The proof parallels that of Construction 3.1 closely. We must ensure that no element is spanned twice. Inequality (1) guarantees that we need not span any element of the subsystem twice, and inequality (2) guarantees that we need not span any element of $S \times \{i\}$ twice. ▮

Before applying these constructions in general, we give some small examples. We consider the small cases in which $U(v) = L(v)$, as then both constructions perform identically. Consider $v = 21$. We take $s = 7$ and $w = 0$. Note that $L(21) = U(21) = 6$. We form a $(2, 2)$-scattered latin square of order 7; this is the only case (for scattered STS) in which Lemma 2.1 fails to produce the required square, but the required square was presented earlier.

Now choosing each of the STS(7) to contain the block $\{1, 2, 7\}$, and applying Construction 3.1 gives a spanned and scattered STS(21). It is perhaps important to remark that while any STS(7) has a scattering set of size 3, we required only an STS(7) with a scattering set of size 2. This requirement for a scattering set of less than maximum size is exploited in the recursive construction.

We just tabulate selections of the main parameters in the construction for some further tight cases:

v	$L(v)=U(v)$	s	w	e	d_1	d_2	d_3	a_1	a_2	a_3
45	9	15	0	0	0	0	0	3	3	3
55	10	18	1	1	0	0	0	3	3	3
91	13	30	1	1	0	0	0	4	4	4
105	14	34	3	2	1	1	1	4	4	4
153	17	51	0	0	0	0	0	6	6	5

The verification that all conditions are met is routine. For example, consider $v=105$. We require an STS(37) with a scattering set of size 6. The subsystem is a block containing two points in the scattering set. By Lemma 2.1, a (4, 4)-scattered latin square of order 34 exists. Finally, we must check the numerical conditions. Both conditions (1) and (2) are met (with equality). We leave the verification of the other cases as easy exercises.

4. Applying the Recursion

As we have remarked, Constructions 3.1 and 3.2 leave many parameters at our disposal. We have seen that when $w \in \{0, 1, 3\}$, we are able to select the d_i values as desired. However, there remains the problem of selecting parameters when $w=7$. To simplify matters, whenever we produce or employ an STS(v) with a sub-STS(7), we ensure that the scattering set produced has precisely two elements in the sub-STS(7); we call such a scattering set *proper*.

We produce and employ scattered STS having proper scattering sets whenever possible. While this simplifies the recursive construction, it forces us to settle some small cases directly:

LEMMA 4.1. *There is an* STS(v) *with a sub*-STS(7) *having a proper scattering set of cardinality n which spans d elements of the sub*-STS(7) *for* $(v, n, d) \in \{(15, 4, 2), (15, 3, 1), (21, 5, 3), (21, 5, 4), (21, 4, 1), (21, 4, 2), (27, 5, 2), (27, 6, 4), (33, 6, 3), (33, 6, 4), (33, 5, 1), (33, 5, 2), (39, 6, 3), (39, 6, 4)\}$. *In addition, an* STS($v$) *with a proper scattering set of size n exists for* $(v, n) \in \{(19, 4), (25, 5), (37, 6)\}$.

Proof. Each of the systems required is easily constructed; we outline their construction here. For the two systems with $v = 15$, apply Construction 3.1 with $s = 4$, $w = 3$, $e = 2$, and $a_1 = a_2 = a_3 = 1$. The resulting STS(15) has three sub-STS(7) (at least), and it is straightforward to omit one or two elements from the scattering set to obtain the desired scattering sets of size four or three.

For $v = 19$, applying Construction 3.2 in the usual manner produces a design with a sub-STS(7).

For $(v, n, d) = (21, 5, 3)$, apply Construction 3.1 with $s = 7$, $w = 0$, $a_1 = a_2 = 2$, $a_3 = 1$. Setting $a_1 = a_2 = 2$, $a_3 = 0$ yields $(v, n, d) = (21, 4, 1)$. Setting $a_1 = 2$, $a_2 = a_3 = 1$ yields $(v, n, d) = (21, 4, 2)$.

