Math 242 Notes 12: Infinite products

Our approach to infinite products is a little more streamlined (and, to be frank, clearer) than Apostol's.

Definition. *If $\{a_k\}_1^n \subset \mathbf{C}$ is a finite sequence then*

$$\prod_1^n a_k := a_1 \cdot a_2 \cdot a_3 \cdots a_n,$$

and if $1 \le m \le n$ then

$$\prod_m^n a_k := a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

Definition. *If $\{p_n\}_1^\infty$ is a sequence of complex numbers, we say that the infinite product*

$$\prod_1^\infty p_n$$

converges if there is an $N \ge 1$ such that the sequence of finite products

$$\prod_{k=N+1}^n p_k$$

converges to a **non-zero** *complex number α as $n \to \infty$; otherwise, the product is said to diverge. If the product does converge, we say it converges to*

$$\left(\prod_1^N p_k\right) \cdot \alpha,$$

which we call its **value**. *If $\exists N$ such that $\forall k \ge N$ $p_k \ne 0$ but*

$$\lim_{n\to\infty} \prod_N^n p_k = 0$$

the product is said to **diverge to 0**.

Remark. One consequence of the preceding definition is that, if $\prod_1^\infty p_n$ converges, then the only way it can be 0 (i.e., have a value of 0) is for one of its factors p_n to be 0.

Theorem. *If the infinite product*

$$\prod_1^\infty p_n$$

converges, its value,

$$\left(\prod_1^N p_k\right) \cdot \alpha, \tag{1}$$

is well-defined; i.e., it does not depend on N.

Proof of theorem. If some $p_k = 0$ then either

$$\prod_1^N p_k \qquad \text{or} \qquad \lim_{n\to\infty} \prod_{k=N+1}^n p_k$$

will be zero, because it will contain a zero factor p_k, and the value of the infinite product will be 0, no matter what N is.

If no $p_k = 0$ then, for the right $N \geq 1$, a value of the limit is given by

$$\lim_{n\to\infty} \prod_1^n p_k = \left(\prod_1^N p_k\right)\left(\lim_{n\to\infty} \prod_{k=N+1}^n p_k\right). \tag{2}$$

If $N' \geq 1$ is now arbitrary then, for $n > N'$,

$$\prod_{k=N'+1}^n p_k = \frac{\prod_1^n p_k}{\prod_1^{N'} p_k}. \tag{3}$$

Since the numerator of the right-hand side of (3) has a non-zero limit, so does the left-hand side. The corresponding value of the infinite product,

$$\left(\prod_1^{N'} p_k\right) \cdot \left(\lim_{n\to\infty} \prod_{k=N'+1}^n p_k\right)$$

will be

$$\left(\prod_1^{N'} p_k\right) \cdot \left(\lim_{n\to\infty} \frac{\prod_1^n p_k}{\prod_1^{N'} p_k}\right) = \lim_{n\to\infty} \prod_1^n p_k,$$

which is the same as (2). Thus the infinite product's value is well-defined. ♣

Remark. We note in passing that if

$$\lim_{n\to\infty} \prod_{k=N+1}^n p_k$$

exists and is non-zero for some N, it will exist and be non-zero for all $N' \geq N$.

Theorem (Cauchy Criterion for Infinite Products). *Let* $\{p_n\}_1^\infty$ *be a sequence of complex numbers. The infinite product*

$$\prod_1^\infty p_n$$

converges iff $\forall \epsilon > 0 \; \exists N \in \mathbf{N} \; \forall N \leq m \leq n$

$$\left| 1 - \prod_m^n p_k \right| < \epsilon.$$

Proof of theorem. Suppose the product converges. WLOG we can assume that no $p_k = 0$. (Why?) Define

$$\pi_n := \prod_1^n p_k.$$

Then $\{\pi_n\}$ is Cauchy in $\mathbf{C}$ and $\to$ a non-zero L. Given $\epsilon > 0$, let N be so large that $n \geq N \Rightarrow |\pi_n| > |L|/2$ and $N < m < n \Rightarrow |\pi_n - \pi_{m-1}| < \epsilon$. Then, for these m and n,

