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The Spectrum of Room Cubes

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A *Room cube* of side n is an n by n by n cube such that each 2-dimensional projection is a Room square. We show that there exists a Room cube of side n if and only if n is an odd positive integer other than 3 or 5.

1. INTRODUCTION

A *Room square* of side n is an n by n array of cells, whose entries are chosen from a set S of $n+1$ objects called symbols, which satisfies the following conditions:

(i) every cell of the array is either empty or contains an unordered pair of distinct symbols from S;
(ii) each symbol occurs in every row and in every column of the array;
(iii) every unordered pair of symbols occurs precisely once in the array.

If $t \geqslant 2$ is an integer, a Room t-cube of side n is a t-dimensional array, each cell of which is either empty or contains an unordered pair of two distinct elements chosen from a set of size $n+1$, such that each 2-dimensional projection is a Room square of side n. In this paper we consider Room 3-cubes which we refer to as *Room cubes*. Let $R_3 = \{n$: there exists a Room cube of side $n\}$. Room t-cubes can be equivalently described as certain types of Latin squares. Such a formulation has the advantage of being easier to visualize if $t \geqslant 3$.

A *Latin square* L of side n is an n by n array of cells, each of which contains exactly one symbol chosen from a set S of size n, such that each element of S occurs once in each row and once in each column of L. Suppose the rows and columns of L are indexed by the members of S. L is said to be *idempotent* if $L(s, s) = s$ for each $s \in S$. L is said to be symmetric if $L(s, s') = L(s', s)$ for every $\{s, s'\} \subseteq S$. Now impose any linear ordering on S. Two Latin squares L and M, on the symbol set S, are said to be *orthogonal symmetric Latin squares* provided L and M are both symmetric and idempotent, and for every $(s, s') \in S \times S$ there exists at most one cell (s_1, s_2) with $s_1 < s_2$ such that $(s, s') = (L(s_1, s_2), M(s_1, s_2))$. We refer to a set of t orthogonal symmetric Latin squares as t pairwise orthogonal symmetric Latin squares (POSLS) if each pair of squares is orthogonal symmetric.

Horton [9] establishes the following.

THEOREM 1.1. *If $t \geqslant 2$ is an integer then the following are equivalent*:
(i) *there exists a Room t-cube of side n,*
(ii) *there exist t POSLS of side n.*

LEMMA 1.2. *There exists a Room cube of side* 9.

PROOF. We exhibit three POSLS of order 9 in Figure 1 below.

REMARK. In [1], of the purported four POSLS of order 9, no three are orthogonal. Thus the three POSLS of Figure 1 yield the first known Room cube of side 9.

We will also make extensive use of pairwise orthogonal Latin squares (POLS) and orthogonal arrays (OAs). For definitions, see [6]. The following well known theorem indicates the connection between these structures.

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1	3	2	5	4	7	6	9	8
3	2	1	6	7	8	9	4	5
2	1	3	7	8	9	5	6	4
5	6	7	4	9	3	8	1	2
4	7	8	9	5	2	1	3	6
7	8	9	3	2	6	4	5	1
6	9	5	8	1	4	7	2	3
9	4	6	1	3	5	2	8	7
8	5	4	2	6	1	3	7	9

1	4	8	6	7	5	9	3	2
4	2	7	1	3	9	5	6	8
8	7	3	9	6	4	1	2	5
6	1	9	4	8	2	3	5	7
7	3	6	8	5	1	2	9	4
5	9	4	2	1	6	8	7	3
9	5	1	3	2	8	7	4	6
3	6	2	5	9	7	4	8	1
2	8	5	7	4	3	6	1	9

1	5	7	3	8	9	2	4	6
5	2	4	9	6	3	8	7	1
7	4	3	8	9	1	6	5	2
3	9	8	4	2	7	1	6	5
8	6	9	2	5	4	3	1	7
9	3	1	7	4	6	5	2	8
2	8	6	1	3	5	7	9	4
4	7	5	6	1	2	9	8	3
6	1	2	5	7	8	4	3	9

FIGURE 1. Three POSLS of order 9.

