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COMBINATORIA

VOLUME THIRTY-TWO

DECEMBER, 1991

WINNIPEG, CANADA

Trains: An Invariant for One-Factorizations

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1. Introduction

We assume the standard ideas of graph theory. A *one-factor* in a graph G is a set of edges of G which together contain each vertex precisely once. A *one-factorization* is a set of disjoint one-factors whose union is the original graph.

In order to have a one-factor it is necessary that a graph have an even number of vertices, but this is not sufficient. There have been many papers written on the existence of one-factors; Tutte's famous paper [14] presents a necessary and sufficient condition. The problem of whether or not a given graph has a one-factorization is even more difficult. However, it is easy to see that the complete graph K_{2n} has a one-factorization for every positive integer n. For an excellent survey on one-factorizations of the complete graph, we refer the reader to [11].

The standard proof that K_{2n} has a one-factorization goes as follows. First, take the vertices of K_{2n} as $\{\infty, 0, 1, 2, \ldots, 2n-2\}$ where the elements are the integers modulo $2n-1$ except that ∞ is a new element satisfying the law "$\infty + x = \infty$". Then select a particular one-factor F_0 with edges:

$$(\infty, 0), (1, -1), (2, -2), \ldots, (n-1, n).$$

(This factor is easiest to understand geometrically, using the picture in Figure 1.) Finally, factor F_i is constructed from F_0 by the rule "add i to each vertex". F_i has edges

$$(\infty, i), (1+i, -1+i), (2+i, -2+i), \ldots, (n-1+i, n+i).$$

(Again the geometric picture is very simple: rotate the diagram of Figure 1 clockwise through i positions.) It is easy to check that $\{F_0, F_1, F_2, \ldots, F_{2n-2}\}$ constitute a one-factorization of K_{2n}. This particular factorization is called the *patterned factorization*, and denoted GK_{2n}.

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[1] This author's research was supported in part by the National Security Agency under grant No. MDA904-89-H-2048. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation hereon.

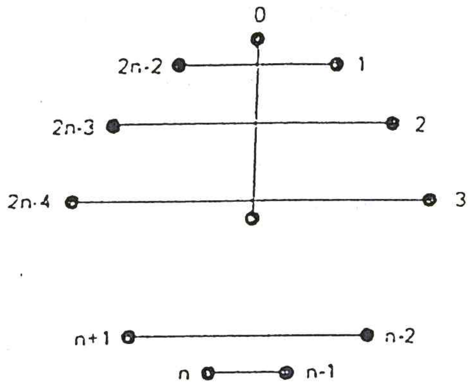

Figure 1

Another one-factorization of K_{2n} is frequently used. The definition varies, depending on whether n is even or odd. In either case the factorization is denoted GA_{2n} [2].

To construct GA_{2n}, one first partitions the vertex-set of K_{2n} into two sets of size n: for convenience, say they are $\{0,1,2,\ldots,n-1\}$ and $\{\overline{0},\overline{1},\ldots\overline{n-1}\}$. For $i = 0,1,\ldots,n-1$, let H_i denote the one-factor with edges $(0,\overline{i}),(1,\overline{1+i})$, $\ldots,(n-1,\overline{n-1+i})$ (where all arithmetic is performed modulo n). Then observe that the n one-factors $H_0, H_1,\ldots, H_{n-1}$, precisely cover all the edges joining the two sets of vertices.

Suppose n is even. Let $\{F_0, F_1,\ldots, F_{n-2}\}$ be the factors of GK_n, on the symbol set $\{0,\ldots,n-1\}$ and $\{F_0^+, F_1^+,\ldots, F_{n-2}^+\}$ be the same factors with each symbol x replaced by the symbol $\overline{x}$. (The symbols $n-1$ and $\overline{n-1}$ can by treated as the ∞ symbols.) Then

$$(F_0 \cup F_0^+),(F_1 \cup F_1^+),\ldots,(F_{n-2} \cup F_{n-2}^+), H_0, H_1,\ldots, H_{n-1}$$

constitute GA_{2n}. For odd n, GA_{2n} is defined similarly. (See Section 6.)

A one-factorization of K_{2n} is called *perfect* if the union of any pair of factors is always a Hamiltonian cycle in K_{2n}. We know that GK_{2n} is perfect when $2n-1$ is a prime, and GA_{2n} is perfect when n is prime. These are the only known infinite families of perfect one-factorizations, although other "sporadic" examples have been found (see [6]). A motivation for our discussion of this new invariant is that a commonly used invariant (using cycle structure) is totally ineffective in discerning nonismorphic perfect one-factorizations.

In this paper we will discuss an invariant of one-factorizations of K_n called the train. In Section 2 we describe this invariant and give some examples. In Section 3 we prove a theorem concerning the length of this invariant. Section 4 introduces another class of one-factorizations as motivation for the use of trains. In Sections 5 and 6 we explicitly compute the trains for GK_{2n} and GA_{2n} (n odd) respectively.

2. Trains

We are interested in describing one-factorizations of K_{2n}, and in distinguishing between nonisomorphic one-factorizations. Initially this was achieved by looking at cycle-structure, in the following sense. Each one-factorization consists of $2n-1$ factors. If these are paired, one constructs $(2n-1)(n-1)$ regular graphs of degree 2. In the case of K_8, one can distinguish between the isomorphism-classes of one-factorizations simply by counting how many of these graphs consist of two 4-cycles. In [17], two factors are called a *pair* if their union is not connected; three factors form a *division* if the union of all three is not connected. The pair and division structure is very useful in computing the automorphism groups of factorizations, and is also used in [17] to calculate automorphism groups of Room squares of side 7.

The cycle structure is not so useful in K_{10}, because of the large number of nonisomorphic one-factorizations (396 of them). Gelling [8] counts the cycles through a vertex. In each of the 36 graphs obtained by taking a union of two factors, a given vertex lies in a 4-cycle, a 6-cycle or a 10-cycle. One can count how many times this occurs for each vertex. Such a count is more useful, but still far from ideal. In particular, cycle structure cannot possibly distinguish between nonisomorphic perfect factorizations.

Kotzig [10] introduced another way of distinquishing between one-factorizations. With each one-factoriza- tion $\mathcal{F}$ of K_{2n} associate $2n$ idempotent quasigroups $Q(\mathcal{F}, v)$, one for each vertex v of K_{2n}. The multiplication in $Q(\mathcal{F}, v)$ is defined as follows: $aa = a$, and if $a \neq b$ then $ab = c$ where (a, c) and (b, v) lie in the same one-factor of $\mathcal{F}$. Then $Q(\mathcal{F}, v)$ induces a set of cycles $(a_1, a_2, a_3, \ldots)$ by the iterative process $a_1 a_2 = a_3,\ a_2 a_3 = a_4, \ldots,\ a_i a_{i+1} = a_{i+2}$.

For a given v, the set of cycles partitions the edges of the K_{2n-1} obtained by deleting v. The lengths of these cycles form an invariant.

Anderson [1] used the Kotzig invariants to study one-factorizations arising from starters in cyclic groups (that is, one-factorizations derived by rotating a diagram in the same way that GK_{2n} is derived by rotating Figure 1). In particular, he applied them to some small perfect one-factorizations.

Another invariant was used by Gross [9]. This invariant was easy to compute, however was only defined for one-factorizations arising from starters in Abelian groups. It also sometimes distinquishes between perfect one-factorizations.

In this paper we will discuss an invariant of one-factorizations called trains. These were first introduced by White [18], and later used by Colbourn, Colbourn and Rosenbaum [4] and by Stinson [13], in discussing Steiner triple systems. Trains of one-factorizations were first discussed by Dinitz [5].

Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_{2n-1}\}$ is a one-factorization of K_{2n}. The *train* of $\mathcal{F}$ is a directed graph whose vertices are the $n(2n-1)^2$ triples $\{x, y, F\}$, where x and y are (an unordered pair of) vertices and F is a factor in $\mathcal{F}$ (i.e. $F = F_i$ for some i). There is exactly one edge leaving each vertex; the edge from $\{x, y, F\}$ goes to $\{z, t, G\}$ where:

(x, z) is an edge in F;
(y, t) is an edge in F;
(x, y) is an edge in G.

We will sometimes think of this edge as a self-mapping ϕ on the vertices of the train where $\phi(x, y, F) = (z, t, G)$. It is obvious that isomorphic one-factorizations have isomorphic trains.

Following [13] we simplify trains by considering only the indegree sequence of the train. That is, with a one-factorization F we associate the sequence $t_0, t_1, t_2, \ldots$ where t_i equals the number of vertices in the train of F which have i edges directed into them. The sequence is normally written so as to terminate with the last nonzero element.

The sequence of indegrees allows us to separate many factorizations. In particular, Dinitz [5] uses it to prove that two perfect factorizations of K_{12} are nonisomorphic– the two sequences were $(330, 176, 165, 0, 55)$ and $(110, 506, 110)$– and to separate three nonisomorphic perfect factorizations of K_{20} and five of K_{24}.

In the Appendix we list the indegree sequence for the train of each one-factorization of K_{2n} for $2n = 8$ and 10. Note that these trains form a complete invariant for $2n = 8$ and almost a complete invariant when $2n = 10$. (Only the trains of one-factorizations numbered 16 and 26 have the same indegree sequence.)

3. The Maximum Length of a Train

One might think that the length of (the indegree sequence of) a train could be extremely long. There are $n(2n-1)^2$ vertices in the train of a one-factorization of K_{2n}, so an arbitrary digraph with all outdegrees 1 could have a vertex with indegree as large as $n(2n-1)^2$. However, the maximum indegree is much smaller.

Theorem 1. *The train of any one-factorization of* K_{2n} *has maximum indegree* $2n-1$.

Proof: Given a vertex $\{z, t, G\}$, assume there are edges directed into it from both $\{x, y, F\}$ and $\{u, v, F\}$. Then z is joined to one of $\{x, y\}$ and to one of $\{u, v\}$ by an edge of F, and so is t. So $\{x, y, F\} = \{u, v, F\}$ and thus there can be at most one edge into $\{z, t, G\}$ for each factor F in the one-factorization. Therefore, each vertex in the train has maximum indegree $2n-1$.

In the case of a perfect one-factorization we can say more.

Theorem 2. *The train of a perfect one-factorization of* K_{2n} *has maximum indegree* n, *for* $n > 2$.

Proof: Suppose there is an edge $\{x, y, F\} \to \{z, t, G\}$ in the train of a perfect one-factorization of K_{2n}. Then xy is an edge of G, and either $\{xz, yt\}$ or $\{xt, yz\}$ belong to F — say the former case. If there were an edge from $\{x, y, F_1\}$ to $\{z, t, G\}$ where $F_1 \neq F$, then xt and yz must be edges of F_1. Then $F \cup F_1$ contains a 4-cycle, which is impossible when $n > 2$. So there is at most one edge into $\{z, t, G\}$ for each edge xy of G.

The maximum indegree $2n - 1$ can be realized for nearly all orders $2n$. To prove this we need a definition. Suppose $R = \{1, 2, \dots, r\}$ and $S_1, S_2, \dots, S_k$ are sets which form a partition of R. By an *incomplete Latin square* of side r with holes $S_1, S_2, \dots, S_k$ we mean an $r \times r$ array with the following properties:

(i) if $\{i, j\} \subseteq S_k$ for some k, then the (i, j) cell is empty; otherwise it contains an element of R;

(ii) if $i \in S_k$, then both row i and column i contain every element of $R \backslash S_k$ precisely once.

Suppose $A = (a_{ij})$ is an incomplete Latin square of side $2n - 2$ with holes $\{1, 2\}, \{3, 4\}, \dots \{2n-3, 2n-2\}$, and further suppose A is symmetric. Define

$$F_0 = \{\infty 0,\ 12,\ 34, \dots, (2n-3)(2n-2)\}$$

and when $i > 0$

$$F_{2i-1} = \{\infty(2i-1),\ 0(2i)\} \cup \{xy : a_{xy} = 2i - 1\},$$
$$F_{2i} = \{\infty(2i),\ 0(2i-1)\} \cup \{xy : a_{xy} = 2i\}.$$

Then $\{F_0, F_1, \dots F_{2n-2}\}$ is a one-factorization of K_{2n} and $\{\infty, 0, F_0\}$ has indegree $2n-1$ in the train. Fu [7] has proven the existence of a symmetric incomplete Latin square of side $2n-2$ with $n-1$ holes of size 2 whenever $2n-2 \neq 4$, so we have

Theorem 3. *There is a one-factorization of K_{2n} whose train has a vertex of indegree $2n - 1$ whenever $2n \neq 6$.*

The unique one-factorization of K_6 has no vertex of indegree 5 in its train, so the situation is completely determined.

Also note that in K_{10}, the trains of one-factorizations numbered 1 through 5 all have a vertex with indegree 9. (See Appendix)

4. Staircase Factorizations

A practical example of the use of trains occurred in the case of staircase factorizations. Following the appearance of [15], F. Bileski [3] outlined a new way to construct a one-factorization of K_{2n}. His technique is as follows. (See also [16]).

First, write down a diagram of cells, with $2n-1$ rows and columns. If this were a square array, only the cells on or above the back diagonal would be included.

Rows are labeled $2n, 2n-1, \ldots, 2$ and columns $1, 2, \ldots, 2n-1$. (This can be done with a single labeling, as shown in Figure 2: the label encompasses the column above it and the row to its left – there is, for example, no *row* labelled 1 because label 1 is below all rows.)

Now construct n paths of cells. Path 1 is vertical, pointing north, on the west side. Path 2 starts from the extreme east, moves one square west, then southwest as far as possible without meeting Path 1, then south for one cell.

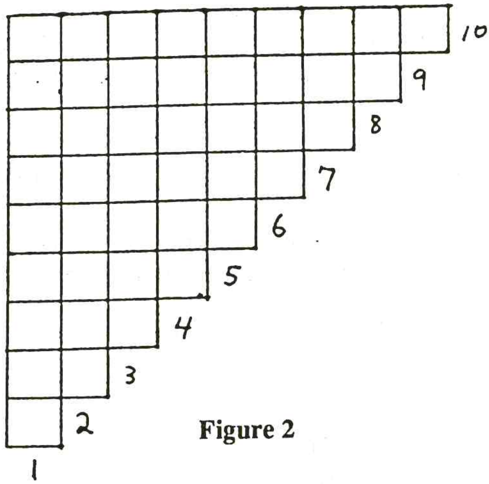

Figure 2

Every subsequent path meets the following description. Path i starts in the cell diagonally adjacent to the end of path $i-1$. Proceed northwest until it is impossible to proceed further (one would either cross Path 1 or escape from the diagram). It turns $45°$ and proceeds one step (it will either be possible to proceed north or to proceed west or north, but not both, so this instruction is unambiguous). Then turn a further $45°$ and go as far as possible. Turn a further $45°$ and move one cell. Turn a final $45°$ and go as far as possible.

These instructions sound complicated but are easy to follow. Path 1 is special. After that, odd-numbered paths go

NW as far as possible
N one step
NE as far as possible
E one step
SE as far as possible.