$$|\pi_n - \pi_{m-1}| = |\pi_{m-1}| \left| 1 - \prod_m^n p_k \right| < \epsilon,$$

implying

$$\left| 1 - \prod_m^n p_k \right| < \frac{\epsilon}{|\pi_{m-1}|} < \frac{2\epsilon}{|L|},$$

which is the Cauchy criterion. Now suppose the criterion holds. Again, we can assume that no $p_k = 0$. Let N_1 be so large that $N_1 < m \leq n$ implies

$$\left| 1 - \prod_m^n p_k \right| < 1/2.$$

Then, for all $n > N_1$,

$$|\pi_{N_1} - \pi_n| = |\pi_{N_1}| \left| 1 - \prod_{N_1+1}^n p_k \right| < (1/2)|\pi_{N_1}|,$$

which implies that the partial products π_n are all bounded by some number M and bounded away from 0: $\exists 0 < r < M < \infty$ such that, $\forall n \in \mathbf{N}$,

$$r < |\pi_n| < M.$$

Now, given $\epsilon > 0$, let N_2 be so big that $N_2 < m \le n$ implies

$$\left| 1 - \prod_m^n p_k \right| < \epsilon.$$

Then $N_2 \le m < n$ implies

$$|\pi_m - \pi_n| = |\pi_m| \left| 1 - \prod_{m+1}^n p_k \right| \le M\epsilon,$$

implying that $\{\pi_n\}$ is a Cauchy sequence. It has a non-zero limit (because $r < |\pi_n|$ for all n.) Therefore the infinite product converges. ♣

Corollary. *If the infinite product*

$$\prod_1^\infty p_n$$

converges then $p_k \to 1$.

Proof of corollary. By the Cauchy criterion, $|1 - \prod_n^n p_k| = |1 - p_n| \to 0$.♣

Remark. The converse fails. Consider $p_k = 1 + 1/k$. Then $\pi_n := \prod_1^n p_k = n + 1$.

Because of the Cauchy criterion, and because we are mainly interested in convergent products, we normally write the factors of an infinite product as $p_k := 1 + a_k$, and note that we can have convergence only if $a_k \to 0$ (but that $a_k \to 0$ doesn't imply convergence).

The inequality proved in the following lemma simplifies some problems with infinite products.

Simplifying Lemma. *If* $\{\zeta_n\}_1^N$ *are complex numbers then*

$$\left| \left(\prod_1^N (1 + \zeta_n) \right) - 1 \right| \le \exp\left(\sum_1^N |\zeta_n| \right) - 1.$$

Proof of lemma.

$$\prod_1^N (1 + \zeta_n) = 1 + \sum_1^N \zeta_n + \sum_{1 \le n_1 < n_2 \le N} \zeta_{n_1}\zeta_{n_2} + \sum_{1 \le n_1 < n_2 < n_3 \le N} \zeta_{n_1}\zeta_{n_2}\zeta_{n_3} + \cdots + \prod_1^N \zeta_n.$$

That is, we get 1 plus the sum of the ζ_n's, the sum of the two-term products, the sum of three-term products, etc., all the way to the product of all the ζ_n's. Therefore,

$$\left(\prod_1^N (1+\zeta_n)\right) - 1 = \sum_1^N \zeta_n + \sum_{1\le n_1<n_2\le N} \zeta_{n_1}\zeta_{n_2} + \sum_{1\le n_1<n_2<n_3\le N} \zeta_{n_1}\zeta_{n_2}\zeta_{n_3} + \cdots + \prod_1^N \zeta_n, \tag{4}$$

and

$$\left|\left(\prod_1^N (1+\zeta_n)\right) - 1\right| \le \sum_1^N |\zeta_n| + \sum_{1\le n_1<n_2\le N} |\zeta_{n_1}||\zeta_{n_2}| + \sum_{1\le n_1<n_2<n_3\le N} |\zeta_{n_1}||\zeta_{n_2}||\zeta_{n_3}| + \cdots + \prod_1^N |\zeta_n|.$$

