Thwarts in Transversal Designs

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Abstract

A subset of points in a transversal design is a *thwart* if each block in the design has one of a small number of intersection sizes with the subset. Applications to the construction of mutually orthogonal latin squares are given. One particular case involves inequalities for the minimum number of distinct symbols appearing in an $\alpha \times \beta$ subarray of a $n \times n$ latin square. Using thwarts, new transversal designs are determined for orders 408, 560, 600, 792, 856, 1046, 1059, 1368, 2164, 2328, 2424, 3288, 3448, 3960, 3992, 3994, 4025, 4056, 4824, 5496, 6264, 7768, 7800, 8096, and 9336.

1 Thwarts

A *transversal design* of *order* n and *blocksize* k, or $TD(k;n)$, is a triple $(X, \mathcal{G}, \mathcal{B})$, where X is a set of kn elements. $\mathcal{G} = \{G_1, \ldots, G_k\}$ is a partition of X into k sets each of size n; each class of the partition is a *group*. $\mathcal{B}$ is a set of k-subsets of X, with the property that each $B \in \mathcal{B}$ satisfies $|B \cap G_i| = 1$ for each $1 \leq i \leq k$; sets in $\mathcal{B}$ are *blocks*. Finally, each unordered pair of elements in X occurs together either in a group, or in a single block, but not both.

A *incomplete transversal design* of *order* n and *blocksize* k with *holes* of sizes $h_1, \ldots, h_\ell$, or $TD(k;n) - \sum_{i=1}^{\ell} TD(k;h_i)$, is a quadruple $(X, \mathcal{H}, \mathcal{G}, \mathcal{B})$. X and $\mathcal{G}$ are as before. $\mathcal{H} = \{H_1, \ldots, H_\ell\}$ is a set of pairwise disjoint subsets of X, with the property that $|H_j \cap G_i| = h_j$ for $1 \leq j \leq \ell$ and $1 \leq i \leq k$; each H_i is a *hole*. Then $\mathcal{B}$ is a set of k-subsets of X as before, with the property that every unordered pair of elements from X is either in a hole or group together, or in exactly one block of $\mathcal{B}$.

Naturally, if there exist $TD(k;h_i)$ for $1 \leq i \leq \ell$ and also a $TD(k;n) - \sum_{i=1}^{\ell} TD(k;h_i)$, then there exists a $TD(k;n)$. This observation is used heavily in the construction of transversal designs [4]. The main application is the use of incomplete transversal designs in conjunction with Wilson's theorem [6].

Theorem 1.1 (Wilson) *Assume there exists a $TD(k+x;n)$ with groups $G_1, \ldots, G_k$, $H_1, \ldots, H_x$ containing a subset S of $H_1, \ldots, H_x$ where $|S| = s$. Let $s_j = |S \cap H_j|$ for each $1 \leq j \leq x$, and assume also that for each block $B \in \mathcal{B}$ there exists a $TD(k; m + u_B) - u_B TD(k;1)$, where $u_B = |B \cap S|$, then there exists a*

$$TD(k; mn + s) - \sum_{j=1}^{x} TD(k; s_j).$$

If, in addition a $TD(k;s_j)$ exists for each $1 \leq j \leq x$, then a $TD(k; mn+s)$ exists.

We shall use a restricted form of Wilson's theorem. Let x be a nonnegative integer, and let $\mathcal{I} = \{i_1, \ldots, i_s\}$ with $0 \leq i_1 < i_2 < \cdots < i_s \leq x$. Further suppose that $0 \leq s_1 \leq s_2 \leq \cdots \leq s_x \leq n$. Let $(X, \mathcal{G}, \mathcal{B})$ be a $TD(k+x;n)$ with $\mathcal{G} = \{G_1, \ldots, G_k, H_1, \ldots, H_x\}$. Then an $(x, \mathcal{I}, s_1, s_2, \ldots, s_x)$ *-thwart* is a set $S = \bigcup_{j=1}^{x} S_j$, where $S_j \subseteq H_j$ with $|S_j| = s_j$ for each $1 \leq j \leq x$, such that for every $B \in \mathcal{B}$, $|B \cap S| \in \mathcal{I}$.

