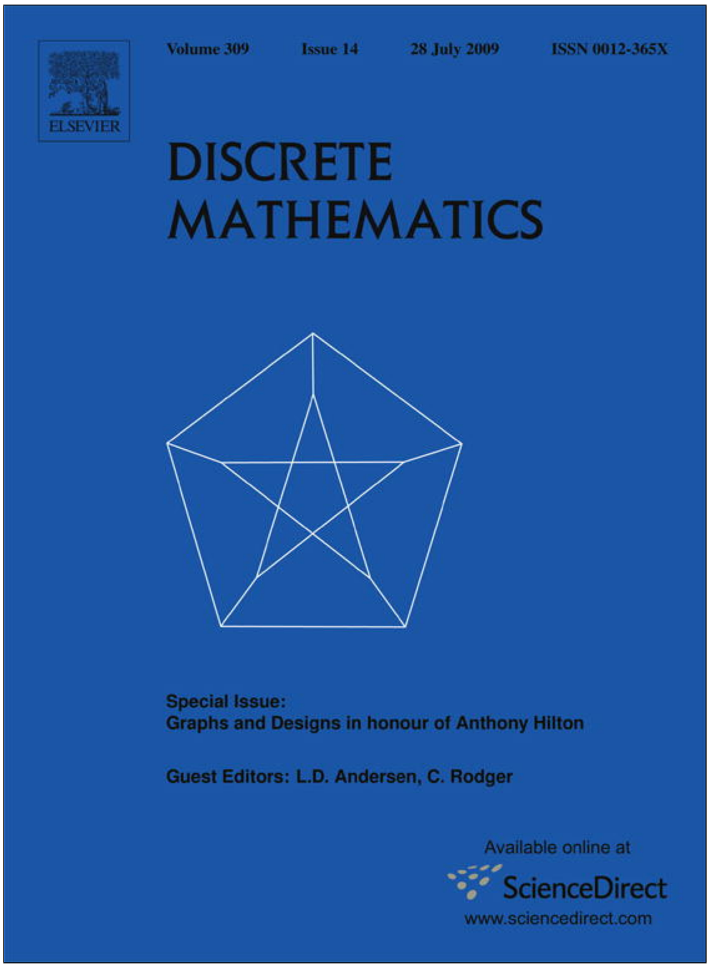
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Maximum uniformly resolvable designs with block sizes 2 and 4

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ABSTRACT

A central question in design theory dating from Kirkman in 1850 has been the existence of resolvable block designs. In this paper we will concentrate on the case when the block size $k = 4$. The necessary condition for a resolvable design to exist when $k = 4$ is that $v \equiv 4 \mod 12$; this was proven sufficient in 1972 by Hanani, Ray-Chaudhuri and Wilson [H. Hanani, D.K. Ray-Chaudhuri, R.M. Wilson, On resolvable designs, Discrete Math. 3 (1972) 343–357]. A resolvable pairwise balanced design with each parallel class consisting of blocks which are all of the same size is called a uniformly resolvable design, a URD. The necessary condition for the existence of a URD with block sizes 2 and 4 is that $v \equiv 0 \mod 4$. Obviously in a URD with blocks of size 2 and 4 one wishes to have the maximum number of resolution classes of blocks of size 4; these designs are called maximum uniformly resolvable designs or MURDs. So the question of the existence of a MURD on v points has been solved for $v \equiv 4 (\mathrm{mod}\ 12)$ by the result of Hanani, Ray-Chaudhuri and Wilson cited above. In the case $v \equiv 8 (\mathrm{mod}\ 12)$ this problem has essentially been solved with a handful of exceptions (see [G. Ge, A.C.H. Ling, Asymptotic results on the existence of 4-RGDDs and uniform 5-GDDs, J. Combin. Des. 13 (2005) 222–237]). In this paper we consider the case when $v \equiv 0 (\mathrm{mod}\ 12)$ and prove that a MURD$(12u)$ exists for all $u \geq 2$ with the possible exception of $u \in \{2, 7, 9, 10, 11, 13, 14, 17, 19, 22, 31, 34, 38, 43, 46, 47, 82\}$.