THEOREM 1.3. *The following are equivalent*:
(i) *there exists an* $OA(n, t)$
(ii) *there exist* $t-2$ *POLS of side* n.

Define $oa(t) = \{n$: there exists an $\mathrm{OA}(n, t)\}$ The following is a well known theorem of MacNeish [10].

THEOREM 1.4. *Let* n *have prime power factorization* $n = \Pi p_i^{\alpha_i}$. *Then* $n \in oa(t)$ *if* $t \leq \min\{p_i^{\alpha_i} + 1\}$.

The work of severval mathematicians resulted in the following theorem concerning Room squares. A condensed proof is given in [17].

THEOREM 1.5. *There exists a Room square of side n if and only if n is an odd positive integer other than* 3 *or* 5.

The following is immediate.

THEOREM 1.6. *If* n *is an even integer, or if* $n = 3$ *or* 5, *then there is no Room cube of side* n.

In order to determine the spectrum for Room cubes, we use a variety of techniques, most importantly, PBD closure.

For the definitions of pairwise balanced design (PBD), and PBD-closed set, see [11]. The following is shown in [5].

THEOREM 1.7. *Let* $t \geq 2$ *be a positive integer, and let* $R_t = \{n$: *there exists a Room t-cube of side* $n\}$. *Then* R_t *is PBD closed.*

We will determine the spectrum of Room cubes by constructing suitable PBD's. Our methods are very similar to those used in examining the spectrum of skew Room squares. Thus, we will often refer to the following four papers: [11, 13, 15, 16].

Briefly, we proceed as follows. In Section 2 constructions for Room t-cubes are described, and it is shown that if $n \geq 7$ is odd and if $n \notin R_3$, then $n = 3m$ where $(m, 15) = 1$. In Section 3 we construct Room cubes of all sides $8t + 1$, a necessary ingredient in our main PBD construction. In Section 4 we use PBD-closure to establish that $n \in R_3$ if $n \geq 10\,355$. Finally, in Section 5, Room cubes of sides $<10\,355$ are investigated.

2. SOME CONSTRUCTIONS FOR ROOM CUBES

An important tool for constructing Room cubes is the strong starter, which is now defined. Let G be an abelian group of odd order $2k+1$, written additively. A *starter* (of order $2k+1$) in G is a collection $A=\{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_k, y_k\}\}$ such that:

(i) $\bigcup_{i=1}^{k} \{x_i, y_i\} = G\backslash\{0\}$,

(ii) $\{\pm(y_i - x_i): 1 \leq i \leq k\} = G\backslash\{0\}$.

If, further, the sums $x_i + y_i$ are all distinct and non-zero, then A is said to be a *strong starter*. The following is shown in [9].

LEMMA 2.1. *If a strong starter exists in an abelian group of order* $2k+1$, *then* $2k+1 \in R_3$.

Two infinite families of strong starters exist by the following theorems. For proofs, see Mullin and Nemeth [14] and Chong and Chan [2], respectively.

LEMMA 2.2. *If* $q = 2^k t + 1$ *is an odd prime power with* $t > 1$ *an odd integer, then there is a strong starter of order* q.

LEMMA 2.3. *If* $q = 2^{2^n} = 1$ *is a Fermat prime greater than* 5, *then there exists a strong starter of order* q.

REMARK. It is not necessary that q be prime in the Chong and Chan construction. However, the theorem as stated is sufficient for our purposes.

The following is a simple exercise in elementary number theory.

LEMMA 2.4. *If* $q = 2^n + 1$ *is a prime power, then either* q *is a Fermat prime or* $q = 9$.

Now, using the above Lemmata 2.2, 2.3 and 2.4 we have that if q is a prime power not equal to 3, 5 or 9, then there exists a strong starter of order q, and hence $q \in R_3$. By Lemma 1.2, $9 \in R_3$. Therefore we summarize the above as the following theorem.

THEOREM 2.5. *If* q *is a prime power greater than* 5, *then* $q \in R_3$.