Using this definition of thwarts, a restricted form of Wilson's theorem is:

Theorem 1.2 *If a $TD(k+x;n)$ exists having an $(x, \mathcal{I}, s_1, s_2, \ldots, s_x)$-thwart, and if for every $i \in \mathcal{I}$ there exists a $TD(k; m+i) - iTD(k;1)$, then there exists a*

$$TD(k; mn + \sum_{j=1}^{x} s_j) - \sum_{j=1}^{x} TD(k; s_j).$$

If, in addition, a $TD(k;s_j)$ exists for each $1 \leq j \leq x$, then a $TD(k; mn + \sum_{j=1}^{x} s_j)$ exists.

We remark that a $TD(k; m+i)$ always has i disjoint blocks for $i \leq 3$ unless $k = m+i+1$ [2], and hence the hypothesis can be relaxed to require only a $TD(k; m+i)$ when $i \leq 3$. When $i > 3$, we can delete one group from a $TD(k+1; m+i)$ to obtain the i disjoint blocks. We also note that value of the parameter m in Theorem 1.2 often is referred to as the *weight* of the construction.

A more general form using thwarts can be obtained by employing weights other than one on the points of the thwart. Of course, Theorem 1.2 is only productive when one can find ingredients with sufficiently large blocksizes. When $\mathcal{I} = \{0, 1, \ldots, x\}$, Theorem 1.2 is just the usual application of Wilson's theorem obtained by truncating x levels, and is used extensively when $x \in \{1, 2\}$ [3]. Our goal is to find thwarts when $\mathcal{I}$ is a proper subset of $\{0, 1, \ldots, x\}$.

The most obvious case where thwarts exist is the case of a transversal design $TD(k; n)$ which contains a sub–transversal design $TD(k; m)$. We have the following lemma in this situation..

Lemma 1.3 *If there exists a $TD(k+x; n)$ containing a sub $TD(k+x; m)$, then there exists a $(x, \{0, 1, x\}, m, m, \ldots, m)$-thwart in the $TD(k+x; n)$.*

Proof: Any block of the transversal design contains either 0 or 1 point of the subdesign or is completely contained in the subdesign. □

Let us mention one well–known instance of such thwarts, corresponding to the presence of a sub–transversal design in the transversal design arising from the projective plane:

Lemma 1.4 *If p is prime, and α and β are nonnegative integers with $\beta \mid \alpha$, then there exists a $TD(p^\alpha + 1; p^\alpha)$ containing a $(p^\beta + 1, \{0, 1, p^\beta + 1\}, p^\beta, p^\beta, \ldots, p^\beta)$-thwart.*

This lemma appears to be of limited use in the manufacture of transversal designs, however.

Before treating some classes of thwarts, we establish a simple (but useful) preliminary result. Given a set $\mathcal{I}$, let $\overline{\mathcal{I}}_x = \{x - i : i \in \mathcal{I}\}$.

Lemma 1.5 *If a $TD(k + x; t)$ contains an $(x, \mathcal{I}, s_1, \ldots, s_x)$-thwart, it also contains an $(x, \overline{\mathcal{I}}_x, t - s_1, \ldots, t - s_x)$-thwart.*

Proof: Simply complement the sets $S_1, \ldots, S_x$ with respect to the respective groups containing them. □

2 Subsquares in latin squares

A *latin square* of side n is a $n \times n$ array, in which each entry contains a single symbol from a set S of size n; moreover, every symbol occurs precisely once in each row and in each column. Evidently a $TD(3; n)$ is equivalent to an $n \times n$ latin square. Moreover, truncating blocks of a $TD(k;n)$ by eliminating all elements in $k-3$ groups yields a $TD(3,n)$. Thus the presence of thwarts when $\mathcal{I} \subseteq \{0,1,2,3\}$ amounts to a question on latin squares arising from three levels of a transversal design.