Even-numbered paths have the same description, except that the order of directions is NW, W, SW, S, SE. If we numbered the back-diagonal cells from 1(SW) to $2n-1$(NE), then if $i \geq 2$

for i odd, path i goes <u>from</u> cell i <u>to</u> cell $2n+1-i$
for i even, path i goes <u>to</u> cell i <u>from</u> cell $2n+1-i$.

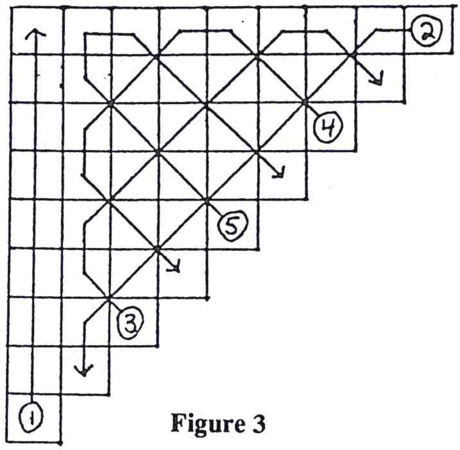

Figure 3

The labeling and the construction of paths is illustrated in Figure 3 for the case $2n = 10$. Paths are shown as lines joining the centers of the relevant cells.

To form a one-factorization, identify each cell of the staircase diagram with the unordered pair given by its coordinates. The first factor consists of the n pairs which are the first elements of the n paths, the second elements give the second factor, and so on.

It is naturally of interest to determine whether or not this factorization is different from the two main classes discussed earlier. Initially we could not decide this, so some experimental work was done. It was found that the trains of the staircased factorization and the patterned factorization are the same for all $2n \leq 36$. Using information from the calculation we constructed the following proof.

Theorem 4. *The staircase factorization is always isomorphic to the patterned factorization.*

Proof: For convenience we decreased the row and column numbers by 1. Then reverse the order of the odd labels, as shown in Figure 4. One can interpret the labels as $0, -0, 2, -2, \ldots (\text{mod } 2n - 1)$ if preferred.

The starting point of a typical path (other than the first) is in row k, column $-k$. As one moves along the path, one gets cells and in general after $2h - 1$ steps comes cell $(2h - k, k + 2h)$; after $2h$ it is $(k - 2h, -k - 2h)$. So the sum of the two coordinates of the cell after $2h - 1$ steps is $4h$; the sum after $2h$ steps is $-4h$.

So the one-factor consisting of entry i from each path is the one-factor consisting of all the pairs with the same sum ($2i + 2$ if i is odd, $-2i$ if i is even). The contribution from path 1 is consistent ($i+1$, $-i$ respectively). These are the factors of GK_{2n}. So the staircased factorization is isomorphic to GK_{2n}.

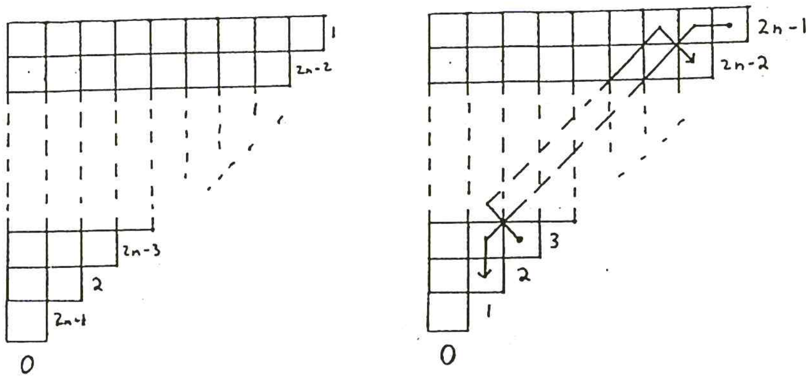

Figure 4

5. The Train of GK_{2n}

In this section we will determine the indegree sequence of the train of the patterned one-factorization GK_{2n} on the graph K_{2n}. This one-factorization was described in the introduction, but we will define it again below in an equivalent (but different looking) manner. The patterned one-factorization on K_{2n} may be defined by

$$GK_{2n} = \{F_1, F_2, \ldots, F_{2n-1}\}, \text{ where}$$
$$F_i = \{\infty, i\} \cup \{\{x, y\} : x + y = 2i\}.$$

(All arithmetic is modulo $2n-1$.) We will use the following notation: let $I = \{1, 2, \ldots, 2n-1\}$, $J = I \cup \{\infty\}$, and $I_x = I \backslash \{x\}$. Also let $S = \{(x, y, f) : x, y, f \in I, y > x\}$, $S' = \{(\infty, y, f) : y, f \in I\}$, and $V = S \cup S'$. We will let f denote the 1-factor F_f, where $f \in I$.

The train of GK_{2n} is a digraph whose vertices are the elements of V and whose outdegrees are all 1. The arcs can therefore be interpreted as the diagram of a mapping $V \to V$. The map is $\phi_1 \cup \phi_2 \cup \phi_3 \cup \phi_4$ where:

$$\phi_1 : (\infty, x, x) \to (\infty, x, x) \quad x \in I,$$
$$\phi_2 : (\infty, x, f) \to (f, 2f - x, x) \quad f \in I, x \in I_f,$$
$$\phi_3 : (x, y, x) \to \left(\infty, 2x - y, \frac{x+y}{2}\right) \quad x \in I, y \in I_x, \text{ and}$$
$$\phi_4 : (x, y, f) \to \left(2f - x, 2f - y, \frac{x+y}{2}\right) \quad f \in I, x \in I_f, y \in I_f.$$

Observe that ϕ_4 is a $1-1$ map from S onto itself. So the image of ϕ_4 on its domain is $S \backslash T$, where $T = \{\phi_4(x, y, f) : x \text{ or } y = f, 1 \le x < y \le 2n-1\}$.

Two easy alternative descriptions of T are

$$T = \{(f,\, 2f - y,\, \frac{f+y}{2}) : f \in I,\, y \in I_f\}$$
$$= \{(3f - 2g,\, f,\, g) : g \in I,\, f \in I_g\}.$$

In what follows we use the words "image of ϕ_i" to mean the collection of all $\phi_i(x,\, y,\, f)$ which are defined, with multiple elements counted multiply. We obtain the following images for the mappings ϕ_1, ϕ_2 and ϕ_3.

$$\text{Image of } \phi_1 = P = \{(\infty,\, x,\, x) : x \in I\}.$$
$$\text{Image of } \phi_2 = Q = \{(2f - x,\, f,\, x) : f \in I,\, x \in I_f\}$$
$$= \{(2f - x,\, f,\, x) : x \in I,\, f \in I_x\}.$$
$$\text{Image of } \phi_3 = R = \{\left(\infty,\, 2x - y,\, \frac{x+y}{2}\right) : x \in I,\, y \in I_x\}$$
$$= \{(\infty,\, 4g - 3y,\, g) : g \in I,\, y \in I_g\}.$$

The train has image $Z = P + Q + R + S - T$. If we let $P_x = \{$elements of P with last entry $x\}$. Then clearly $Z_x = P_x + Q_x + R_x + S_x - T_x$.

If $3 \nmid 2n-1$, then for x fixed $\{4x - 3y : y \neq x\} = I \backslash \{x\}$. So $P_x + R_x = \{(\infty, y, x) : y \in I\}$ and thus every member of S' appears exactly once in $P + R$. If $2n - 1 = 3t$,then $(\infty, 4x - 3y, x) = (\infty, 4x - 3z, x)$ whenever $3y - 3z$ (i.e. $y = z \pm t$). In that case $P_x \cup R_x$ contains t different elements three times each. So $P + R$ contains $t(2n-1) = \frac{1}{3}(2n-1)^2$ elements, each of frequency 3.