The following is our main recursive construction for Room cubes. It is a straightforward modification of Mullin's indirect product construction for skew Room squares [11]. We state it in terms of POSLS.

THEOREM 2.6. *Suppose there exist* t *POSLS of side* u, *and* t *POSLS of side* v *containing* t *sub-POSLS of side* w. *Further, suppose* $0 \leq a \leq w$, *and there exist* t *POLS of side* $v-a$ *containing* t *sub-POLS of side* $w-a$. *Finally, suppose there exist* t *POSLS of side* $u(w-a)+a$. *Then there exist* t *POSLS of side* $u(v-a)+a$, *containing* t *sub-POSLS of sides* u *and* $u(w-a)+a$.

We may now obtain Horton's construction [9] as a corollary.

THEOREM 2.7. *Suppose there exist* t *POSLS of side* u, *and* t *POSLS of side* v *containing* t *sub-POLS of side* w. *Further, suppose there exist* t *POLS of side* $v-w$. *Then there exist* t *POSLS of side* $u(v-w)+w$ *containing* t *sub-POSLS of sides* u, v *and* w.

PROOF. Let $a = w$ in Theorem 2.6. Then it suffices to check that the resulting squares contain sub-POSLS of order v. This is an easy verification.

In order to apply Theorems 2.6 and 2.9 with $t=3$ it is important to have three POLS of various sides.

LEMMA 2.8. *If $n \neq 2, 3, 6, 10$ or 14, then there exist three POLS of side n.*

PROOF. See [7, 19].

We can now derive a simple multiplication theorem.

THEOREM 2.9. *If $u, v \in R_3$, then $uv \in R_3$.*

PROOF. Apply Theorem 2.7 ($w=0$) and Lemma 2.8. Note that $3, 5 \notin R_3$.

The following quintuplication theorem is due to Horton [8].

LEMMA 2.10. *Suppose there exists a strong starter of order n, where $(n, 3)=1$. Then there exists a strong starter of order $5n$. Hence, $5n \in R_3$.*

Before proving the main theorem of this section, we record the existence of several of the strong starters which we will need in this paper.

LEMMA 2.11. *There exist strong starters of orders* 15, 21, 33, 35, 39, 45, 51, 57, 69, 87, 93, 111, 123, 129 *and* 321.

PROOF. See [3, 4 and 18].

THEOREM 2.12. *If $n \notin R_3$, n odd, and $n \geqslant 7$, then $n=3m$ where $(m, 15)=1$.*

PROOF. Write $n=3^\alpha 5^\beta m$, with $(m, 15)=1$. If $\alpha=\beta=0$ the result follows by Theorems 2.5 and 2.9.

If $\alpha=0$, $\beta=1$, then $n=5m$ with $(m, 15)=1$ and $m>1$ since $n \geqslant 7$. Let p be any prime divisor of m. Then there exists a strong starter of order p by Theorem 2.5, and so $5p \in R_3$ by Lemma 2.10. Now $m/p \in R_3$ by the above, so $5m \in R_3$ by Theorem 2.9.

Next, we have 15, 45 and $75 \in R_3$ by Lemma 2.11, and 9, 25, 27, $125 \in R_3$ by Theorem 2.5. Thus, if $\alpha+\beta=2$ or 3, then $n=m \cdot n/m$ with $m, n/m \in R_3$, so $n \in R_3$.

If $\alpha+\beta \geqslant 4$, then $n=st$ with $s=9$ or 25, and $t \in R_3$ by induction on $\alpha+\beta$. Hence $n \in R_3$.

This completes the proof, since the only possible exceptions have $\alpha=1$ and $\beta=0$.

3. CUBES OF SIDE $8t+1$

In this section we establish the existence of Room cubes of side $8t+1$. We use Theorem 2.7 in conjunction with some pairwise balanced designs.