Consider a thwart with $\mathcal{I} = \{0,1,3\}$ and $x = 3$. Examine the latin square corresponding to the three groups of the thwart. A thwart is then a selection of rows, columns and symbols of the latin square so that whenever two of row, column or symbol are among the chosen values, the third is also among them. Thus a thwart is simply a latin subsquare in this case.

To apply Theorem 1.2, it remains to find latin subsquares on three levels of a $TD(k+3;n)$, preferably with k "large". Suppose then that n is a prime power, and consider the $TD(n+1;n)$ arising from the finite field construction (see, for example, [2]). In the corresponding set of $n-1$ mutually orthogonal latin squares, one square L is defined by $L(i,j) = i+j$ in $GF(n)$ for $1 \leq i,j \leq n$. It follows that a $TD(p^\alpha+1;p^\alpha)$ contains three groups corresponding to a latin square with subsquares of order p^β for each $0 \leq \beta \leq \alpha$, and hence:

Lemma 2.1 *For $0 \leq k \leq p^\alpha - 2$ and $0 \leq \beta < \alpha$, there exists a $TD(k+3;p^\alpha)$ containing a $(3,\{0,1,3\},p^\beta,p^\beta,p^\beta)$-thwart.*

Corollary 2.2 *If there exists a $TD(k;m)$, $TD(k;m+1)$ and $TD(k;m+3)$ and $k \leq p^\beta + 1$, then there exists a $TD(k, mp^\alpha + 3p^\beta)$.*

In particular, taking $p = 2$, $\beta = 4$, $\alpha = 5$, $m = 16$ and $k = 17$, we find by this Corollary a $TD(17;560)$ (Brouwer [3] reports a $TD(9;560)$).

One can also consider the complementary thwart using Lemma 1.5, to obtain a $(3,\{0,2,3\},p^\alpha - p^\beta, p^\alpha - p^\beta, p^\alpha - p^\beta)$-thwart. Again taking $p = 2$ and $\beta = \alpha - 1$, we obtain a $TD(k;(2m+3)p^\beta)$ when $k \leq p^\beta + 1$ and $TD(k;m)$, $TD(k;m+2)$ and $TD(k;m+3)$ all exist. Taking, for example, $m = 125$, we have $TD(126;125)$, $TD(126;127)$ and $TD(126;128)$ since all are prime powers. Thus with $\beta = 5$ and $k = 33$ we obtain a $TD(33;8096)$.

We have seen that a $TD(p^\beta+1;p^\beta)$ is contained in a $TD(p^\alpha+1;p^\alpha)$ if and only if $\beta \mid \alpha$. However, this condition is not necessary for the containment of a $TD(k;p^\beta)$ with $k < p^\beta + 1$. In particular, we can find a $TD(6;25)$ in a $TD(126;125)$, and hence a $(6,\{0,1,6\},25,25,25,25,25,25)$–thwart. Applying Theorem 1.2 with weight 31 (using $\{31,32,37\}$), we produce a $TD(26;4025)$.

3 Subarrays of latin squares

In this section, we examine thwarts with $\mathcal{I} = \{0,1,2\}$ and $x = 3$. A $(3,\{0,1,2\},a,b,c)$-thwart in a $TD(k+3;n)$ also appears in a suitably truncated $TD(3;n)$, or latin square.

In fact, such a thwart arises from a set of a rows, b columns, and c symbols, with the property that none of the c symbols occurs in an array element that is both in a chosen row and in a chosen column. In other words, there is an $a \times b$ subarray containing at most $n-c$ symbols.

Let $\mathcal{L}_n$ be the set of all distinct latin squares of side n. An $\alpha \times \beta$ *subarray* of a latin square L is an array obtained by retaining the entries in a fixed set of α rows and β columns of L. Let $S_{\alpha,\beta}(L)$ be the set of all $\alpha \times \beta$ subarrays of L. For an array A, let $\sigma(A)$ denote the number of distinct symbols in the entries of A.