Similar applications of Construction 3.2 handle the cases when $v = 39$. To settle the remaining cases, we constructed examples by employing a hill-climbing algorithm which completes partial triple systems [7]. Since each case is straightforward to produce (even by hand), we omit them here. ∎

Now we apply the recursions to prove:

MAIN THEOREM. (I) *For every* $v \equiv 1, 3 \pmod 6$, *there is a scattered* STS(v). *Moreover, when* $v = 31$ *or* $v \geqslant 43$, *there is an* STS(v) *having a sub*-STS(7) *and a scattering set of maximum cardinality which is proper and spans at least four elements of the sub*-STS(7).

(II) *For every* $v \equiv 1, 3 \pmod 6$, *there is a spanned* STS(v). *Moreover, there is such an* STS *in which the spanning set of minimum cardinality is an independent set.*

Proof. We let $x = L(v)$ or $U(v)$ depending on whether we are to produce a scattered or spanned system. We consider congruence classes for $v \pmod{18}$ and for $x \pmod 3$. In each case, we identify a selection for the main parameters in Construction 3.1 or Construction 3.2. The cases and appropriate selections are detailed in Table 4.1. In this table, we have marked with an asterisk the cases in which $L(v) = U(v)$ is possible. We give an example to clarify the presentation of the table. Given an order v, say $v = 91$, we write $91 = 18t + 1$, whence $t = 5$. Now write $U(91) = 13 = 3y + 1$, whence $y = 4$. The table then prescribes the parameters for the construction as follows: $s = 30$, $w = 1$, $e = 1$, $a_1 = a_2 = a_3 = 4$. The table does not prescribe the d_i values; we introduce appropriate selections for the d_i's later.

First we prove statement (I) of the Main Theorem (the proof of statement (II) is essentially the same). In general, we apply Construction 3.2 using the values specified in Table 4.1. We must determine when the necessary conditions of Construction 3.2 are met.

The existence of the (a_1, a_3)-scattered latin square follows from Lemma 2.1, except in two cases: $v = 13$ and $v = 21$. This can be seen by

TABLE 4.1

Parameters for the Recursion

v	x	s	w	a_1	a_2	a_3	e	Tight?
$18t+1$	$3y$	$6t$	1	y	y	$y-1$	1	
$18t+1$	$3y+1$	$6t$	1	y	y	y	1	*
$18t+1$	$3y+2$	$6t$	1	$y+1$	y	y	1	
$18t+3$	$3y$	$6t+1$	0	y	y	y	0	*
$18t+3$	$3y+1$	$6t+1$	0	$y+1$	y	y	0	
$18t+3$	$3y+2$	$6t+1$	0	$y+1$	$y+1$	y	0	*
$18t+7$	$3y$	$6t+2$	1	y	y	$y-1$	1	
$18t+7$	$3y+1$	$6t+2$	1	y	y	y	1	
$18t+7$	$3y+2$	$6t+2$	1	$y+1$	y	y	1	
$18t+9$	$3y$	$6t+3$	0	y	y	y	0	*
$18t+9$	$3y+1$	$6t+3$	0	$y+1$	y	y	0	
$18t+9$	$3y+2$	$6t+3$	0	$y+1$	$y+1$	y	0	*
$18t+13$	$3y$	$6t+2$	7	y	$y-1$	$y-1$	2	
$18t+13$	$3y+1$	$6t+2$	7	y	y	$y-1$	2	
$18t+13$	$3y+2$	$6t+2$	7	y	y	y	2	
$18t+15$	$3y$	$6t+4$	3	y	y	y	0	*
$18t+15$	$3y+1$	$6t+4$	3	y	y	y	1	
$18t+15$	$3y+2$	$6t+4$	3	y	y	y	2	*