LEMMA 3.1. *Suppose $2^\alpha+1 \in R_3$. If n is odd and $n \neq 5$, then $n \cdot 2^\alpha+1 \in R_3$.*

PROOF. If $n \in R_3$, then set $u=n$, $v=2^\alpha+1$ and $w=1$ in Theorem 2.7. Note that there exist three POLS of side 2^α if $\alpha>1$, and that $3 \notin R_3$.

If $n=3$ then $3 \cdot 2^\alpha+1 \in R_3$ by Theorem 2.12.

If $n \notin R_3$, $n \geqslant 7$, then $n=3m$ with $m \in R_3$ by Theorem 2.12.

Set $u = m$, $v = 3 . 2^{\alpha} + 1$, $w = 1$ in Theorem 2.7 to obtain $n \in R_3$. Note that there exist three POLS of side $3 . 2^{\alpha}$ unless $\alpha = 1$. However $3 \notin R_3$, so the hypothesis is not satisfied in this case.

We now define a *9-head* (see [13]) to be a PBD whose block sizes are 7, 9 or 17 (all members of R_3), which further contains a distinguished variety occurring only in blocks of size 9. A 9-head evidently contains $8r+1$ varieties where r is the number of blocks containing the distinguished variety. We refer to this r as the *generalized replication number* of the 9-head. Let GR denote the set of all generalized replication numbers of 9-heads. Since R_3 is PBD-closed and $\{7, 9, 17\} \subseteq R_3$, we have the following.

LEMMA 3.2. *If* $r \in GR$, *then* $8r+1 \in R_3$.

The following is established in [13].

LEMMA 3.3. *If* $k > 5$, $k \neq 7, 8, 9, 10, 11, 13$ *or* 15, *then* $2^k \in GR$.

LEMMA 3.4. *If* $\alpha \geq 3$, *then* $2^{\alpha} + 1 \in R_3$.

PROOF. If $\alpha \geq 19$ or $\alpha = 9, 15$ or 17, then Lemmata 3.2 and 3.3 imply $2^{\alpha} + 1 \in R_3$. If α is even then $(2^{\alpha} + 1, 3) = 1$ so $2^{\alpha} + 1 \in R_3$ by Theorem 2.12. If $\alpha = 5$ or 7 then Lemma 2.11 applies. If $\alpha = 11$, consider Theorem 2.7 and the fact that $2049 = 15(145-9)+9$ and $145 = 9(17-1)+1$. If $\alpha = 13$ then $8193 = 61(153-19)+19$ and $153 = 19(9-1)+1$.

Thus by application of Lemmata 3.1 and 3.4 we have the following theorem.

THEOREM 3.5. *If* $\alpha \geq 3$, *n is odd and* $n \neq 5$, *then* $n . 2^{\alpha} + 1 \in R_3$.

It now remains to construct squares of orders $5 . 2^{\alpha} + 1$. We need the following multiplication theorem for GR which was proven in [13].

LEMMA 3.6. *If* $\{r, s\} \subseteq GR$ *and there exists an* $OA(r, s)$, *then* $rs \in GR$.

LEMMA 3.7. *If* $2^k \in GR$ *and* $k \geq 4$, *then* $5 . 2^{k+1} \in GR$.

PROOF. Since there exists an affine plane of order 9, we have $10 \in GR$. Thus take $r = 10$, $s = 2^k$ in Lemma 3.6. The required $OA(10, 2^k)$ exists by Theorem 1.3.

LEMMA 3.8. *If* $k \geq 1$, $k \neq 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 14$ *or* 16, *then* $5 . 2^k \in GR$.

PROOF. As already noted, $10 \in GR$. Apply Lemmata 3.3 and 3.7 to obtain the result.

THEOREM 3.9. *If* $\alpha \geq 3$, $\alpha \neq 12$, *then* $5 . 2^{\alpha} + 1 \in R_3$.

