Uniform Room Frames with Five Holes

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ABSTRACT

In 1981, Dinitz and Stinson [2] proved that the necessary conditions were sufficient for the existence of a Room frame of type t^u for all $u \geq 6$. Very recently, Zhu Lie and Ge Gennian [5] constructed all t^5 Room frames except for four missing orders. In this article we construct t^5 Room frames for $t = 11, 13, 17$, and 19; this completes the proof that the necessary conditions are sufficient for the existence of a Room frame of type t^5. © 1993 John Wiley & Sons, Inc.

Let S be a set, and let $\{S_1, \ldots, S_n\}$ be a partition of S. An $\{S_1, \ldots, S_n\}$-*Room frame* is an $|S| \times |S|$ array, F, indexed by S, which satisfies the following properties:

1. every cell of F either is empty or contains an unordered pair of symbols of S,
2. the subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are referred to as *holes*),
3. each symbol $x \notin S_i$ occurs once in row (or column) s, for any $s \in S_i$,
4. the pairs occurring in F are those $\{s, t\}$, where $(s, t) \in (S \times S) \backslash \cup_{i=1}^{n} (S_i \times S_i)$.

As is usually done in the literature, we shall refer to a Room frame simply as a *frame*. The *type* of a frame F is defined to be the multiset $\{|S_i|: 1 \leq i \leq n\}$. We use an "exponential" notation to describe types: a frame has type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ if there are u_i S_j's of cardinality t_i, $1 \leq i \leq k$. The *order* of the frame is $|S|$. A frame of type t^u (one hole size) is called a *uniform* frame.

To illustrate these definitions, a frame of type 2^6 is presented in Figure 1.

For a survey of results on Room frames the reader is referred to [3]. In the following theorem we summarize existence results for uniform Room frames. (See [4] for a similar summary in the nonuniform case.)

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CCC 1063-8539/93/040323-06
Journal of Combinatorial Designs, Vol. 1, No. 4 (1993)

		59		6b		24		3a		87	
			48		7b		35		2a		96
5a				71		8b		46		09	
	4a				06		9b		57		18
68		7a				93		0b		12	
	79		6a				28		1b		03
2b		08		9a				15		34	
	3b		19		8a				04		25
37		4b		02		1a				56	
	26		5b		13		0a				47
49		16		38		05		27			
	58		07		29		14		36		

FIG. 1. A frame of type 2^6 on the symbols $\{0, 1, \ldots, 9, a, b\}$.

Theorem 1. *There exist uniform frames of the following types:*

(1) t^4 *for all even* $t \geq 4$, *except possibly for* $t \in \{14, 22, 26, 34, 38, 46, 62, 74, 82, 86, 98, 122, 134, 146\}$, [5]
(2) t^5 *for all* $t \geq 2$, *except possibly for* $t = 11, 13, 17$ *or* 19, [5]
(3) t^u *for* $u \geq 6$ *and both* t *and* u *even,* [2]
(4) t^u *for all* t *and all odd* $u \geq 7$, [2].

We also summarize nonexistence results.

Theorem 2. *There does not exist a frame of type* $T = t^u$ *in any of the following cases:*

(1) $u = 1, 2$ *or* 3 (*i.e., if the number of holes is* 1, 2 *or* 3)
(2) $T = 2^4$, [7]
(3) $T = 1^5$, [6]
(4) t *is odd and* u *is even,* [2].

Thus, the existence problem for uniform frames of type t^u is completely solved when $u \geq 6$ and there are only a handful of cases with $u = 4$ or 5 that are not solved. The purpose of this article is to provide constructions for the remaining four frames of type t^5. Our main result will be

Theorem 3. *There exist Room frames of type* t^5 *for* $t = 11, 13, 17$ *and* 19.