Next consider Q_x. Does it contain repetitions? Now $\phi_2(\infty, x, f) = \phi_2(\infty, x, g)$ implies that either $(2f - x, f, x) = (2g - x, g, x)$ (which implies $g = f$) or $(2f - x, f, x) = (g, 2g - x, x)$ for some $f \neq g$. This means that $g = 2f - x$ and $f = 2g - x$, so $f = 4f - 3x$. This occurs if and only if $3f = 3x$. Since $f \neq x$, the necessary and sufficient condition is that 3 divides $2n-1$. If $2n - 1 = 3t$, then $f = x + t$ or $x + 2t$. So to summarize, we have that if $3 \nmid 2n-1$, then Q_x contains $2n - 2$ distinct elements. But if $3 | 2n - 1$, say $2n - 1 = 3t$, then Q_x contains one element twice $((x + t, x + 2t, x))$ and $2n - 4$ elements once each.

Clearly T_x contains no repetitions. But can Q_x and T_x have common elements? There are two possibilities, either $(2f - x, f, x) = (3f - 2x, f, x)$ or $(2f - x, f, x) = (g, 3g - 2x, x)$. The first of these is clearly impossible. For the second one we get $g = 2f - x$ and $f = 3g - 2x$ so $f = 6f - 5x$. Equality is impossible when $(5, 2n-1) = 1$; if $2n - 1 = 5u$, then the entries of Q_x with $f = x + iu, i = 1, 2, 3, 4$, all occur in T_x. So there is a 4-element overlap: $(x + 2iu, x + iu, x)$ for $i = 1, 2, 3, 4$. We can see then that $Q_x \backslash T_x$ has $2n - 6$ elements.

Can this overlap be among the repeats in the case $3 | 2n-1$ (i.e. $2n-1 = 15v$)? Then $u = 3v$ and $t = 5v$. The overlap is $(x + 3iv,\, x + 6iv,\, x)$: $i = 1, 2, 3, 4$, while the repeats are $(x + 5v,\, x + 10v,\, x)$. So they are distinct.

Now we can write out the indegree sequence of the train of GK_{2n}. There are four cases:

(i) $(2n-1, 15) = 1$.
Every element of V has indegree 1, except for the elements of T (indegree 0) and those of R (indegree 2). Since $|T| = |R| = (2n-1)(2n-2)$, the sequence is

$$(2n-1)(2n-2),\ (2n-1)(2n^2-5n+4),\ (2n-1)(2n-2).$$

(ii) $(2n-1, 15) = 3$.
S' contains $\frac{1}{3}(2n-1)^2$ elements of indegree 3 and $\frac{2}{3}(2n-1)^2$ elements of indegree 0. Also Q contains $2n-1$ elements twice and $(2n-1)(2n-4)$ elements once each. As none of these are elements of T, they give rise to vertices of indegree 3 and 2 respectively. So the sequence is

$$\begin{aligned}
&(\frac{2}{3}(2n-1)^2 + (2n-1)(2n-2), *, (2n-1)(2n-4), \frac{1}{3}(2n-1)^2 \\
&\quad + 2n-1) \\
&= (\frac{2}{3}(2n-1)(5n-4), (2n-1)(2n^2-7n+6), (2n-1)(2n-4), \\
&\quad \frac{2}{3}(2n-1)(n+1))
\end{aligned}$$

(iii) $(2n-1, 15) = 5$.
We know that $Q \cap T$ has $4(2n-1)$ elements. On comparing with case (i), this means that the sequence is

$$\begin{aligned}
&((2n-1)(2n-2) - 4(2n-1), *, (2n-1)(2n-2) - 4(2n-1)) \\
&= ((2n-1)(2n-6), (2n-1)(2n^2-5n+12), (2n-1)(2n-6))
\end{aligned}$$

(iv) $(2n-1, 15) = 15$.
Similarly to the above, we modify the solution (ii), obtaining

$$\begin{aligned}
&(\frac{2}{3}(2n-1)(5n-4) - 4(2n-1), *, (2n-1)(2n-4) - 4(2n-1), \\
&\quad \frac{2}{3}(2n-1)(n+1)) \\
&= (\frac{10}{3}(2n-1)(n-2), (2n-1)(2n^2-7n+14), (2n-1)(2n-8), \\
&\quad \frac{2}{3}(2n-1)(n+2)).
\end{aligned}$$

As examples of the above trains we get:
The train of GK_4 is $12,0,0,6$.
The train of GK_6 is $0,75$.
The train of GK_8 is $12,0,0,6$.
The train of GK_{10} is $0,75$.
The train of GK_{12} is $12,0,0,6$.
The train of GK_{14} is $0,75$.
The train of GK_{16} is $12,0,0,6$.
The train of GK_{22} is $0,75$.
The train of GK_{36} is $0,75$.

6. The Train of GA_{2n}, n Odd and $n \geq 5$

In this section we will determine the train for the 1-factorization GA_{2n} of K_{2n}, with n odd, $n \geq 5$. As discussed in the introduction, the vertices of K_{2n} are $0, 1, \ldots, n-1, \overline{0}, \overline{1}, \ldots, \overline{n-1}$

When n is odd, we construct GK_{n+1} on symbols $\{\infty, 0, 1, \ldots, n-1\}$ and call the factors $F_0, F_1, \ldots, F_{n-1}$. Also construct GK_{n+1} on the symbols $\{\infty, \overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ and call the factors $F_0^+, F_1^+, \ldots, F_{n-1}^+$. Now $F_i \cup F_i^+$ is not a one-factor, because ∞ appears twice. But, if we delete edges (∞, i) and $(\infty, \overline{i})$ and add $(i, \overline{i})$ we obtain a one-factor; call it K_i. In general, we will denote this new one-factor K_f simply by f where $f \in \{0, 1, \ldots, n-1\}$.

We define a further set of one-factors by $g_i = \{(x, \overline{x+i}) | x \in Z_n\}$ for $1 \leq i \leq n-1$. Then $K_0, K_1, \ldots, K_{n-1}, g_1, g_2, \ldots, g_{n-1}$ is a one-factorization of K_{2n} called GA_{2n}.

From the definition above it is easy to compute the images of all vertices in the train. (Note again that each vertex in the train is a triple containing two vertices in K_{2n} and a one-factor in GA_{2n}). The following is a list of **all** possible cases of mappings depending on the structure of the triple in the domain. By convention we assume $\{x, y, f\} \in \{0, 1, \ldots, n-1\}, x \neq y, f \neq x, f \neq y$.

$$\phi_1 : (x, y, f) \rightarrow \left(2f - x, 2f - y, \frac{1}{2}(x + y)\right) \quad (x < y)$$

$$\phi_2 : (x, y, x) \rightarrow \left(2x - y, \overline{x}, \frac{1}{2}(x + y)\right)$$

$$\phi_3 : (\overline{x}, \overline{y}, f) \rightarrow \left(\overline{2f - x}, \overline{2f - y}, \frac{1}{2}(x + y)\right) \quad (x < y)$$

$$\phi_4 : (\overline{x}, \overline{y}, x) \rightarrow \left(x, \overline{2x - y}, \frac{1}{2}(x + y)\right)$$

$$\phi_5 : (x, y, g_i) \rightarrow \left(\overline{x+i}, \overline{y+i}\frac{1}{2}(x+y)\right) \quad (x < y)$$

$$\phi_6 : (\overline{x}, \overline{y}, g_i) \rightarrow \left(x - i, y - i, \frac{1}{2}(x+y)\right) \quad (x < y)$$

$$\phi_7 : (x, \overline{x}, f) \rightarrow (2f - x, \overline{2f - x}, x)$$

$$\phi_8 : (x, \overline{x}, x) \rightarrow (x, \overline{x}, x)$$

$$\phi_9 : (x, \overline{x}, g_i) \rightarrow (x - i, \overline{x + i}, x)$$

$$\phi_{10} : (x, \overline{y}, f) \rightarrow (2f - x, \overline{2f - y}, g_{y-x})$$

$$\phi_{11} : (x, \overline{y}, x) \rightarrow (\overline{x}, 2f - y\frac{1}{2}(x+y))$$

$$\phi_{12} : (x, \overline{y}, y) \rightarrow (2f - x, \overline{2x - y}, g_{y-x})$$

$$\phi_{13} : (x, \overline{y}, g_i) \rightarrow (y - i, \overline{x + i}, g_{y-x})$$