PROOF. If $\alpha \geq 20$, or $\alpha = 10, 16$ or 18, then Lemmata 3.2 and 3.8 imply $5 . 2^{\alpha} + 1 \in R_3$. If α is odd then $(5 . 2^{\alpha} + 1, 3) = 1$, so $5 . 2^{\alpha} + 1 \in R_3$ by Theorem 2.12. To handle the cases $\alpha = 8$ and $\alpha = 14$ we note that $21 \in R_3$ (Lemma 2.11). If $\alpha = 8$, consider $1281 = 21 . 61$, and if $\alpha = 14$, consider $81\,921 = 21 . 3901$. For $\alpha = 6$, apply Lemma 2.11.

The theorem stated below follows immediately from Theorems 3.5 and 3.9.

THEOREM 3.10. *If* $n \equiv 1 \bmod 8$, $n \neq 20\,481$, *then* $n \in R_3$.

4. A PRELIMINARY BOUND

By constructing PBDs with suitable block sizes, we show that Room cubes exist for all odd orders exceeding 10 355. The following PBD construction is found in [13].

LEMMA 4.1. *Suppose* Y *is a PBD closed set, and* $m \in oa(10)$ *with* $0 \leq t \leq m$. *If* $\{7, 9, 6t+1, 6m+1, 8m+1\} \subseteq Y$ *then* $56m+6t+1 \in Y$.

THEOREM 4.2. *If* $n \equiv 1 \bmod 8$, *then* $n \in R_3$.

PROOF. In view of Theorem 3.10, only $n = 20\,481$ need be considered. However $20\,481 \in R_3$ since R_3 is PBD closed and $20\,481 = 56 \cdot 361 + 6 \cdot 44 + 1$. $361 \in oa(10)$ by Theorem 1.4 and the necessary Room cubes all exist.

Lemma 4.1 is adapted to Room cubes as follows.

LEMMA 4.3. *If* $0 \leq t \leq m$ *and* $m \in oa(10)$, *then* $56m+6t+1 \in R_3$.

PROOF. The result follows from Theorems 1.4 and 2.12 and Lemmata 4.1 and 4.2.

We are now ready to obtain a preliminary bound.

THEOREM 4.4. *If* $n \geq 10\,355$ *is odd, then* $n \in R_3$.

PROOF. We need only consider $n \equiv 3 \bmod 6$. Mullin *et al.* [14] establish that for $n \equiv 3 \bmod 6$, $n \geq 10\,355$, one can write $n = 56m+6t+1$ with $0 \leq t \leq m$ and $m \in oa(10)$. Thus Lemma 4.3 implies the result.

5. ROOM CUBES WITH SMALL SIDES

We use a variety of techniques in order to construct Room cubes with small sides ($<10\,355$). By Theorem 2.12, only orders $n \equiv 3 \bmod 6$ need be considered. The following is obtained from Lemma 4.3.

LEMMA 5.1. *If* $m \in oa(10)$, $m \equiv 1 \bmod 3$, $n \equiv 3 \bmod 6$, and $56m+1 \leq n \leq 62m+1$, *then* $n \in R_3$.

The next four lemmata are analogous to Lemmata 4.1 and 5.1 and will be used to show certain Room cubes exist. See [16] for proofs.

LEMMA 5.2. *Suppose* Y *is a PBD closed set, and* $m \in oa(18)$, $0 \leq t \leq m$. *If* $\{7, 17, 6t+1, 6m+1, 16m+1\} \subseteq Y$ *then* $112m+6t+1 \in Y$.

LEMMA 5.3. *If* $m \in oa(18)$, $n \equiv 3 \bmod 6$, $m \equiv 2 \bmod 3$, *and* $112m+1 \leq n \leq 118m+1$, *then* $n \in R_3$.

LEMMA 5.4. *Suppose* Y *is a PBD closed set,* $m \in oa(9)$ *and* $0 \leq t \leq m$. *If* $\{7, 9, 7m, m+6t\} \subseteq Y$ *then* $57m+6t \in Y$.