In order to prove this theorem we will need an algebraic object whose existence insures the existence of a Room frame. Let G be an additive abelian group of order g, and let H be a subgroup of order h of G, where $g - h$ is even (i.e., g and h are both even or both odd). A *frame starter* in $G \backslash H$ is a set of unordered pairs $S = \{\{s_i, t_i\}: 1 \leq i \leq (g - h)/2\}$ such that the following two properties are satisfied:

1. $\{s_i\colon 1 \le i \le (g-h)/2\} \cup \{t_i\colon 1 \le i \le (g-h)/2\} = G \backslash H$
2. $\{\pm(s_i - t_i)\colon 1 \le i \le (g-h)/2\} = G \backslash H$

A frame starter in $G\backslash H$ is referred to as a frame starter of *type* $h^{g/h}$.

Let $S = \{\{s_i, t_i\}\colon 1 \le i \le (g-h)/2\}$ and $T = \{\{u_i, v_i\}\colon 1 \le i \le (g-h)/2\}$ be two frame starters in $G\backslash H$. Without loss of generality, we may assume that $s_i - t_i = u_i - v_i$, for all i. Then S and T are said to be *orthogonal* frame starters if $u_i - s_i = u_j - s_j$ implies $i = j$, and if $u_i - s_i \notin H$ for all i.

The following theorem gives the connection between orthogonal frame starters and Room frames. The proof, which first appeared in [1], is straightforward.

Theorem 4. *The existence of two orthogonal frame starters of type* t^u *implies the existence of a Room frame of type* t^u.

The easiest and most common way to construct two orthogonal frame starters is by constructing a *strong* frame starter. A starter $S = \{\{s_i, t_i\}\colon 1 \le i \le (g-1)/2\}$ is said to be *strong* if $s_i + t_i = s_j + t_j$ implies $i = j$, and for any i, $s_i + t_i \notin H$. If $S = \{\{s_i, t_i\}\colon 1 \le i \le (g-h)/2\}$ is a frame starter, then $-S = \{\{-s_i, -t_i\}\colon 1 \le i \le (g-h)/2\}$ is also a starter.

The following result concerning strong starters is also easy to prove [1].

Theorem 5. *If S is a strong frame starter, then S and $-S$ are orthogonal frame starters.*

Unfortunately, by a theorem of Dinitz and Stinson [1], there does not exist a strong frame starter of type t^5 for any odd value of t. This result does not, however, preclude the existence of two orthogonal frame starters of type t^5.

Using the hill-climbing algorithm for finding frame starters (see [2]), we easily found (on a personal computer) pairs of orthogonal frame starters of type t^5 for $t = 11$, 13, 17, and 19. Thus we have:

Theorem 6. *There exist Room frames of type t^5 for all $t \ge 2$.*

Proof. We list below pairs of orthogonal frame starters of type t^5 in the additive groups $\mathbb{Z}_{5t}\backslash\mathbb{Z}_t$ for $t = 11$, 13, 17, and 19. This in conjunction with the results of Ge and Zhu [5] prove the theorem. □

Two orthogonal 11^5 frame starters

41,42	47,49	8,11	48,52	16,22	2,9	26,34	53,7	18,29	21,33	4,17
24,38	12,28	37,54	43,6	27,46	23,44	14,36	51,19	32,1	13,39	31,3

48,49	6,8	31,34	52,1	33,39	4,11	54,7	17,26	2,13	32,44	16,29
53,12	21,37	24,41	9,27	23,42	22,43	36,3	28,51	14,38	47,18	19,46

Two orthogonal 13^5 frame starters

38,39	17,19	28,31	7,11	46,52	64,6	18,26	48,57	51,62	4,16	23,36
33,47	27,43	56,8	49,2	44,63	37,58	12,34	1,24	54,13	3,29	59,21
14,42	32,61	22,53	9,41							

62, 63	39, 41	24, 27	64, 3	2, 8	31, 38	1, 9	17, 26	33, 44	6, 18	36, 49
7, 21	16, 32	42, 59	43, 61	28, 47	48, 4	29, 51	34, 57	53, 12	11, 37	52, 14
56, 19	58, 22	23, 54	46, 13							