Let P_i = image of Q_i. There are six types of vertices in the train and we partition the P_i accordingly:

Type 1 vertices are of the form (x, y, f) and arise from the sets P_1 and P_6;

Type 2 vertices are of the form $(x, \overline{y}, f)$ and arise from the sets P_2, P_4, P_7, P_8, P_9;

Type 3 vertices are of the form $(\overline{x}, \overline{y}, f)$ and arise from the sets P_3 and P_5;

Type 4 vertices are of the form $(x, \overline{y}, g_i)$ and arise from the sets P_{10} and P_{13};

Type 5 vertices are of the form $(\overline{x}, \overline{y}, g_i)$ and arise from the set P_{11};

Type 6 vertices are of the form (x, y, g_i) and arise from the set P_{12}.

To compute the indegrees of all **Type 1** vertices, we explicitly compute P_1 and P_6. We get

$$P_1 = \left\{(a, b, h) \,|\, a \neq h, h \neq \frac{3a - b}{2}, h \neq \frac{3b - a}{2}\right\}$$

$$P_6 = \left\{(a, b, h) \,|\, a \neq b \text{ and } h \neq \frac{a + b}{2}\right\}.$$

So there are three types of Type 1 vertices.

1) : $\left\{(a,b,h)|h \neq \frac{3a-b}{2}, h \neq \frac{3b-a}{2}, h \neq \frac{a+b}{2}\right\}$.

These will have indegree 2.

2) : $\left\{(a,b,h)|h = \frac{3a-b}{2}, \text{ or } h = \frac{3b-a}{2}, \text{ and } h = \frac{a+b}{2}\right\}$.

These will have indegree 1.

3) : $\left\{(a,b,h)|h \neq \frac{3a-b}{2} \text{and } h \neq \frac{3b-a}{2}, \text{ and } h = \frac{a+b}{2}\right\}$.

These will have indegree 1.

The indegrees were deduced by noting that both ϕ_1 and ϕ_6 are $1-1$. Also, since it cannot be that $\frac{1}{2}(3a-b) = \frac{1}{2}(a+b)$, there is no fourth case.

Now by counting, we find $\binom{n}{2}(n-3)$ vertices with indegree 2 and $\binom{n}{2}\cdot 2 + \binom{n}{2}\cdot 1$ vertices with indegree 1.

For **Type 2** we again note that $\phi_2, \phi_4, \phi_7, \phi_8$ and ϕ_9 are all $1-1$. Now computing the P_i we see that

$$\begin{aligned} P_2 &= \left\{\{a,\bar{b},\frac{3b-a}{2}\}|a \neq b\right\}, \\ P_4 &= \left\{\{a,\bar{b},\frac{3a-b}{2}\}|a \neq b\right\}, \\ P_7 &= \{\{a,\bar{a},h\}|a \neq h\}, \\ P_8 &= \{\{a,\bar{a},a\}\}, \quad \text{and} \\ P_9 &= \left\{\{a,\bar{b},\frac{a+b}{2}\}|a \neq b\right\}. \end{aligned}$$

If $\frac{1}{2}(3b-a) = \frac{1}{2}(3a-b)$, then $a=b$, a contradiction. So $P_2 \cap P_4 = \phi$. Also, if $\frac{1}{2}(3b-a) = \frac{1}{2}(a+b)$, then again, $a=b$. So, $P_2 \cap P_9 = \phi$ and $P_4 \cap P_9 = \phi$. Clearly P_7 and P_8 are disjoint from each other and from P_2, P_4, and P_9.

So these are all distinct sets and thus every element in $P_2 \cup P_4 \cup P_7 \cup P_8 \cup P_9$ has indegree 1. Now $|P_2| + |P_4| + |P_7| + |P_8| + |P_9| = n(n-1) + n(n-1) + n(n-1) + n + n(n-1) = 4n^2 - 3n$ more vertices with indegree 1.

Type 3 is analogous to Type 1. We get $\binom{n}{2}(n-3)$ additional vertices with indegree 2 and $\binom{n}{2}\cdot 3$ additional vertices with indegree 1.

to do **Type 4** we must first note that ϕ_{10} is *not* $1-1$. Since $\phi_{10}(x,\overline{y},f) = 2f-x, 2f-y, g_{y-x})$, then for any k, $0 \leq k \leq n-1$,

$$\phi_{10}\left(x+k, \overline{y+k}, f+\frac{k}{2}\right) = (2f-x)\overline{2f-y}, g_{y-x}).$$

So ϕ_{10} appears to be n to 1. However, if $x+k=f+\frac{k}{2}$ or if $y+k=f+\frac{k}{2}$, then we are in cases corresponding to mappings ϕ_{11} or ϕ_{12}, respectively. Thus, if $k=2(f-x)$ or $k=2(f-y)$, then we are not in this case and so these images must be different. Except for these two possibilities, all $(x+k,\overline{y+k},f+\frac{k}{2})$ have the same image under ϕ_{10}. Therefore ϕ_{10} is $n-2$ to 1.

Now, to finish Type 4, from the above discussion we have that every vertex in $P_{10}=\{(a,\overline{b},g_i)|a\neq b, i=a-b\}$ has indegree at least $n-2$. Note that if $P_{13}=\{(a,\overline{b},g_i)|a\neq b, i\neq a-b\}$, then necessarily $i\neq a-b$ and so $P_{13}\cap P_{10}=\phi$. Thus in P_{10} there are $n(n-1)$ vertices, each with indegree $n-2$. Since ϕ_{13} is $1-1$, there are also $n(n-1)(n-1)$ vertices in P_{13} which each have indegree 1.

The **Type 5** vertices are just the image under ϕ_{11}. Since $P_{11}=\{\{\overline{a},\overline{b},g_{a-b}\}|a\,/\!= b\}$, then there are $n(n-1)$ additional vertices of degree 1.

Type 6 is analogous to Type 5, so there are $n(n-1)$ more vertices of degree 1.

The total is

- $\binom{n}{2}3+(4n^2-3n)+\binom{n}{2}3+n(n-1)(n-1)+n(n-1)+n(n-1)$ $=n(n^2+7n-7)$ vertices of indegree 1
- $\binom{n}{2}(n-3)+\binom{n}{2}(n-3)=n(n-1)(n-3)$ vertices of indegree 2
- $n(n-1)$ vertices of indegree $n-2$.

By subtracting from the total number of vertices, which is $\binom{2n}{2}(n-1)$, we find that there are $2n(n-1)(n-3)$ vertices of indegree 0.

So the vector for GA_{2n} is

$$2n(n-1)(n-3),n(n^2+2n-7),n(n-1)(n-3),\overbrace{0,0,\ldots,0}^{n-5},n(n-1).$$

As examples of the above trains we get:

The trains of GA_{10} is 80, 265, 40, 20.

The trains of GA_{14} is 336, 637, 168, 0, 0, 42.

The trains of GA_{18} is 864, 1233, 432, 0, 0, 0, 0, 72.