LEMMA 5.5. *If $m \in oa(9)$, $m \equiv 1$ or $5 \bmod 6$, $n \equiv 3 \bmod 6$ and $57m \leqslant n \leqslant 63m$, then $n \in R_3$.*

LEMMA 5.6. *If $n \equiv 3 \bmod 6$ and $6777 \leqslant n \leqslant 11\,161$ then $n \in R_3$.*

PROOF. Apply Lemma 5.1 with $m = 121$, 127, 139, 151, 157, 163, 169, 181. Each such value of m is in $oa(10)$ by Theorem 1.4. Also, for any two consecutive values $m_1 < m_2$ in the above list, $62m_1 + 1 \geqslant 56m_2 + 1$. Finally, $56 . 121 + 1 = 6777$ and $62 . 181 + 1 = 11\,161$.

Thus we have improved the original bound.

THEOREM 5.7. *If n is odd and $n \geqslant 6777$, then $n \in R_3$.*

It is convenient now to establish that some small values are in R_3.

LEMMA 5.8. *If $7 \leqslant n \leqslant 145$, n odd, then $n \in R_3$.*

PROOF. By Theorem 2.12 and Lemma 2.11, only 141 need be considered. But, by Theorem 2.7, $141 \in R_3$ since $141 = 7(21-1)+1$.

With the aid of Lemma 5.8 the following can be established.

LEMMA 5.9. *If n is odd but $n \notin R_3$ then $n = 3p$, where p is a prime and $53 \leqslant p \leqslant 2251$.*

PROOF. Suppose $n \notin R_3$. By Theorems 2.12 and 5.7, $n \leqslant 6771$ and $n = 3m$ where $(m, 15) = 1$. Suppose m is not a prime. Let p be the smallest prime divisor of m. Then $p \leqslant 47$, since $n \leqslant 6771$ and $3 . 53^2 > 6771$. So $3p \leqslant 141$, thus by Lemma 5.8, $3p \in R_3$. By Theorem 2.12, $m/p \in R_3$, hence by the multiplication Theorem 2.9, $n \in R_3$, a contradiction. Therefore, m is a prime and $n = 3m$.

LEMMA 5.10. *If $n \geqslant 513$, n odd, then $n \in R_3$ unless $n \in \{591, 597, 699, 717, 831, 843, 879, 1203, 1227, 1263, 2019\}$.*

PROOF. In Table 1 below, we list intervals covered by applications of Lemmata 5.1–5.5. The orthogonal arrays exist by Theorem 1.4 unless otherwise noted.

The only intervals not covered are 6765–6771, 2009–2071, 1199–1309, 821–895, 695–727 and 569–627. By Lemma 5.9, we need only consider orders $3p$ where p is a prime. These orders contained in the uncovered intervals are $n = 2049$, 2031, 2019, 1299, 1293, 1263, 1257, 1227, 1203, 879, 849, 843, 831, 723, 717, 699, 597, 591, 579, 573.

Room cubes of side $8t+1$ exist by Theorem 4.2, and the remaining cubes are obtained by applying Theorem 2.7: $2031 = 29(71-1)+1$, $1299 = 59(23-1)+1$, $1293 = 17(77-1)+1$, $723 = 19(39-1)+1$, $579 = 17(35-1)+1$ and $573 = 13(45-1)+1$.

Several of the possible exceptions over 500 can be eliminated by using the indirect product construction, Theorem 2.6. In order to apply this theorem we need POLS containing sub-POLS. These are obtained by use of the following theorems.

THEOREM 5.11. *Suppose there exist t POLS of orders u and v. Then there exist t POLS of order uv, containing t sub-POLS of orders u and v.*

PROOF. See [6].

TABLE 1

m	Lemma	Interval covered	
109	5.1	6105–6759	
103	5.1	5769–6387	
97	5.1	5433–6015	
89	5.5	5073–5607	
82	5.1	4593–5085	(1)
79	5.1	4425–4899	
73	5.1	4089–4527	
67	5.1	3753–4155	
61	5.1	3417–3783	
29	5.3	3249–3423	
53	5.5	3021–3339	
49	5.1	2745–3039	
47	5.5	2679–2961	
43	5.5	2451–2709	
43	5.1	2409–2667	
41	5.5	2337–2583	
37	5.5	2109–2339	
37	5.1	2073–2295	
17	5.3	1905–2007	
31	5.1	1737–1923	
29	5.5	1653–1827	
27	5.4	1539–1659	(2)
25	5.5	1425–1575	
23	5.5	1311–1449	
19	5.5	1083–1197	
19	5.1	1065–1179	
17	5.5	969–1071	
16	5.1	897–993	
13	5.5	741–819	
13	5.1	729–807	
11	5.5	627–693	
9	5.4	513–567	(3)