Two orthogonal 17^5 frame starters

18, 19	84, 1	78, 81	13, 17	16, 22	49, 56	23, 31	64, 73	77, 3	36, 48	39, 52
57, 71	21, 37	12, 29	28, 46	72, 6	47, 68	32, 54	53, 76	69, 8	41, 67	34, 61
59, 2	83, 27	43, 74	26, 58	9, 42	62, 11	63, 14	7, 44	51, 4	79, 33	82, 38
24, 66										

76, 77	81, 83	26, 29	44, 48	17, 23	32, 39	34, 42	58, 67	11, 22	54, 66	6, 19
49, 63	68, 84	69, 1	9, 27	14, 33	43, 64	56, 78	51, 74	38, 62	47, 73	71, 13
3, 31	72, 16	82, 28	4, 36	8, 41	18, 52	21, 57	24, 61	59, 12	53, 7	46, 2
37, 79										

Two orthogonal 19^5 frame starters

43, 44	9, 11	54, 57	32, 36	68, 74	7, 14	91, 4	3, 12	67, 78	21, 33	24, 37
38, 52	63, 79	17, 34	48, 66	89, 13	61, 82	42, 64	58, 81	53, 77	1, 27	2, 29
56, 84	94, 28	62, 93	86, 23	26, 59	49, 83	51, 87	76, 18	8, 46	72, 16	6, 47
31, 73	71, 19	92, 41	88, 39	22, 69						

47, 48	66, 68	28, 31	18, 22	61, 67	89, 1	54, 62	14, 23	38, 49	74, 86	21, 34
84, 3	72, 88	59, 76	9, 27	32, 51	37, 58	69, 91	79, 7	2, 26	17, 43	19, 46
24, 52	78, 12	56, 87	39, 71	73, 11	77, 16	63, 4	94, 36	6, 44	53, 92	42, 83
82, 29	93, 41	64, 13	57, 8	81, 33						

We note that the uniform Room frame problem is now completely solved for five or more holes. In the case of four holes, there are 14 cases which are not known. Only six of these cases, $t \in \{14, 22, 26, 34, 38, 46\}$, are essential. If there exists a 14^4 frame, then Lemma 3.2 in [5] can be used to construct frames of type t^4 for $t = 62, 74, 98, 122$, and 146. Similarly, a frame of type 26^4 can be used to construct frames of type 86^4 and 134^4, and a frame of type 22^4 can be used to construct a frame of type 82^4. We suspect that all of these frames do indeed exist. However, D. R. Stinson [8] has now shown that they *cannot* be found using orthogonal frame starters in the cyclic group.

Theorem 7 (Stinson [8]). *If $t \equiv 2$ mod 4, then there does not exist a pair of orthogonal frame starters of type t^4 in the group $\mathbb{Z}_{4t} \backslash \mathbb{Z}_t$.*

Proof. Suppose we have a frame starter S in $\mathbb{Z}_{4t} \backslash \mathbb{Z}_t$. We can write the $3t$ pairs in S as

$$S = \{\{a_1, a_1 + 1\}, \{a_2, a_2 + 2\}, \{a_3, a_3 + 3\}, \{a_5, a_5 + 5\}, \ldots \{a_{2t-1}, a_{2t-1} + 2t - 1\}\}.$$

where all arithmetic is modulo $4t$

Let $s = a_1 + a_2 + a_3 + a_5 + \ldots + a_{2t-1} \pmod{4t}$. Then $2s + 1 + 2 + 3 + 5 + 6 + 7 + \cdots + (2t - 3) + (2t - 2) + (2t - 1) = 1 + 2 + 3 + 5 + 6 + 7 + \cdots + (4t - 3) + (4t - 2) + (4t - 1)$ and solving for s we get $s = t(9t)/4$. Since $t \equiv 2 \pmod 4$, $s \equiv 1 \pmod 4$.