Now formula (4), applied to the sequence $\{|\zeta_n|\}_1^N$, implies that

$$\sum_1^N |\zeta_n| + \sum_{1\le n_1<n_2\le N} |\zeta_{n_1}||\zeta_{n_2}| + \sum_{1\le n_1<n_2<n_3\le N} |\zeta_{n_1}||\zeta_{n_2}||\zeta_{n_3}| + \cdots + \prod_1^N |\zeta_n|$$

equals

$$\left(\prod_1^N (1+|\zeta_n|)\right) - 1. \tag{5}$$

Since $1+x \le \exp(x)$ for all x and

$$\prod_1^N \exp(|\zeta_n|) = \exp\left(\sum_1^N |\zeta_n|\right),$$

the expression in (5) is less than or equal to

$$\exp\left(\sum_1^N |\zeta_n|\right) - 1,$$

which was to be proved. ♣

Corollary. *Let $\{\zeta_n\} \subset \mathbf{C}$ and $\sum |\zeta_n| < \infty$. Then $\prod_1^\infty (1+\zeta_n)$ converges.*

Proof of corollary. Given $\epsilon > 0$, let $\delta > 0$ be so small that $e^x - 1 < \epsilon \; \forall 0 < x < \delta$. Let N be so big that $\forall N \le m \le n \; \sum_m^n |\zeta_k| < \delta$. Now the lemma shows that, for such m and n,

$$\left|1 - \prod_m^n (1+\zeta_k)\right| \le \exp\left(\sum_m^n |\zeta_k|\right) - 1 < \epsilon,$$

which is the Cauchy criterion. ♣

Theorem. *Let $\{a_n\}_1^\infty$ be non-negative real numbers. Then $\prod_1^\infty(1+a_n)$ converges iff $\sum a_n < \infty$.*

Proof of theorem. The preceding corollary proves one direction. For the other, we use the fact that $\prod_1^n(1+a_k) \nearrow$ with n and is always positive, so the product converges iff $\{\prod_1^n(1+a_k)\}$ is bounded. Then observe that

$$\sum_1^n a_k \le \prod_1^n (1+a_k)$$

for all n: if $\sum_1^n a_k \to \infty$, so does $\prod_1^n(1+a_k)$. ♣

Corollary. *Let $\{a_n\}_1^\infty$ be non-negative real numbers. Then $\prod_1^\infty(1-a_n)$ converges iff $\sum a_n < \infty$.*

Proof. We only need to prove one direction. WLOG every $a_k < 1/2$, so $1 - a_k > 0$. Suppose $\sum a_n = \infty$. Then $\prod_1^n(1+a_k) \to \infty$. Now look at the following:

$$\begin{aligned}
1 - a_k^2 &\le 1 \\
1 - a_k &\le (1+a_k)^{-1} \\
\prod_1^n (1-a_k) &\le \left(\prod_1^n (1+a_k)\right) \to 0,
\end{aligned}$$

and $\prod_1^\infty(1-a_k)$ diverges to 0. ♣

Definition. *An infinite product $\prod_1^\infty(1+a_n)$ (with $\{a_n\} \subset \mathbf{C}$) is said to converge absolutely if $\prod_1^\infty(1+|a_n|)$ converges.*

The preceding theorem and corollary imply

Corollary. *If $\prod_1^\infty(1+a_n)$ converges absolutely then it converges.*

Proof of corollary. $\prod_1^\infty(1+|a_n|)$ converges$\Rightarrow \sum_1^\infty |a_n| < \infty \Rightarrow \prod_1^\infty(1+a_n)$ converges. ♣

There is another way to get convergence of $\prod_1^\infty(1+a_n)$ from $\sum_n |a_n| < \infty$. Let $\{a_n\} \subset \mathbf{C}$ and suppose $\sum |a_n| = M < \infty$. Define

$$\begin{aligned}
\pi_0 &:= 1 \\
\pi_n &:= \prod_1^n (1+a_k) \qquad \text{if } n \ge 1.
\end{aligned}$$