The trains of GA_{22} is 1760, 2101, 880, 0, 0, 0, 0, 0, 0, 110.

The trains of GA_{26} is 3120, 3289, 1560, 0, 0, 0, 0, 0, 0, 0, 0, 156.

The trains of GA_{30} is 5040, 4845, 2520, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 210.

From the results of this and the previus section we can see that for n odd, $n\geq 5$ that GA_{2n} is not isomorphic to GK_{2n} (since their trains are different). In fact it is interesting to note that while the train of GK_{2n} has length at most 4, the train of GA_{2n} has length $n-1$.

Acknowledgment

The author wish to thank the referee for the careful reading of this fairly technical paper and for making several useful suggestions.

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Appendix

Listed below are the indegree sequences of the trains of the 6 nonisomorphic one-factorizations of K_8. These are ordered according to Wallis et.al.[17] with the one-factorization number followed by the indegree sequence of the train of that one-factorization.

1	168	0	0	0	0	0	0	28
2	144	0	16	8	8	16	0	4
3	112	16	36	24	4	4	0	0
4	108	48	0	12	28	0	0	0
5	72	64	48	12	0	0	0	0
6	42	112	42	0	0	0	0	0

Listed below are the indegree sequences of the trains of the 396 nonisomorphic one-factorizations of K_{10}. These are ordered according to Gelling [8] with the one-factorization number followed by the indegree sequence of the train of that one-factorization.

1	144	216	0	36	0	0	0	0	0	9
2	148	180	42	26	0	0	2	6	0	1
3	157	156	57	26	0	3	2	3	0	1
4	160	156	45	29	9	3	2	0	0	1
5	152	168	48	20	8	8	0	0	0	1
6	152	152	73	17	4	3	3	1	0	0
7	153	156	63	18	12	0	0	3	0	0
8	178	110	78	24	8	7	0	0	0	0
9	162	134	78	18	8	3	0	2	0	0
10	183	98	85	25	8	6	0	0	0	0
11	135	198	36	27	0	0	9	0	0	0
12	180	90	108	18	0	9	0	0	0	0
13	192	84	84	30	12	3	0	0	0	0
14	165	141	55	31	7	3	3	0	0	0
15	161	144	60	25	12	1	1	1	0	0
16	156	150	63	22	9	4	0	1	0	0
17	158	152	56	21	14	3	0	1	0	0
18	169	122	77	27	4	4	2	0	0	0
19	156	144	73	20	7	4	0	1	0	0
20	149	156	70	19	5	5	0	1	0	0
21	172	108	92	24	4	5	0	0	0	0
22	136	173	64	24	8	0	0	0	0	0
23	152	150	72	19	7	4	1	0	0	0
24	160	144	58	33	5	4	1	0	0	0
25	141	160	81	15	5	1	1	1	0	0
26	156	150	63	22	9	4	0	1	0	0
27	167	131	65	28	11	2	1	0	0	0
28	163	137	66	24	12	2	1	0	0	0

29	159	139	70	26	9	1	0	1	0	0
30	179	111	69	31	12	3	0	0	0	0
31	166	136	62	23	16	0	2	0	0	0
32	160	139	70	22	11	2	1	0	0	0
33	133	175	70	20	5	2	0	0	0	0
34	170	119	79	24	9	4	0	0	0	0
35	153	156	58	25	9	2	2	0	0	0
36	146	165	60	20	12	0	2	0	0	0
37	173	114	78	29	8	2	1	0	0	0
38	155	144	75	20	6	3	2	0	0	0
39	164	137	62	29	10	1	2	0	0	0
40	163	130	78	22	7	5	0	0	0	0
41	156	147	63	29	5	5	0	0	0	0
42	158	143	69	23	7	3	2	0	0	0
43	143	163	67	22	9	0	1	0	0	0
44	138	176	64	18	1	6	1	1	0	0
45	152	153	67	15	17	1	0	0	0	0
46	155	151	61	24	11	2	1	0	0	0
47	163	134	68	29	8	2	1	0	0	0
48	153	155	56	29	11	0	0	1	0	0
49	128	187	60	24	4	2	0	0	0	0
50	167	131	65	27	12	3	0	0	0	0
51	147	160	63	26	5	3	1	0	0	0
52	162	128	82	22	8	3	0	0	0	0
53	160	135	72	30	4	4	0	0	0	0
54	153	142	80	21	6	2	1	0	0	0
55	164	127	76	28	8	2	0	0	0	0
56	172	117	72	34	8	2	0	0	0	0

57	178	103	84	26	14	0	0	0	0	0
58	155	134	85	25	4	2	0	0	0	0
59	164	134	64	32	8	3	0	0	0	0
60	148	151	76	23	3	3	1	0	0	0
61	149	158	65	22	6	3	2	0	0	0
62	145	160	68	22	8	1	1	0	0	0
63	157	150	58	26	10	3	1	0	0	0
64	153	152	63	25	8	4	0	0	0	0
65	142	164	66	25	7	0	1	0	0	0
66	158	145	63	26	10	2	1	0	0	0
67	150	158	62	21	11	2	1	0	0	0
68	152	156	58	27	8	4	0	0	0	0
69	157	148	57	33	7	2	1	0	0	0
70	155	150	59	31	7	2	1	0	0	0
71	155	147	70	19	10	3	1	0	0	0
72	161	136	69	28	8	3	0	0	0	0
73	165	126	75	29	8	2	0	0	0	0
74	158	142	68	23	12	2	0	0	0	0
75	162	136	65	30	11	1	0	0	0	0
76	178	109	72	33	12	1	0	0	0	0
77	159	138	71	24	12	1	0	0	0	0
78	162	131	71	33	7	1	0	0	0	0
79	149	150	72	26	7	1	0	0	0	0
80	151	155	61	27	8	3	0	0	0	0
81	155	147	63	30	8	2	0	0	0	0
82	162	128	77	30	7	1	0	0	0	0
83	161	133	73	27	10	1	0	0	0	0
84	175	105	86	29	9	1	0	0	0	0
85	166	121	83	23	11	1	0	0	0	0
86	168	122	77	25	11	2	0	0	0	0
87	148	147	81	22	5	2	0	0	0	0
88	149	157	61	29	7	1	1	0	0	0
89	162	133	73	26	9	0	2	0	0	0
90	165	120	90	19	9	0	2	0	0	0
91	148	152	77	17	7	4	0	0	0	0
92	151	148	68	34	1	3	0	0	0	0
93	147	162	61	23	8	4	0	0	0	0
94	154	155	54	30	8	4	0	0	0	0
95	162	128	78	30	4	3	0	0	0	0
96	149	159	58	29	8	1	1	0	0	0
97	152	156	58	28	6	5	0	0	0	0
98	144	166	60	25	8	1	0	1	0	0
99	140	166	70	21	5	2	1	0	0	0
100	150	157	58	31	8	0	0	1	0	0
101	145	173	44	30	11	2	0	0	0	0
102	158	139	74	23	7	3	1	0	0	0
103	158	140	70	27	7	2	1	0	0	0
104	156	144	66	31	5	2	1	0	0	0
105	149	147	85	9	14	1	0	0	0	0