Next, we partition the pairs of S into 6 types:

x_1 pairs $\{a_i, a_i + d\}$ where $a_i \equiv 1 \pmod 4, d \equiv 1 \pmod 4, d \leq 2t - 1$,

x_2 pairs $\{a_i, a_i + d\}$ where $a_i \equiv 2 \pmod 4, d \equiv 1 \pmod 4, d \leq 2t - 1$,

y_1 pairs $\{a_i, a_i + d\}$ where $a_i \equiv 1 \pmod 4, d \equiv 2 \pmod 4, d \leq 2t - 1$,

y_2 pairs $\{a_i, a_i + d\}$ where $a_i \equiv 3 \pmod 4, d \equiv 2 \pmod 4, d \leq 2t - 1$,

z_1 pairs $\{a_i, a_i + d\}$ where $a_i \equiv 2 \pmod 4, d \equiv 3 \pmod 4, d \leq 2t - 1$, and

z_2 pairs $\{a_i, a_i + d\}$ where $a_i \equiv 3 \pmod 4, d \equiv 3 \pmod 4, d \leq 2t - 1$.

Since there are exactly $\frac{t}{2}$ pairs with difference d for each $d \equiv 1$, 2 or 3 (mod 4), we get the following three equations in these 6 unknowns: $x_1 + x_2 = \frac{t}{2}$, $y_1 + y_2 = \frac{t}{2}$, and $z_1 + z_2 = \frac{t}{2}$. Furthermore, by counting all the pairs that contain an element congruent to 1 modulo 4 we get $x_1 + y_1 + y_2 + z_1 = t$. Hence $x_1 + z_1 = \frac{t}{2}$ and $x_2 + z_2 = \frac{t}{2}$.

Now we compute s modulo 4, $s = x_1 + 2x_2 + y_1 + 3y_2 + 2z_1 + 3z_2$. Using the equations above, this is $s \equiv \frac{t}{2} + x_2 + \frac{t}{2} + 2y_2 + t + z_2 = \frac{5t}{2} + 2y_2 \pmod 4$. There are two cases: 1) if $t \equiv 2 \pmod 8$, then $\frac{5t}{2} + 2y_2 \equiv 1 + 2y_2 \pmod 4$ and 2) if $t \equiv 6 \pmod 8$, then $\frac{5t}{2} + 2y_2 \equiv 3 + 2y_2 \pmod 4$.

But we showed above that $s \equiv 1 \pmod 4$: hence if $t \equiv 2 \pmod 8$, then $2y_2 \equiv 0 \pmod 4$ and y_2 must be even. Since $y_1 + y_2 = \frac{t}{2}, y_1$ is odd. In the other case, if $t \equiv 6 \pmod 8$, then $2y_2 \equiv 2 \pmod 4$ and y_2 is odd. Since $y_1 + y_2 = \frac{t}{2}, y_1$ is even. We conclude from this that in either case, y_1 and y_2 have different parity.

Now suppose that T is a starter which is orthogonal to S. The pairs of type y_1 and y_2 must have adders that are congruent to 2 modulo 4. Hence a y_1 pair in S is mapped into a y_2 pair in T, and a y_2 pair in S is mapped into a y_1 pair in T. But the number of y_1 pairs in S is of different parity than the number of y_2 pairs in T, so we have a contradiction. □

Assume the frames of type t^4 that we wish to construct, namely $t \in \{14, 22, 26, 34, 38, 46\}$, were to come from frame starters in some abelian group other than $\mathbb{Z}_{4t} \backslash \mathbb{Z}_t$. Then necessarily we would need to find a group G of order $8p$, p a prime which contains a subgroup H of index 4 such that $G \backslash H$ contains no element of order 2. But it is obvious that such a group must be $\mathbb{Z}_{4t}$. In conjunction with Theorem 7, this says that Room frames of type t^4 with $t \in \{14, 22, 26, 34, 38, 46\}$ can not be found by using a pair of orthogonal frame starters in an abelian group.

Finally, we note that the frames of type 6^4 and 10^4 were constructed directly using the hill climbing algorithm for frames given in [4]. That algorithm does not appear to be feasible for finding frames of type t^4 for $t > 14$.

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Received February 28, 1993
Accepted March 10, 1993