1. Introduction and definitions

Let K be a subset of positive integers. A *pairwise balanced design* PBD(v, K) of order v with block sizes from K is a pair $(\mathcal{V}, \mathcal{B})$, where $\mathcal{V}$ is a finite set (the *points*) of cardinality v and $\mathcal{B}$ is a family of subsets (the *blocks*) of $\mathcal{V}$ which satisfy the properties:

1. If $B \in \mathcal{B}$, then $|B| \in K$.
2. Every pair of distinct elements of $\mathcal{V}$ occurs in exactly one block of $\mathcal{B}$.

A *parallel class* in a pairwise balanced design is a subset of blocks $\mathcal{A} \subset \mathcal{B}$ such that each point in $\mathcal{V}$ is contained in exactly one block in $\mathcal{A}$. A pairwise balanced design is *resolvable* if the set of blocks $\mathcal{B}$ can be partitioned into parallel classes.

A parallel class in a PBD is *uniform* if every block in the parallel class is of the same size. A *uniformly resolvable design*, URD(v, K, R), is a resolvable PBD(v, K) such that all of the parallel classes are uniform. R is a multiset, where $|R| = |K|$ and for each $k \in K$ there corresponds a positive $r_k \in R$ such that there are exactly r_k parallel classes of size k.

In this paper, we are interested in the case when $K = \{2, 4\}$. Since a block of size 4 can be decomposed into three parallel classes of size 2, our interest is to construct URD$(v, \{2, 4\}, \{r_2, r_4\})$ where r_4 is maximized.

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Evidently, for a URD(v, {2, 4}, {r_2, r_4}) with $r_4 > 0$ to exist, v must be a multiple of 4. The following lemma gives upper bounds on the value of r_4 in the three cases modulo 12 when $v \equiv 0 (\text{mod } 4)$. The proof is obvious.

Lemma 1.1. $r_4 \leq \frac{v-\alpha_v}{3}$ *where* $\alpha_v = \begin{cases} 1, & \text{if } v \equiv 4(\text{mod } 12) \\ 2, & \text{if } v \equiv 8(\text{mod } 12) \\ 3, & \text{if } v \equiv 0(\text{mod } 12). \end{cases}$

A URD(v, {2, 4}, {r_2, r_4}) with r_4 meeting the upper bound in Lemma 1.1 is said to be *maximum URD*. For the purposes of this paper we will denote a maximum URD(v, {2, 4}, {r_2, r_4}) as simply a MURD(v). In this paper we will use standard objects from combinatorial design theory such as group divisible designs, transversal designs, and frames. The reader is referred to [2] or [3] for definitions and results concerning these objects.

When $v \equiv 4(\text{mod } 12)$, the necessary condition for the existence of resolvable BIBD(v, 4, 1) (a resolvable PBD(v, {4}) was shown to be sufficient in 1972 by Hanani, Ray-Chaudhuri and Wilson [5]. Hence, the existence of a MURD(v) is known for all $v \equiv 4(\text{mod } 12)$. When $v \equiv 8(\text{mod } 12)$, it is clear that a MURD(v)is a resolvable group divisible design with blocks of size 4 and $\frac{v}{2}$ groups all of size 2 a MURD(v). It has recently been shown in [4] that the necessary conditions for the existence of a resolvable group divisible design with blocks of size 4 and $u = \frac{v}{2}$ groups all of size 2 (namely that $u \geq 4$ and $u \equiv 4(\text{mod } 6)$) are sufficient except when $u = 4$ and $u = 10$ and possibly when $u \in \{34, 46, 52, 70, 82, 94, 100, 118, 130, 142, 178, 184, 202, 214, 238, 250, 334, 346\}$.

In view of the results above we will concentrate on the case where $v \equiv 0(\text{mod } 12)$ in the remainder of this paper. We should note that when $v \equiv 0(\text{mod } 12)$ a resolvable group divisible design of type 3^h with $h \equiv 0(\text{mod } 4)$ is a uniformly resolvable design on $v = 3h$ points with $\frac{v-3}{3}$ parallel classes of blocks of size 4 and one parallel class of blocks of size 3 (the groups in the GDD). Such RGDDs exist for all orders except when $h = 4$ (see [4] or [2]); however, clearly the blocks of size 3 cannot be divided into two parallel classes of blocks of size 2 and so these do not yield MURD($12u$) in any straightforward way.