We consider the following question: Determine

$$\tau_n(\alpha, \beta) = \max_{L \in \mathcal{L}_n} \min_{A \in S_{\alpha,\beta}(L)} \sigma(A).$$

In other words, $\tau_n(\alpha,\beta)$ is the smallest number of distinct symbols so that one can guarantee that every $n \times n$ latin square contains an $\alpha \times \beta$ subarray on at most this many symbols.

Evidently $\tau_n(\alpha,\beta) = \tau_n(\beta,\alpha)$. Our interest in this function is the following:

Lemma 3.1 *If* $c \le n - \tau_n(a,b)$, *then* every $TD(k+3;n)$ *contains a* $(3,\{0,1,2\},a,b,c)$*-thwart, and its complementary* $(3,\{1,2,3\},n-a,n-b,n-c)$*-thwart.*

Wojtas [5] used the inequality $\tau_n(a,b) \le ab - \max(a,b) + 1$ in applying Theorem 1.2. It is natural to ask for improvements to the simple bound, both for latin squares in general, and for latin squares arising from specific transversal designs.

3.1 Some Inequalities

We start with some trivial computations:

Lemma 3.2

$$\begin{array}{rcll} \tau_n(0,\beta) & = & 0 & (0 \le \beta \le n) \\ \tau_n(1,\beta) & = & \beta & (1 \le \beta \le n) \\ \tau_n(2,\beta) & = & \beta+1 & (2 \le \beta \le n-1) \\ \tau_n(2,n) & = & n & \end{array}$$

Proof: For $\tau_n(2,\beta)$, proceed as follows. In an arbitrary $n \times n$ latin square $L = (\ell_{ij})$, choose two rows arbitrarily. Now choose a set of β columns, taking the first arbitrarily. In choosing the jth column, the first $j-1$ choices account for either j or $j-1$ symbols in the entries of the two fixed rows. If they contain only $j-1$, choose the jth column arbitrarily. Otherwise, one of the j symbols appears in a column not yet chosen, so include such a column. □

Next we consider an inequality from a simple greedy strategy:

Lemma 3.3 *If $\alpha + \beta \leq n$, then*

$$\tau_n(\alpha+1,\beta) \leq \tau_n(\alpha,\beta) + \left\lfloor \frac{\beta(n-\tau_n(\alpha,\beta))}{n-\alpha} \right\rfloor.$$

Proof: Consider an $\alpha \times \beta$ subarray with γ distinct symbols. Consider the occurrences of symbols in the chosen β columns and the remaining $n-\alpha$ rows. Of these $\beta(n-\alpha)$ entries, at least $\beta(\gamma-\alpha)$ are occupied by symbols already in the subarray. Thus we may choose a row to append that contains at least $\frac{\beta(\gamma-\alpha)}{n-\alpha}$ entries with symbols already seen; the remaining symbols in the new row are all distinct since the square is latin. Thus the chosen row could add as many as $\beta - \frac{\beta(\gamma-\alpha)}{n-\alpha}$ symbols to those already seen. Thus we have

$$\tau_n(\alpha+1,\beta) \leq \max\max_{(\alpha,\beta)\leq\gamma\leq\tau_n(\alpha,\beta)} \left\{ \gamma + \left\lfloor \frac{\beta(n-\gamma))}{n-\alpha} \right\rfloor \right\}.$$

(the number of additional symbols must be integer.) Since $\beta \leq n-\alpha$, this is maximized when $\gamma = \tau_n(\alpha,\beta)$, and the proof is complete. □

Finally, we examine the use of conjugates of the square:

Lemma 3.4 $\tau_n(\alpha, n-\tau_n(\alpha,\beta)) \leq n-\beta$.