106	162	124	90	20	4	5	0	0	0	0
107	155	146	66	28	7	3	0	0	0	0
108	155	142	73	25	8	2	0	0	0	0
109	165	120	90	19	8	2	1	0	0	0
110	162	126	85	22	8	1	1	0	0	0
111	151	146	71	31	6	0	0	0	0	0
112	156	140	73	25	11	0	0	0	0	0
113	152	150	67	23	13	0	0	0	0	0
114	152	140	81	25	7	0	0	0	0	0
115	149	151	74	21	8	1	1	0	0	0
116	162	131	75	25	11	1	0	0	0	0
117	161	137	69	25	10	3	0	0	0	0
118	143	155	79	21	6	1	0	0	0	0
119	162	127	81	25	9	1	0	0	0	0
120	159	137	69	31	8	1	0	0	0	0
121	136	168	74	22	2	3	0	0	0	0
122	150	150	74	20	8	3	0	0	0	0
123	158	144	63	27	11	2	0	0	0	0
124	147	151	79	17	10	1	0	0	0	0
125	146	164	56	30	6	3	0	0	0	0
126	160	141	61	33	7	3	0	0	0	0
127	164	131	66	34	10	0	0	0	0	0
128	140	161	78	18	6	2	0	0	0	0
129	154	137	85	19	9	1	0	0	0	0
130	143	161	71	19	10	1	0	0	0	0
131	141	162	71	23	8	0	0	0	0	0
132	165	132	60	42	3	3	0	0	0	0
133	153	146	74	19	12	0	1	0	0	0
134	153	146	71	26	7	1	1	0	0	0
135	163	122	85	29	4	2	0	0	0	0
136	155	138	77	29	4	2	0	0	0	0
137	151	149	71	23	10	1	0	0	0	0
138	155	132	85	29	4	0	0	0	0	0
139	161	123	90	22	9	0	0	0	0	0
140	151	142	82	22	7	1	0	0	0	0
141	154	146	65	31	9	0	0	0	0	0
142	165	127	70	34	9	0	0	0	0	0
143	158	137	76	22	10	2	0	0	0	0
144	165	120	85	25	10	0	0	0	0	0
145	159	131	79	29	6	1	0	0	0	0
146	164	114	99	19	9	0	0	0	0	0
147	152	148	68	28	8	1	0	0	0	0
148	136	180	55	24	8	1	1	0	0	0
149	158	129	86	24	8	0	0	0	0	0
150	164	125	80	24	12	0	0	0	0	0
151	147	159	63	26	8	2	0	0	0	0
152	152	148	74	16	14	1	0	0	0	0
153	146	168	52	27	9	2	1	0	0	0
154	140	161	77	20	5	2	0	0	0	0

155	161	130	81	22	8	3	0	0	0	0
156	136	171	70	20	6	2	0	0	0	0
157	152	149	66	30	7	0	1	0	0	0
158	152	149	69	25	7	3	0	0	0	0
159	140	163	76	16	8	2	0	0	0	0
160	146	156	69	26	7	1	0	0	0	0
161	147	150	79	20	8	1	0	0	0	0
162	147	154	70	28	4	1	1	0	0	0
163	152	147	71	26	7	2	0	0	0	0
164	144	164	63	23	9	2	0	0	0	0
165	150	149	71	28	6	0	1	0	0	0
166	157	141	69	29	6	3	0	0	0	0
167	132	175	70	24	2	2	0	0	0	0
168	139	164	74	21	5	2	0	0	0	0
169	155	143	68	31	7	1	0	0	0	0
170	151	150	68	26	9	1	0	0	0	0
171	148	155	69	22	9	2	0	0	0	0
172	119	197	62	24	3	0	0	0	0	0
173	137	160	86	16	5	1	0	0	0	0
174	148	150	73	28	5	1	0	0	0	0
175	153	143	75	26	6	2	0	0	0	0
176	151	151	67	27	6	3	0	0	0	0
177	151	143	77	30	2	2	0	0	0	0
178	161	122	93	19	10	0	0	0	0	0
179	142	154	80	25	4	0	0	0	0	0
180	155	141	70	32	7	0	0	0	0	0
181	154	147	64	30	10	0	0	0	0	0
182	154	142	70	35	3	0	1	0	0	0
183	153	145	72	24	11	0	0	0	0	0
184	164	122	80	34	4	1	0	0	0	0
185	154	143	71	28	9	0	0	0	0	0
186	151	147	72	27	7	1	0	0	0	0
187	155	131	91	20	8	0	0	0	0	0
188	152	145	73	29	4	1	1	0	0	0
189	146	145	87	23	3	1	0	0	0	0
190	156	132	88	20	8	1	0	0	0	0
191	149	154	69	21	10	2	0	0	0	0
192	148	160	63	21	9	4	0	0	0	0
193	150	158	59	27	7	4	0	0	0	0
194	158	144	63	29	7	4	0	0	0	0
195	143	159	72	24	5	2	0	0	0	0
196	141	168	62	26	5	3	0	0	0	0
197	149	152	68	30	3	3	0	0	0	0
198	135	170	72	22	5	1	0	0	0	0
199	142	157	74	28	4	0	0	0	0	0
200	152	145	71	31	5	1	0	0	0	0
201	138	174	57	27	9	0	0	0	0	0
202	151	148	73	23	8	2	0	0	0	0
203	134	166	80	22	2	1	0	0	0	0

204	134	167	76	26	2	0	0	0	0	0
205	136	167	77	17	7	1	0	0	0	0
206	141	165	67	23	8	1	0	0	0	0
207	150	149	75	21	7	3	0	0	0	0
208	130	177	72	20	6	0	0	0	0	0
209	145	160	62	32	5	1	0	0	0	0
210	138	170	61	31	5	0	0	0	0	0
211	148	148	78	24	6	1	0	0	0	0
212	146	154	71	29	3	2	0	0	0	0
213	147	154	69	29	4	2	0	0	0	0
214	144	149	82	28	2	0	0	0	0	0
215	145	157	69	27	6	1	0	0	0	0
216	161	130	77	27	10	0	0	0	0	0
217	139	160	79	23	2	2	0	0	0	0
218	153	152	56	36	7	1	0	0	0	0
219	148	147	79	25	5	1	0	0	0	0
220	153	141	77	27	6	1	0	0	0	0
221	145	156	70	28	5	1	0	0	0	0
222	144	155	77	23	3	3	0	0	0	0
223	151	147	71	29	6	1	0	0	0	0
224	153	144	70	32	5	1	0	0	0	0
225	139	154	87	23	2	0	0	0	0	0
226	156	135	80	28	4	2	0	0	0	0
227	150	148	74	24	8	1	0	0	0	0
228	156	136	78	28	6	1	0	0	0	0
229	141	157	80	22	3	2	0	0	0	0
230	146	155	71	25	7	1	0	0	0	0
231	145	153	72	32	3	0	0	0	0	0
232	151	142	79	27	6	0	0	0	0	0
233	138	165	69	31	1	1	0	0	0	0
234	150	146	72	34	2	1	0	0	0	0
235	155	136	80	28	5	1	0	0	0	0
236	154	138	80	26	6	1	0	0	0	0
237	156	140	70	32	6	1	0	0	0	0
238	154	137	83	23	7	1	0	0	0	0
239	151	143	80	24	5	2	0	0	0	0
240	155	135	82	26	7	0	0	0	0	0
241	159	130	82	26	7	1	0	0	0	0
242	157	130	85	27	6	0	0	0	0	0
243	158	144	62	31	6	4	0	0	0	0
244	141	163	70	22	9	0	0	0	0	0
245	144	157	73	23	7	1	0	0	0	0
246	150	147	73	28	7	0	0	0	0	0
247	148	151	71	29	5	1	0	0	0	0
248	143	153	82	20	7	0	0	0	0	0
249	155	142	71	30	4	3	0	0	0	0
250	151	147	73	26	6	2	0	0	0	0
251	142	159	74	22	8	0	0	0	0	0
252	143	147	93	17	4	1	0	0	0	0