In Section 2 we present direct constructions for MURD($12u$) with small u and in Section 3 we give some recursive constructions and prove asymptotic existence. Finally, in Section 4 we provide construction for some smaller orders and prove our main theorem. We will prove the following theorem.

Theorem 1.2. *A* MURD($12u$) *exists for all* $u \geq 2$ *with the possible exception of* $u \in \{2, 7, 9, 10, 11, 13, 14, 17, 19, 22, 31, 34, 38, 43, 46, 47, 82\}$.

2. Direct constructions

In this section, we present direct constructions for uniformly resolvable designs with block sizes 2 and 4 for some small values of v.

Lemma 2.1. *There exists a* MURD(36)

Proof. Let $\mathcal{V} = \mathbb{Z}_{18} \times \{0, 1\}$. Two parallel classes are generated by the base block $\{0_0, 3_0, 1_1, 6_1\}$ by first taking the odd translates then taking the even translates. Next, generate a parallel classes by taking the following base blocks:

$$\{1_0, 2_0, 13_0, 15_0\}, \{7_0, 17_0, 11_1, 12_1\}, \{3_0, 10_1, 14_1, 17_1\}, \{5_0, 7_1, 13_1, 15_1\}, \{0_0, 9_0, 0_1, 9_1\}.$$

Add 9 to each of the first four blocks; these eight blocks together with the last block form the parallel class. Adding i for $i = 0, 1, \ldots, 8$ to this first parallel class produces nine parallel classes. The two unused mixed differences, 15 and 17, generate two 1-factors on $\mathcal{V}$. ■

Lemma 2.2. *There exists a* MURD(48).

Proof. Let $\mathcal{V} = \mathbb{Z}_{24} \times \{0, 1\}$. One parallel class is generated by the base block $\{0_0, 12_0, 0_1, 12_1\}$ by adding i for $1 \leq i \leq 11$. Two more parallel classes are generated by the base block $\{0_0, 1_0, 2_1, 5_1\}$ by taking the odd or even translates. Finally, consider the following base blocks:

$$\{0_1, 1_1, 5_1, 11_1\}, \{0_0, 3_1, 18_1, 20_1\}, \{3_0, 8_0, 14_1, 22_1\},$$
$$\{5_0, 7_0, 13_0, 4_1\}, \{9_0, 18_0, 22_0, 7_1, \}, \{2_0, 16_0, 23_0, 9_1\}.$$

Add 12 to the above six blocks and these twelve blocks become a parallel class. Use this parallel class to obtain twelve parallel classes by adding i for $0 \leq i \leq 11$ to each block. The two unused mixed differences, 10 and 16, generate two 1-factors on $\mathcal{V}$. ■

Lemma 2.3. *There exists a* MURD(60).

Proof. Let $\mathcal{V} = \mathbb{Z}_{30} \times \{0, 1\}$. Each of the blocks $\{0_0, 7_0, 2_1, 19_1\}$ and $\{2_0, 19_0, 0_1, 7_1\}$ generates two parallel classes by taking the odd or even translates modulo 30. Consider the following base blocks:

$$\{1_0, 2_0, 5_0, 23_0\}, \{6_0, 2_1, 16_1, 27_1\}, \{5_1, 8_1, 9_1, 29_1\}, \{7_0, 12_0, 11_1, 13_1\},$$
$$\{9_0, 11_0, 3_1, 25_1\}, \{10_0, 29_0, 7_1, 19_1\}, \{3_0, 13_0, 19_0, 6_1\}, \{0_0, 15_0, 0_1, 15_1\}.$$

Add 15 to each block (except the last one) and these 15 blocks form a parallel class. Take the next 14 consecutive translates modulo 30 to generate an additional 14 parallel classes. The two unused mixed differences, 7 and 13, form two 1-factors. ■

Lemma 2.4. *There exists a* MURD(72).