Proof: Consider an $\alpha \times \beta$ subarray on $\tau(\alpha,\beta)$ symbols. Conjugate the square by interchanging the roles of columns and symbols. The conjugate of the chosen subarray is an $\alpha \times \tau(\alpha,\beta)$ subarray containing all occurrences of β symbols. Considering then the complementary set of columns yields the inequality. □

To illustrate the strength (weakness?) of these lemmas, we give below their consequences for $n = 17$. We present a 17×17 matrix whose (i, j) entry is an upper bound on $\tau_{17}(i, j)$:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	17
3	4	6	8	9	11	12	12	13	14	14	15	15	16	17	17	17
4	5	8	10	11	12	13	14	14	15	15	15	16	17	17	17	17
5	6	9	11	13	13	14	14	15	15	15	16	17	17	17	17	17
6	7	11	12	13	14	15	15	15	15	16	17	17	17	17	17	17
7	8	12	13	14	15	15	15	15	16	17	17	17	17	17	17	17
8	9	12	14	14	15	15	15	16	17	17	17	17	17	17	17	17
9	10	13	14	15	15	15	16	17	17	17	17	17	17	17	17	17
10	11	14	15	15	15	16	17	17	17	17	17	17	17	17	17	17
11	12	14	15	15	16	17	17	17	17	17	17	17	17	17	17	17
12	13	15	15	16	17	17	17	17	17	17	17	17	17	17	17	17
13	14	15	16	17	17	17	17	17	17	17	17	17	17	17	17	17
14	15	16	17	17	17	17	17	17	17	17	17	17	17	17	17	17
15	16	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17
16	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17
17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17

These bounds may be quite weak for $\alpha, \beta \geq 3$. In fact, determining $\tau_n(3, 3)$ already appears to be difficult. One can prove that $\tau_n(3, 3) \geq 5$ for infinitely many values of n (for example, in the cyclic squares of prime order at least 5). However, determining whether $\tau_n(3, 3) \leq 5$ in general is open.

3.2 Primes and primepowers

As with subsquares, in certain cases we need not treat arbitrary latin squares. Again we consider the addition latin square in the $TD(p^\alpha + 1; p^\alpha)$ to obtain:

Lemma 3.5 *For p a prime, $0 \leq k \leq p^\alpha - 2$, there is a $TD(k + 3; p^\alpha)$ containing a $(3, \{0, 1, 2\}, a, b, c)$-thwart and its complementary $(3, \{1, 2, 3\}, n-a, n-b, n-c)$-thwart, whenever $a + b + c \leq n + 1$.*

Proof: Consider the subarray formed by the first a rows and the first b columns of the cyclic latin square. This $a \times b$ subarray has at most $a + b - 1$ distinct symbols. Thus if these a rows and b columns are deleted, it suffices to retain $n - c \geq a + b - 1$ symbols in order to obtain a $(3, \{1, 2, 3\}, n - a, n - b, n - c)$-thwart. Thus $a + b + c \leq n + 1$ as required. $\square$

This lemma leads to new constructions for transversal designs. In Theorem 1.2 use $TD(k+3;n)$ with the following $(3,\{1,2,3\},a,b,c)$–thwarts to obtain $TD(k;mn+a+b+c)$.

k	n	m	$mn+a+b+c$	a	b	c
11	27	78	2164	16	17	25
12	19	208	3992	11	13	16
12	19	208	3994	13	13	16

Abel [1] recently found a $TD(11;80)$ that is employed in the first case here. This implies in addition the existence of $TD(10;x)$ for each $x \in \{79,80,81,82,83\}$. Thus one can consider truncations of more than three levels.

A $(3,\{1,2,3\},a,b,c)$-thwart in a $TD(k;n)$ ensures the presence of a $(4,\{1,2,3,4\},a,b,c,d)$-thwart for all $0 \leq d \leq n$. Using this simple observation, we obtain again by use of Theorem 1.2 two additional new results for existence of transversal designs:

k	n	m	$mn+a+b+c+d$	a	b	c	d
10	13	78	1046	9	9	13	1
10	13	79	1059	9	9	13	1

One can exploit further properties of the squares to obtain better bounds. For example, in the cyclic squares of primepower (but not prime) order, subsquares appear for each divisor of the order. Naturally we obtain subarrays with fewer contained symbols from such subsquares.