Then, as we've seen, $|\pi_n| \le e^M$ for all n. For any $n \ge 1$ we can write

$$\pi_n - \pi_0 = \sum_1^n (\pi_k - \pi_{k-1}).$$

But

$$\pi_k - \pi_{k-1} = (1 + a_k)\pi_{k-1} - \pi_{k-1} = a_k\pi_{k-1},$$

and

$$\sum_1^\infty |a_k||\pi_{k-1}| \le Me^M.$$

Therefore

$$\lim_{n\to\infty} \pi_n = 1 + \lim_{n\to\infty} \sum_1^n (\pi_k - \pi_{k-1})$$

exists in **C**. If the a_ks are so small that $Me^M < 1$ then the limit of the π_ns will be non-zero. This means that, if $\sum_k |a_k| < \infty$, and we start our partial products at an n for which $\sum_{k\ge n} |a_k| =: M$ satisfies $Me^M < 1$, then that sequence of partial products will converge to a non-zero limit. In other words, if $\sum_k |a_k| < \infty$ then $\prod_1^\infty (1 + a_k)$ converges.

An Application: the Riemann Zeta Function. If $s > 1$ we define

$$\zeta(s) := \sum_1^\infty \frac{1}{n^s}.$$

$\zeta(s)$ is called the Riemann zeta function. Our treatment, which won't be entirely rigorous, will try to give you an idea of why $\zeta(s)$ is a big deal in the study of prime numbers.

Theorem (Euler). *Let the set of primes be $\{p_1 < p_2 < p_3 < \cdots\}$. If $s > 1$ then*

$$\frac{1}{\zeta(s)} = \prod_1^\infty (1 - p_k^{-s}),$$

where the product on the right converges absolutely.

Proof of theorem. Since

$$\sum_1^\infty p_k^{-s} \le \sum_1^\infty \frac{1}{n^s} < \infty$$

when $s > 1$, the absolute convergence is assured. Since none of the factors is zero, the product is non-zero, and we can flip it and look at

$$\prod_1^\infty (1 - p_k^{-s})^{-1} = \lim_n \prod_1^n (1 - p_k^{-s})^{-1}.$$

By the geometric series formula, for any k,

$$(1 - p_k^{-s})^{-1} = \frac{1}{1 - p_k^{-s}} = \sum_{j=0}^\infty p_k^{-sj} = 1 + p_k^{-s} + p_k^{-2s} + p_k^{-3s} + \cdots$$

Therefore

$$\prod_{1}^{n}(1-p_k^{-s})^{-1} = \prod_{k=1}^{n}\left(1+p_k^{-s}+p_k^{-2s}+p_k^{-3s}+\cdots\right)$$
$$= 1+\sum_{r\in S_n} r^{-s},$$

where S_n is the set of natural numbers > 1 whose prime factors are all $\leq p_n$. You can see that this is so by writing out what a number $L \in S_n$ must look like. Say,

$$L := p_1^{a_1}\cdot p_2^{a_2}\cdot p_3^{a_3}\cdots p_n^{a_n},$$

where each $a_k \in \{0,1,2,3,\cdots\}$ and at least one of them ≥ 1. Then

$$L^{-s} = p_1^{-sa_1}\cdot p_2^{-sa_2}\cdot p_3^{-sa_3}\cdots p_n^{-sa_n}$$

arises by taking the a_1^{th} term from

$$1+p_1^{-s}+p_1^{-2s}+p_1^{-3s}+\cdots$$

(if we count the leading 1 as the zeroth term), the a_2^{th} term from

$$1+p_2^{-s}+p_2^{-2s}+p_2^{-3s}+\cdots,$$

the a_3^{th} term from

$$1+p_3^{-s}+p_3^{-2s}+p_3^{-3s}+\cdots,$$

and so on, up to n factors, and multiplying them together. By letting $n \nearrow \infty$ we have

$$1+\sum_{r\in S_n} r^{-s} \to \sum_{1}^{\infty}\frac{1}{r^{-s}} = \zeta(s),$$

while at the same time

$$1+\sum_{r\in S_n} r^{-s} \to \lim_{n}\prod_{1}^{n}(1-p_k^{-s})^{-1},$$

proving the result. ♣