Proof. Let $\mathcal{V} = \mathbb{Z}_{36} \times \{0, 1\}$. One parallel class is generated by the two base blocks $\{0_0, 9_0, 18_0, 27_0\}$ and $\{0_1, 9_1, 18_1, 27_1\}$. Four more parallel classes are generated by the two base blocks $\{0_0, 1_0, 3_0, 14_0\}$ and $\{0_1, 1_1, 3_1, 14_1\}$ as the points are distinct modulo 4. Consider the following blocks:

$$\{0_0, 0_1, 4_1, 10_1\}, \{1_0, 5_0, 24_1, 31_1\}, \{2_0, 31_0, 9_1, 33_1\}, \{3_0, 24_0, 12_1, 32_1\}, \{4_0, 30_0, 16_1, 21_1\},$$
$$\{7_0, 35_0, 5_1, 20_1\}, \{8_0, 14_0, 11_1, 19_1\}, \{9_0, 29_0, 8_1, 25_1\}, \{10_0, 15_0, 34_0, 35_1\}.$$

Add 18 to all blocks, and these 18 blocks form a parallel class. Take the next 17 consecutive translates modulo 36 to obtain an additional 17 parallel classes. The two unused mixed differences, 18 and 28, form two 1-factors. ■

Lemma 2.5. *There exists a* MURD(96).

Proof. Let $\mathcal{V} = \mathbb{Z}_{24} \times \mathbb{Z}_4$. Three parallel classes are generated by three short orbits $\{0_0, 6_0, 12_0, 18_0\}$, $\{0_0, 0_1, 0_2, 0_3\}$ and $\{0_0, 6_1, 12_2, 18_3\}$ respectively. Four parallel classes are generated by the base blocks $\{0_0, 1_0, 3_0, 10_0\}$ by taking add $(4i)_j$ for $i = 0, 1, \ldots, 5$ and $j = 0, 1, 2, 3$. These 24 blocks form a parallel class. Translate them to obtain a total of four parallel classes. The base blocks

$$\{0_0, 4_0, 1_1, 9_1\}, \{2_0, 7_0, 6_1, 20_2\}, \{3_0, 16_0, 13_2, 5_3\}, \{8_0, 21_1, 23_2, 14_3\}, \{10_0, 17_1, 12_2, 22_3\}, \{11_0, 19_1, 15_2, 18_3\}$$

are distinct in the first component modulo 24. Cycle them in the second component to obtain a parallel class. 24 parallel classes can then be generated by cycling them modulo 24. ■

We now give the definition of an *incomplete MURD*, an *IMURD*. Let $v, h \equiv 0 \pmod{12}$. An IMURD$(v+h, h)$ is a $\{2, 4\}$-GDD of type $1^v h^1$ such that the blocks can be partitioned into three types of resolution classes as follows:

1. Two classes of blocks, with all blocks of size 2, where each class consists of $\frac{v}{2}$ blocks covering all v points not in the group of size h.
2. $\frac{h-3}{3}$ classes of block, with all blocks of size 4, where each class consists of $\frac{v}{4}$ blocks covering all v points not in the group of size h.
3. $\frac{v}{3}$ classes of blocks with all blocks of size 4, where each class consists of $\frac{v+h}{4}$ blocks covering all $v + h$ points.

Lemma 2.6. *There exists a* IMURD$(48 + 12, 12)$.

Proof. Let $\mathcal{V} = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \{0, 1, 2\} \cup \{x_0, x_1, y_0, y_1, \ldots, y_9\}$. We first construct three parallel classes from short orbits. From the three base blocks $\{(0, 0, i), (0, 1, i), (0, 2, i), (0, 3, i)\}$ with $i = 0, 1, 2$, construct a parallel class of 12 blocks by adding $(x, 0, 0)$ for each $x \in \mathbb{Z}_4$. From the three base blocks $\{(0, 0, i), (1, 0, i), (2, 0, i), (3, 0, i)\}$, with $i = 0, 1, 2$, construct a parallel class of 12 blocks by adding $(0, x, 0)$ for each $x \in \mathbb{Z}_4$. And finally, from the three base blocks $\{(0, 0, i), (1, 1, i), (2, 2, i), (3, 3, i)\}$, with $i = 0, 1, 2$, construct a parallel class of 12 blocks by adding $(x, 0, 0)$ for each $x \in \mathbb{Z}_4$. Now, consider the following base blocks:

$$\{x_0, (0, 2, 0), (0, 3, 2), (2, 0, 2)\}, \{x_1, (2, 0, 0), (0, 3, 1), (2, 2, 1)\}, \{y_0, (0, 3, 0), (2, 3, 1), (1, 3, 2)\},$$
$$\{y_1, (1, 1, 0), (1, 0, 1), (3, 2, 2)\}, \{y_2, (1, 3, 0), (3, 0, 1), (2, 1, 2)\}, \{y_3, (2, 1, 0), (1, 2, 1), (0, 1, 2)\},$$
$$\{y_4, (2, 2, 0), (3, 2, 1), (3, 1, 2)\}, \{y_5, (2, 3, 0), (3, 1, 1), (1, 1, 2)\}, \{y_6, (3, 0, 0), (2, 0, 1), (3, 0, 2)\},$$
$$\{y_7, (3, 1, 0), (0, 2, 1), (1, 0, 2)\}, \{y_8, (3, 2, 0), (2, 1, 1), (2, 3, 2)\}, \{y_9, (3, 3, 0), (1, 1, 1), (2, 2, 2)\}$$
$$\{(0, 0, 0), (1, 2, 0), (0, 0, 1), (1, 3, 1)\}, \{(0, 1, 0), (1, 0, 0), (0, 0, 2), (1, 2, 2)\},$$
$$\{(0, 1, 1), (3, 3, 1), (0, 2, 2), (3, 3, 2)\}.$$

The parallel classes (consisting of 15 blocks each) are generated by adding elements (i, j) for $i, j \in \mathbb{Z}_4$ to the first two coordinates of each point in the design. If the added element is of the form $(i, 1)$ or $(i, 3)$ permute the two infinite points x_0 and x_1. The y_i's stay fixed all the time. It is straightforward to check that the blocks form the required designs. ■

3. Recursive constructions

The first lemma is a standard construction which uses 4-frames and IMURDs to construct MURDs. It should be pointed out however, that the IMURDs are essential in this construction and that the construction does not work with just the use of MURDs. Again we refer the reader to [2] or [3] for definitions and results concerning the objects such as frames and RGGDs that are used in this section.

Lemma 3.1. *If there exists a 4-frame of type* $(12h_1)(12h_2)\dots(12h_n)$, *an* $\text{IMURD}(12h_i+u,u)$ *for all* $i=1,2,\dots,n-1$ *and a* $\text{MURD}(12h_n+u)$, *then there exists a* $\text{MURD}(u+\sum 12h_i)$.

Proof. First add u infinite points to the frame and then for each $i=1,2,\dots,n-1$ fill in an $\text{IMURD}(12h_i+u,u)$ on the points of the ith group plus the infinite points. Fill in a $\text{MURD}(12h_n+u)$ on the points of the last group plus the infinite points. The parallel classes come from the frame and the IMURDs and the MURD. Note that the number of holey parallel classes missing a group of size $(12h_i)$ is $(12h_i)/3$ which is precisely the number of type 3 resolution classes in the $\text{IMURD}(12h_i+u,u)$. ■

To apply this lemma we will use the following theorem concerning the existence of 4-frames of type h^n.

Theorem 3.2 ([4,6]). *There exists a 4-frame of type* $(12h)^u$ *if and only if* $u\ge 5$, *except possibly when* $h=3$ *and* $u=12$.

Theorem 3.3. *If* $u\equiv 1 \pmod 4$, *and* $u\ge 21$, *then there exists a* $\text{MURD}(12u)$.

Proof. Begin with a 4-frame of type 48^n which exists by Theorem 3.2 with $n\ge 5$ and add 12 infinite points. Then use Lemma 3.1 with $h_i=4$ for $1\le i\le n$ and $u=12$ to get a $\text{MURD}(12+48n)=\text{MURD}(12(1+4n))$ for all $n\ge 5$. ■

The following is a general recursive construction using resolvable GDDs that is similar to Lemma 3.1.

Lemma 3.4. *If there exists a 4-RGDD of type* $(12h)^n$ *and a* $\text{MURD}(12h)$, *then there exists a* $\text{MURD}(12hn)$ *and an* $\text{IMURD}(12h(n-1)+12h,12h)$.

Proof. Just fill in each group in the RGDD with the $\text{MURD}(12h)$ to get a $\text{MURD}(12hn)$. The IMURD is constructed by leaving the last group unfilled. ■

To apply this construction we will need to know the existence of 4-RGDD of type $(12h)^n$. This is provided in the next theorem.