determining, for each case, the relation between t and y. In each case, s is approximately $6t$ while y is approximately $2\sqrt{t}$. We treat one case in detail, and leave the rest to the diligent reader. Suppose that $v=18t+3$ and $x=3y$. For the (y,y)-scattered latin square of order s to exist, Lemma 2.1 requires that $s \geqslant y^2+2y$. Since $v \geqslant \binom{x+1}{2}$,

$$36t+6 \geqslant (3y+1)(3y) = 9y^2+3y.$$

Hence we have that

$$y^2+y/3 \leqslant 4t+2/3.$$

But then the scattered latin square exists provided that $y \leqslant \frac{6}{5}t+\frac{1}{5}$, which holds provided $t>1$ (the cases for $t=2$ and 3 do not have $x \equiv 0 \pmod 3$). The remaining cases are quite similar. To handle the two exceptions, we proceed as follows. For $v=21$, the required $(2,2)$-scattered latin square of order 7 was given earlier. For $v=13$, it is easily verified that both STS(13) have scattering sets of size 4.

Now we choose the three STS. For $v \geqslant 43$, we must ensure that the STS produced has the appropriate proper scattering set (and hence the sub-STS(7) as well). For $b \geqslant 43$ and $v=31$, we have $s+w \geqslant 15$. Now if $s+w<43$, $s+w \neq 31$, Lemma 4.1 ensures the existence of the required

STS$(s+w)$. Otherwise, we may suppose by induction that an STS$(s+w)$ exists with a proper scattering set of maximum cardinality. Provided a_i is less than the size of the scattering set in this STS, we may omit one or more elements from the scattering set to obtain the desired value of d_i. Moreover, if any one of the STS has the required proper scattering set, the application of Construction 3.2 preserves it.

Finally we must check the inequalities. We always choose the d_i so that (1) holds. We must then check that, with suitable selection of the d_i, condition (2) can be met. We do not explicitly check all eighteen cases here, but rather check three in detail and leave the remainder as an exercise.

Consider $v=18t+1$, $x=3y+1$. Rewriting condition (2), it reduces to checking that

$$y(y+1)\leqslant 4t.$$

Since $v\geqslant\binom{x+1}{2}$, we obtain

$$y\leqslant \tfrac{1}{3}(\lfloor\tfrac{1}{2}(\sqrt{144t+9}-1)\rfloor-1),$$

and hence we have

$$y\leqslant \tfrac{1}{2}\sqrt{16t+1}-\tfrac{1}{2},$$

which satisfies the required inequality. Note that it actually satisfies it with equality; this is necessary since the case chosen is one of the possibly "tight" cases.

Next consider $v=18t+3$, $x=3y+2$. Condition (2) reduces to checking that

$$3y^2+5y\leqslant 12t.$$

Now we have

$$y\leqslant \tfrac{1}{6}\sqrt{144t+25}-\tfrac{5}{6}$$

by construction, and this satisfies the requirement with equality (note again that this in potentially a "tight" case).

Finally, consider $v=18t+13$, $x=3y$. Choosing $d_1=4$ and $d_2=d_3=1$ reduces condition (2) to verifying that

$$3y^2+y\leqslant 12t+6.$$

The verification of this is similar.

We finish the description of the cases for which the d_i values are not all zero, but omit the verification. For $v\equiv 1$, 3, 7, 9 (mod 18), all d_i are zero. In the remaining cases, we choose the d_i values as follows:

v(mod 18)	x(mod 3)	d_1	d_2	d_3
13	0	4	1	1
13	1	2	2	2
13	2	2	2	2
15	0	1	1	1
15	1	0	0	0
15	2	1	1	1

Finally, one must verify that applying Construction 3.2 using the base cases from Lemma 4.1 not only produces scattered STS with proper scattering sets, but also guarantees that at least four elements of the sub-STS(7) are spanned. The verification of this is routine.