253	158	139	68	31	8	1	0	0	0	0
254	146	151	76	26	6	0	0	0	0	0
255	143	161	67	27	6	1	0	0	0	0
256	144	161	63	31	5	1	0	0	0	0
257	144	155	76	23	6	1	0	0	0	0
258	142	159	70	31	2	1	0	0	0	0
259	166	117	87	28	6	0	1	0	0	0
260	139	161	76	24	5	0	0	0	0	0
261	151	152	61	33	8	0	0	0	0	0
262	155	133	83	30	4	0	0	0	0	0
263	151	140	85	22	6	1	0	0	0	0
264	150	141	84	25	4	1	0	0	0	0
265	149	144	81	25	6	0	0	0	0	0
266	131	176	70	24	3	1	0	0	0	0
267	135	164	81	22	2	1	0	0	0	0
268	130	172	78	23	2	0	0	0	0	0
269	142	164	64	29	5	0	1	0	0	0
270	134	170	74	22	4	1	0	0	0	0
271	142	149	90	20	4	0	0	0	0	0
272	138	167	74	16	8	2	0	0	0	0
273	145	150	77	31	2	0	0	0	0	0
274	155	142	72	25	11	0	0	0	0	0
275	150	153	65	26	11	0	0	0	0	0
276	147	154	68	29	7	0	0	0	0	0
277	144	152	81	21	7	0	0	0	0	0
278	149	149	71	30	6	0	0	0	0	0
279	141	158	78	22	5	1	0	0	0	0
280	147	153	70	29	5	1	0	0	0	0
281	147	146	80	29	3	0	0	0	0	0
282	152	134	91	23	5	0	0	0	0	0
283	140	159	80	19	6	1	0	0	0	0
284	145	150	79	27	4	0	0	0	0	0
285	138	168	66	29	3	0	1	0	0	0
286	150	142	79	31	3	0	0	0	0	0
287	141	155	81	24	4	0	0	0	0	0
288	148	140	91	21	5	0	0	0	0	0
289	147	144	84	27	3	0	0	0	0	0
290	126	189	54	36	0	0	0	0	0	0
291	135	162	81	27	0	0	0	0	0	0
292	140	159	76	26	4	0	0	0	0	0
293	133	170	75	23	4	0	0	0	0	0
294	145	161	61	31	6	1	0	0	0	0
295	140	162	71	27	5	0	0	0	0	0
296	143	156	74	28	3	1	0	0	0	0
297	140	161	74	24	6	0	0	0	0	0
298	125	180	77	21	2	0	0	0	0	0
299	124	188	70	16	6	1	0	0	0	0
300	146	156	67	29	7	0	0	0	0	0
301	141	160	72	28	3	1	0	0	0	0

302	142	155	81	21	5	1	0	0	0	0
303	151	135	95	17	6	1	0	0	0	0
304	162	126	81	27	9	0	0	0	0	0
305	158	133	81	23	9	1	0	0	0	0
306	138	157	86	20	4	0	0	0	0	0
307	127	180	73	21	4	0	0	0	0	0
308	154	131	90	26	4	0	0	0	0	0
309	135	165	80	20	5	0	0	0	0	0
310	134	167	78	22	4	0	0	0	0	0
311	138	162	77	23	5	0	0	0	0	0
312	135	161	86	20	3	0	0	0	0	0
313	132	171	77	21	3	1	0	0	0	0
314	135	162	86	18	3	1	0	0	0	0
315	140	154	86	22	2	1	0	0	0	0
316	145	139	100	18	3	0	0	0	0	0
317	150	145	78	26	4	2	0	0	0	0
318	142	157	75	27	3	1	0	0	0	0
319	142	156	78	24	4	1	0	0	0	0
320	141	153	86	20	5	0	0	0	0	0
321	144	156	71	29	5	0	0	0	0	0
322	145	146	87	23	4	0	0	0	0	0
323	144	156	74	24	6	1	0	0	0	0
324	124	179	82	18	2	0	0	0	0	0
325	140	159	76	26	4	0	0	0	0	0
326	139	157	82	24	3	0	0	0	0	0
327	153	139	81	25	6	1	0	0	0	0
328	112	197	80	16	0	0	0	0	0	0
329	140	153	84	28	0	0	0	0	0	0
330	128	169	88	20	0	0	0	0	0	0
331	136	168	72	23	6	0	0	0	0	0
332	149	152	65	34	4	1	0	0	0	0
333	126	178	80	18	2	1	0	0	0	0
334	132	166	87	15	5	0	0	0	0	0
335	131	172	77	21	4	0	0	0	0	0
336	139	159	79	24	4	0	0	0	0	0
337	132	176	69	21	7	0	0	0	0	0
338	143	161	66	28	7	0	0	0	0	0
339	134	169	74	24	4	0	0	0	0	0
340	144	154	77	23	7	0	0	0	0	0
341	140	161	73	26	5	0	0	0	0	0
342	132	167	82	22	2	0	0	0	0	0
343	141	160	72	27	5	0	0	0	0	0
344	146	147	84	23	4	1	0	0	0	0
345	140	155	83	24	3	0	0	0	0	0
346	151	141	82	24	7	0	0	0	0	0
347	144	145	90	24	2	0	0	0	0	0
348	148	148	75	30	3	1	0	0	0	0
349	134	166	78	25	2	0	0	0	0	0
350	140	157	79	27	1	1	0	0	0	0

351	131	171	76	26	1	0	0	0	0	0
352	129	183	64	23	5	1	0	0	0	0
353	137	162	77	27	2	0	0	0	0	0
354	138	164	70	31	2	0	0	0	0	0
355	144	151	83	22	4	0	1	0	0	0
356	137	164	74	27	3	0	0	0	0	0
357	140	154	87	20	3	1	0	0	0	0
358	140	145	101	18	1	0	0	0	0	0
359	131	172	79	17	6	0	0	0	0	0
360	127	178	79	17	2	2	0	0	0	0
361	144	148	87	21	5	0	0	0	0	0
362	129	180	63	33	0	0	0	0	0	0
363	129	171	84	18	3	0	0	0	0	0
364	154	143	66	38	4	0	0	0	0	0
365	134	170	71	27	3	0	0	0	0	0
366	132	168	81	21	3	0	0	0	0	0
367	134	163	84	22	2	0	0	0	0	0
368	124	183	76	18	4	0	0	0	0	0
369	128	175	80	18	4	0	0	0	0	0
370	117	193	74	20	1	0	0	0	0	0
371	141	151	88	22	3	0	0	0	0	0
372	142	150	88	22	2	1	0	0	0	0
373	138	162	75	27	3	0	0	0	0	0
374	122	186	76	18	2	1	0	0	0	0
375	126	183	68	26	2	0	0	0	0	0
376	128	179	70	26	2	0	0	0	0	0
377	143	155	75	29	2	1	0	0	0	0
378	126	175	82	22	0	0	0	0	0	0
379	137	153	96	16	3	0	0	0	0	0
380	126	180	75	21	3	0	0	0	0	0
381	126	180	78	18	0	3	0	0	0	0
382	156	126	93	27	3	0	0	0	0	0
383	115	197	73	18	2	0	0	0	0	0
384	127	178	75	23	2	0	0	0	0	0
385	141	154	83	23	4	0	0	0	0	0
386	123	178	88	13	3	0	0	0	0	0
387	122	178	88	17	0	0	0	0	0	0
388	136	165	76	24	4	0	0	0	0	0
389	152	134	87	31	1	0	0	0	0	0
390	112	199	78	14	2	0	0	0	0	0
391	128	170	89	15	3	0	0	0	0	0
392	135	165	77	27	0	1	0	0	0	0
393	134	156	96	19	0	0	0	0	0	0
394	108	193	100	4	0	0	0	0	0	0
395	144	145	92	20	4	0	0	0	0	0
396	80	265	40	20	0	0	0	0	0	0