Theorem 3.5 ([4]). *There exists a 4-RGDD of type* $(12h)^u$ *if* $u\ge 4$ *and except possibly when* $h=1$ *and* $u=27$; $h=2$ *and* $u=23$; *and* $h=3$ *and* $u\in\{11,14,15,18,23\}$.

From Lemma 3.4, Theorem 3.5 and Lemmas 2.1–2.4 we get the following theorem.

Theorem 3.6. (a) *There exists an* $\text{IMURD}(36(u-1)+36,36)$ *and a* $\text{MURD}(36u)=\text{MURD}(12(3u))$ *for all* $u\ge 4$ *and* $u\ne 11,14,15,18,23$.
(b) *There exists an* $\text{IMURD}(48(u-1)+48,48)$ *and a* $\text{MURD}(48u)$ *for all* $u\ge 4$.
(c) *There exists an* $\text{IMURD}(60(u-1)+60,60)$ *and a* $\text{MURD}(60u)$ *for all* $u\ge 4$.
(d) *There exists an* $\text{IMURD}(72(u-1)+72,72)$ *and a* $\text{MURD}(72u)$ *for all* $u\ge 4$.

The following two propositions are our final general recursive constructions. Proposition 3.9 will be used to close out the spectrum of MURDs. We first cite a recent result on the existence of 5-GDDs.

Theorem 3.7 ([1]). *A 5-GDD of type* g^5m^1 *exists if* $g\equiv 0 \pmod 4$, $m\equiv 0 \pmod 4$, *and* $m\le 4g/3$, *with the possible exceptions of* $(g,m)=(12,4)$ *and* $(12,8)$.

Proposition 3.8. *Assume* $k\ge 3$, $k\ne 10,13,14,17,22$ *and* $x\le 4k$. *If there exists a* $\text{MURD}(12(x+3))$, *then there exists a* $\text{MURD}(12(15k+x+3))$.

Proof. Begin with a 5-GDD of type $(12k)^5(4x)^1$ which exists by the previous theorem. Give weight 3 to each point in the GDD and then replace each block by a 4-frame of type 3^5 (which exists by Theorem 3.2). The result is a 4-frame of type $(36k)^5(12x)^1$. Now add 36 infinite points. Use Lemma 3.1 with an $\text{IMURD}(36k+36,36)$ on each of the first five groups plus the infinite points and a $\text{MURD}(12x+36)$ on the last group plus the infinite points to obtain a $\text{MURD}(12(15k+x+3))$. Note that IMURDs exists by Theorem 3.6(a). ■

Proposition 3.9. *Assume* $k\ge 3$ *and* $x\le \frac{16}{3}k$. *If there exists a* $\text{MURD}(12(x+4))$, *then there exists a* $\text{MURD}(12(20k+x+4))$.

Proof. The proof is similar to the proof of Proposition 3.8. Begin with a 5-GDD of type $(16k)^5(4x)^1$ and again give each point weight 3 and use the 4-frame of type 3^5 to obtain a 4-frame of type $(48k)^5(12x)^1$. Now add 48 infinite points. The first five groups are then filled in with an IMURD$(48k+48, 48)$ and a MURD$(12x+48)$ goes on the last group plus the infinite points resulting in a MURD$(12(20k+x+4))$. Note that IMURD exists by Theorem 3.6(b) and that now there are no further restrictions on the value of k except that $k \geq 3$. ■

In order to utilize Proposition 3.9 to close the spectrum of MURD$(12u)$'s we need to construct 20 consecutive values u for which there exists a MURD$(12u)$. In the next two lemmas we will construct MURD$(12u)$ for all $48 \leq u \leq 68$.

Lemma 3.10. *If* $48 \leq u \leq 93$ *and* $u \equiv 3, 4, 5, 6, 8 \pmod 9$, *then there exists a* MURD$(12u)$.