4 An affine subspace

Brouwer [3] observes that the desarguesian projective plane $PG(2,q)$ contains an affine plane $AG(2,3)$ whenever $q \equiv 0,1 \pmod 3$ and q is a primepower. Thus we have:

Lemma 4.1 *For $n \equiv 0,1 \pmod 3$ a primepower, there exists a $TD(n+1;n)$ containing a $(4,\{0,1,3\},2,2,2,2)$-thwart.*

Proof: Form the TD by removing any point of the $AG(2,3)$. □

We find more use for the complementary thwart, a $(4,\{1,3,4\},n-2,n-2,n-2,n-2)$ thwart. In particular, when n and $n-2$ are both prime powers, we can apply weight 28 (using $TD(30;x)$ for $x \in \{29,31,32\}$) or 124 (using $TD(126;x)$ for $x \in \{125,127,128\}$). The following new transversal designs $TD(k;t)$ can then be produced by Theorem 1.2:

t	k	m	n
408	10	28	13
600	16	28	19
792	22	28	25
856	24	28	27
1368	30	28	43
2328	30	28	73
2424	16	124	19
3288	30	28	103
3448	24	124	27
3960	28	124	31
4056	30	28	127
4824	30	28	151
5496	40	124	43
6264	46	124	49
7768	30	28	243
7800	58	124	61
9336	70	124	73

One can also exploit the presence of other affine subdesigns, although general results ensuring their presence are not known. For example, the desarguesian plane $PG(2,49)$ contains a subplane $PG(2,7)$ and hence also an $AG(2,7)$. Thus there is a $TD(50;49)$ containing a $(8,\{0,1,7\},6,6,\ldots,6)$–thwart and its complementary $(8,\{1,7,8\},43,43,\ldots,43)$–thwart.

General results ensuring the presence of other affine subdesigns, or indeed any thwarts corresponding to group divisible designs, appear to be of value in the construction of new transversal designs.

5 Conclusion

In this paper we have introduced the notion of thwarts in transversal designs. These are interesting subconfigurations of the transversal design that are particularly useful for finding new transversal designs via a restricted version of Wilson's Theorem.

Let $N(n)$ denote the maximum number of mutually orthogonal latin squares of order n. It is well–known that $N(n) \geq k$ if and only if there exists a transversal design $TD(k+2;n)$. In the table below we summarize our new results on transversal designs in terms of $N(n)$. The old bounds are all from [3]. We also list the section of this paper where the new bound is obtained.

n	$N(n) \geq$	old $N(n)$	Section	n	$N(n) \geq$	old $N(n)$	Section
408	8	7	4	560	15	7	2
600	14	10	4	792	20	16	4
856	22	18	4	1046	8	7	3.2
1059	8	7	3.2	1368	28	7	4
2164	9	7	3.2	2328	28	7	4
2424	14	10	4	3288	28	7	4
3448	22	18	4	3960	26	22	4
3992	10	8	3.2	3994	10	8	3.2
4025	24	15	2	4056	28	8	4
4824	28	10	4	5496	38	8	4
6264	44	40	4	7768	28	8	4
7800	56	52	4	8096	31	30	2
9336	68	11	4				

References

[1] R.J.R. Abel, private communications, 1993.

[2] T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, Cambridge University Press, Cambridge, 1986.

[3] A.E. Brouwer, "The number of mutually orthogonal latin squares", ZW 123, Math. Centre Amsterdam, 1979.

[4] A.E. Brouwer and G.H.J. van Rees, "More mutually orthogonal latin squares", *Discrete Math.* 39 (1982), 263–281.

[5] M. Wojtas, "A note on mutually orthogonal latin squares", Komunikat nr. 236, Instytut Matematyki Politechniki Wroclawskiej, 1978.

[6] R.M. Wilson, "Concerning the number of mutually orthogonal latin squares", *Discrete Math.* 9 (1974), 181–198.