(1) $82 \in oa(10)$ [13].
(2) $189 \in R_3$ by Lemma 5.9. If $0 \leq t \leq 20$ then $27+6t \in R_3$ by Lemma 5.8.
(3) If $0 \leq t \leq 9$ then $9+6t \in R_3$ by Lemma 5.8.

THEOREM 5.12

(1) *Suppose that there exist t POLS of orders wu and $u+1$. Also, suppose that there exist $t+1$ POLS of order v with $0 \leq w \leq v$. Then there exist t POLS of order $uv+w$, containing t sub-POLS of order w.*

(2) *Suppose there exist t POLS of orders u, $u+1$, and $u+2$. Also, suppose there exist $t+2$ POLS of order v, and t POLS of orders w_1 and w_2 with $0 \leq w_1, w_2 \leq v$. Then there exist t POLS of order $uv+w_1+w_2$ containing t sub-POLS of orders w_1 and w_2.*

PROOF. Those are corollaries of Wilson's construction [20]. The existence of the sub-POLS is not stated there, but is easily verified.

LEMMA 5.13. *If $n \geq 513$, n odd, then $n \in R_3$ unless $n \in \{591, 831\}$.*

PROOF. In Table 2 below we apply Theorem 2.6 to all the exceptions listed in Theorem 5.10 except 591 and 831.

TABLE 2

u	v	w	a	$v-a$	$w-a$	$u(w-a)+a$	$u(v-a)+a$	Remarks
11	189	7	6	183	1	17	2019	$189=7.27$
7	183	7	3	180	4	31	1263	$183=7(27-1)+1$ $180=4.45$
11	117	13	6	111	7	83	1227	$117=9.13$ $111=8.13+7$
13	99	11	7	92	4	59	1203	$99=11.9$ $92=4.23$
11	89	11	10	79	1	21	879	$89=11(9-1)+1$
9	99	11	6	93	5	51	843	$99=9.11$ $93=8.11+5$
13	57	7	2	55	5	67	717	$57=7(9-1)+1$ $55=5.11$
7	105	7	6	99	1	13	699	$105=7.15$
11	57	7	3	54	4	47	597	$57=7(9-1)+1$ $54=7.7+4+1$

LEMMA 5.14. *If n is an odd positive integer, then $n \in R_3$ unless $n \in \{3, 5, 159, 213, 219, 237, 291, 303, 327, 411, 447, 453, 471, 591, 831\}$.*

PROOF. Except for $n=3$ and 5, we need only consider $n=3p$ with p a prime and $n \geqslant 153$. Also, if $n \geqslant 513$ Lemma 5.13 applies. We give constructions for Room cubes of sides not listed as possible exceptions in the statement of the lemma: $183=7(27-1)+1$, $267=7(39-1)+1$, $309=11(29-1)+1$, $339=13(27-1)+1$, $381=19(21-1)+1$ and $501=25(21-1)+1$. The remaining sides are all congruent to 1 mod 8.

It appears that the remaining possible exceptions can not be eliminated by any of the previous methods. However, by use of the computer, we have constructed strong starters of all these orders except, of course, 3 and 5 (see [3]).

LEMMA 5.15. *There exist strong starters of orders* 159, 213, 219, 237, 291, 303, 327, 411, 447, 453, 471, 591 *and* 831.

Thus we have our main result and an interesting corollary.

THEOREM 5.16. *There exists a Room cube of side n if and only if n is an odd positive integer other than* 3 *or* 5.

COROLLARY 5.17. *There exists a Room cube of side n if and only if there exists a Room square of side n.*

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