Proof. Take a TD$(n+1, 9)$ for $5 \leq n \leq 9$ and give weight 12 to all the points in first n groups; in the last group give weight 12 to y points and weight 0 to the rest. Replace each block in the TD with a 4-frame of type 12^n or 12^{n+1} (existence guaranteed by Theorem 3.2) to obtain a 4-frame of type $108^n(12y)^1$ for $0 \leq y \leq 9$. Now assume that $y = 0, 1, 2, 3, 5, 9$ and add 36 infinite points. Use Lemma 3.1 and fill in the first n holes with an IMURD$(108+36, 36)$ and the last hole plus the infinite points with a MURD$(12y+36)$ for $y = 0, 1, 2, 3, 5, 9$. The IMURD$(108+36, 36)$ exists by Theorem 3.6(a) and the MURD$(12y+36)$ exists by Lemmas 2.1–2.5 and Theorem 3.6(a). The result is a MURD$(12(9n+3+y))$ for every $5 \leq n \leq 9$ and $y = 0, 1, 2, 3, 5, 9$, completing the proof. ■

Lemma 3.11. *If* $48 \leq u \leq 68$ *there exists a* MURD$(12u)$.

Proof. The set of values of u with $48 \leq u \leq 68$ for which a MURD$(12u)$ is not already constructed in the previous lemma is $\{52, 54, 55, 56, 61, 63, 64, 65\}$. A MURD$(12 \cdot 61)$ exists by Theorem 3.3. For all the other values of u, a MURD$(12 \times u)$ exists by one of the parts of Theorem 3.6. ■

Lemma 3.12. *For all* $k \geq 12$ *and* $48 \leq x \leq 68$ *there exists a* MURD$(12(20k+x))$.

Proof. This is just an application of Proposition 3.9 and Lemma 3.11 since $k \geq 12$ guarantees that $x - 4 \leq \frac{16}{3}k$. ■

We are now in a position to show that a MURD$(12u)$ exists when u is large enough.

Theorem 3.13. *For every* $u \geq 288$ *there exists a* MURD$(12u)$.

Proof. Let $u \geq 288$. Let $20k$ be a multiple of 20 in the interval $[u-68, u-48]$. Then $k \geq 12$ and $u = 20k + x$ where $48 \leq x \leq 68$. Hence by Lemma 3.12 there is a MURD$(12u)$. ■

4. The smaller orders

We begin this section with a construction for some MURD$(12u)$ with $u \leq 48$.

Lemma 4.1. *There exists a* MURD$(12u)$ *for* $u = 23, 24, 25, 26$.

Proof. From Theorem 3.7 there exists a 5-GDD of type 16^5y^1 for $y = 8, 12, 16, 20$. Give weight 3 to each point and fill in each block with a 4-frame of type 3^5 [4] to obtain a 4-frame of type $48^5(3y)^1$ for $y = 8, 12, 16, 20$. Now add 12 infinite points and apply Lemma 3.1 with the ingredients an IMURD$(48+12, 12)$ and a MURD$(12u)$ for $u = 3, 4, 5, 6$ (all of these ingredients were constructed in Section 2) to make a MURD$(12u)$ for $u = 23, 24, 25, 26$, respectively. ■

Now define the set $E = \{2, 7, 9, 10, 11, 13, 14, 17, 19, 22, 31, 34, 38, 43, 46, 47, 82\}$. This will be our eventual set of exceptional cases.

Proposition 4.2. *There exists a* MURD$(12u)$ *for every* $2 \leq u \leq 68$ *except possibly for* $u \in E$.

Proof. This follows from Lemmas 2.1–2.5, Theorems 3.3 and 3.6, Lemmas 3.11 and 4.1. ■

We now construct MURD$(12u)$ for $69 \leq u \leq 287$.

Lemma 4.3. *There exists a* MURD$(12u)$ *for all* $69 \leq u \leq 81$ *and* $83 \leq u \leq 87$.

Proof. For $u = 69, 71, 85$ and 86 there exists a MURD$(12u)$ from Lemma 3.10. A MURD$(12 \cdot 70)$ exists from Theorem 3.6. A MURD$(12 \cdot 79)$ and a MURD$(12 \cdot 83)$ exist from Proposition 3.8 with $k = 5$ and $x = 1$ and 5, respectively.