Turning to statement (II), the proof is essentially the same. In fact, the only difference is that, in this case, the d_i must be chosen so as to ensure that all elements in the sub-STS(w) are spanned. Hence we use the following selections for the d_i.

v(mod 18)	x(mod 3)	d_1	d_2	d_3
13	0	4	2	2
13	1	3	3	2
13	2	3	3	3
15	0	1	1	1
15	1	1	1	1
15	2	1	1	1

The verification of the conditions for Construction 3.1 is similar to that done before. In this case, however, the constraints on the scattered latin square are more severe. Hence, Lemma 2.1 does not provide the required square for $v \in \{7, 9, 13, 21, 27, 39, 75, 81\}$. For $v = 7$ and $v = 9$, taking any four elements, no three on a block, yields the required spanning set. For $v = 13$, both STS(13) have spanning sets of size 5. For $v = 21$, we use the (2, 2)-scattered latin square displayed earlier. For $v = 27$, we apply Construction 3.1 with $s = 9$ and $w = 0$, but choose $a_1 = a_2 = 3$ and $a_3 = 1$. For $v = 39$, we apply Construction 3.1 with $s = 12$, $w = 3$, and $e = 2$. For $v = 75$, we apply Construction 3.1 with $s = 24$, $w = 3$ and $e = 2$. For $v = 81$, we apply Construction 3.1 with $s = 27$ and $w = 0$, but choose $a_1 = a_2 = 5$ and $a_3 = 3$.

The required scattered STS are produced in the proof of statement (I), and by the designs in Lemma 4.1, with the exception of $v = 31$. In this case, we apply Construction 3.1 with $s = 8$, $w = 7$, $e = 3$, and $a_1 = a_2 = a_3 = 1$ (the STS(15) required is easily constructed). Conditions (1) and (2) are met as before. Hence, with the exceptions noted above, we can apply Construction 3.1 with the parameters in Table 4.1 to produce a spanned STS(v). ∎

5. Concluding Remarks

Although the construction presented here is quite involved, it is important to note that the cases are handled in a fairly uniform manner. Moreover, there is typically much latitude in selecting the parameters for the recursion, even in the tight cases. Hence we suspect that scattered and spanned STS are by no means rare. In fact, we tested all eighty Steiner triple systems of order 15 by computer, and found that all but one of them is spanned and scattered. The lone exception is [5, #80]. We also tested Steiner triple systems of order 21 generated "at rondom" using a hill-climbing algorithm [7]. In one thousand trials, we found only two STS(21) which are not spanned; we present one of these systems here, on symbols $\{a\text{–}u\}$:

abs ach ade afp agn aiu ajk alr amq aot bcr bdg bej bfl bhi bkt
bmn bop bqu cdp ceg cfu cil cjm ckn cos cqt dfk dho diq djn
dlu dme dst efr ehl eit ekm ens eoq epu fgy fhq fin fjo fms ghj
gik glo gmu gpr gqs hks hmp hnu hrt ijr imo ips jls jpq jtu klp
kou kqr lmt lnq nor npt rsu

The Main Theorem suggests two interesting directions. In a geometric vein, it suggests the problem of determining the possible sizes of complete arcs. The Main Theorem establishes that the minimum value permitted by numerical conditions is always achievable. In an algebraic vein, it suggests the problem of determining sizes for 3-generating sets in Steiner quasigroups (or, indeed, k-generating sets). A 3-generating set of size x can "span" $x+\binom{x}{2}+x\cdot\binom{x-1}{2}$ elements. It is easy to verify that any three elements of the unique Steiner triple system of order 9 either form a triple or form a 3-generating set; at present, we know of no further "tight" examples for 3-generating sets (the next possible case is $x=5$, $v=45$). However, the techniques developed here appear to be useful in addressing this problem.

In closing, it is interesting to remark that, by modifying the hill-climbing algorithm from [7] to complete a partial triple system, examples of spanned Steiner triple systems for orders less than 100 were easily constructed by computer. We found that in the cases where $U(v) \neq L(v)$, the spanning set could be required to be independent (as in the Main Theorem), or to have at least one triple on it. These computational results support the belief that spanning sets of minimum cardinality are by no means rare.

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