To get all the remaining values in this range, begin with a transversal design TD$(7, 12)$ and give weight 12 to all the points in the first five groups, to a points in the sixth group and to x points in the last group (weight 0 to all other points). Now replace each block with a 4-frame of type 12^n for $5 \leq n \leq 7$, which exist by Theorem 3.2, to obtain a 4-frame of type $144^5(12a)^1(12x)^1$. We restrict to $a = 0, 9, 12$ and $x = 0, 1, 2, 3, 5, 9, 12$ and add 36 infinite points. Note that for each of these values of x there exists a MURD$(12(x+3))$ and in addition there exists an IMURD$(108+36, 36)$ and an IMURD$(144+36, 36)$. Applying Lemma 3.1 gives a MURD$(12(60+a+x+3))$ where $a = 0, 9, 12$ and $x = 0, 1, 2, 3, 5, 9, 12$. Hence we get a MURD$(12u)$ for $u = 63, 64, 65, 66, 68, 72, 73, 74, 75, 76, 77, 78, 80, 81, 84, 87$. ■

Lemma 4.4. *There exists a* MURD$(12u)$ *for* $88 \le u \le 162$.

Proof. Begin with a transversal design TD(14, 13) and remove the points on a block to obtain a 13-GDD of type 12^{14}. Give weight 12 to the points in six groups, weight 12 to 0, 9 or 12 points in seven groups, weight 12 to 0, 1, 2, 3, 5, 9, 12 points in the last group and add 36 infinite points. Let a be the number of such groups where 12 points received weight 12 and b be the number of groups where 9 points received weight 12. Again replacing each block with a 4-frame of type 12^n for $5 \le n \le 14$ yields a 4-frame of type $144^{6+a}108^b(12x)$ where $x = 0, 1, 2, 3, 5, 9$ or 12. Applying Lemma 3.1 gives a MURD$(12(72 + 12a + 9b + x + 3))$ where $a + b \le 7$ and $x = 0, 1, 2, 3, 5, 9, 12$. Substituting appropriate values for a, b and x gives a MURD$(12u)$ for $u = 88, 89, 90$ and all $92 \le u \le 162$.

A MURD$(12 \cdot 91)$ exists via Proposition 3.8 with $k = 5$ and $x = 13$. ■

Lemma 4.5. *There exists a* MURD$(12u)$ *for* $163 \le u \le 243$.

Proof. The proof is the same as Lemma 4.4 except that now we start with TD(17, 16) and remove the points on a block to obtain a 16-GDD of type 15^{17}. Give weight 12 to the points in six groups, weight 12 to 0, 9, 12 or 15 points in ten groups and weight 12 to 0, 1, 2, 3, 5, 9, 12 points in the last group and add 36 infinite points. Assume a of these groups have 15 points receiving weight 12, b groups have 12 points receiving weight 12 and c groups have 9 points getting weight 12. Proceed as before to obtain a MURD$(12(90+15a+12b+9c+x+3))$ where $a+b+c \le 10$ and $x = 0, 1, 2, 3, 5, 9, 13, 15$. Substituting appropriate values for a, b, c and x it is easy to get a MURD$(12u)$ for $163 \le u \le 243$. ■

The next lemma finishes off the constructions for MURD$(12u)$ with $u \le 287$.

Lemma 4.6. *There exists a* MURD$(12u)$ *for all* $244 \le u \le 287$.

Proof. The proof is also the same as Lemma 4.4 except that this time we start with TD(20, 19) and remove the points on a block to obtain a 19-GDD of type 18^{20}. Give weight 12 to the points in twelve groups, weight 12 to 0, 9, 12, 15 or 18 points in seven groups and weight 12 to 0, 1, 2, 3, 5, 9, 12 points in the last group and add 36 infinite points. Assume a of these groups have 18 points receiving weight 12, b groups have 15 points receiving weight 12, c groups have 12 points getting weight 12 and d groups have 9 points receiving weight 12. Proceeding as before we obtain a MURD$(12(216 + 18a + 15b + 12c + 9d + x + 3))$ where $a + b + c + d \le 7$ and $x = 0, 1, 2, 3, 5, 9, 13, 15$. Substituting appropriate values for a, b, c and x one can obtain a MURD$(12u)$ for all $244 \le u \le 287$. ■

We give our main result below. It follows from Theorem 3.13 and the lemmas in this section. Note that clearly there is no MURD(12) as it is not possible to find even two parallel classes of blocks of size 4. The existence of a MURD(24) is unknown, although it should be noted that there does indeed exist a resolvable 4-GDD of type 3^8.

Theorem 4.7. *There exists a* MURD$(12u)$ *for all* $u \ge 2$ *with the possible exception of* $u \in \{2, 7, 9, 10, 11, 13, 14, 17, 19, 22, 31, 34, 38, 43, 46, 47, 82\}$.

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