THE UNIVERSITY OF VERMONT
DEPARTMENT OF MATHEMATICS AND STATISTICS
SIXTY-SEVENTH ANNUAL HIGH SCHOOL PRIZE EXAMINATION
MARCH 12, 2024

1. Express as a rational number in lowest terms: $\dfrac{4^{-3} - 4^{-2}}{4^{-1} - 4^{0}}$

2. Tokens come in four shapes: circles, triangles, squares, and stars. Tokens of the same shape weigh the same. One circle weighs as much as one triangle and one star combined; one square weighs as much as one circle and one triangle combined; and three squares weigh as much as four stars. How many triangles are needed to match the weight of two circles?

3. Jesse goes to the bank and asks the teller to exchange a hundred-dollar bill for some two-dollar bills, one-dollar bills, and five-dollar bills with at least one of each kind of bill and exactly ten times as many one-dollar bills as two-dollar bills. How many two-dollar bills does the teller give Jesse?

4. The vertices of quadrilateral $ABCD$ are at $(0.5, 2.5), (2.5, 4.5), (4.5, 2.5)$ and $(2.5, 0.5)$ as shown in the figure. Find the area of the quadrilateral.
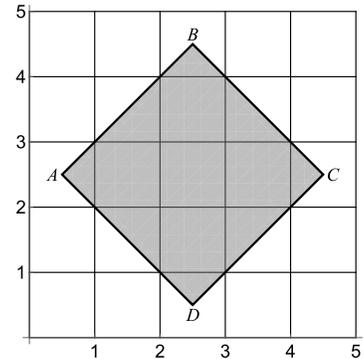
5. Diane delivers newspapers for $8 a day plus $0.06 per newspaper delivered. Jeremy delivers newspapers for $5 a day plus $0.10 per newspaper delivered. One day Diane delivers the same number of newspapers as Jeremy, and they each earn the same amount of pay for the day. How many newspapers did Diane deliver?

6. What is the 2024th term of the arithmetic sequence that begins $\{-2, -1.5, -1, -0.5, \cdots\}$?

7. In polygon $ABCDA$, points E and F lie on edges $\overline{CD}$ and $\overline{AD}$ respectively, $BEDF$ is a square, and ABF and BCE are triangles, as shown in the figure. If $DF = DE = CE = 3$ and $AF = 4$, what is the area of polygon $ABCDA$?
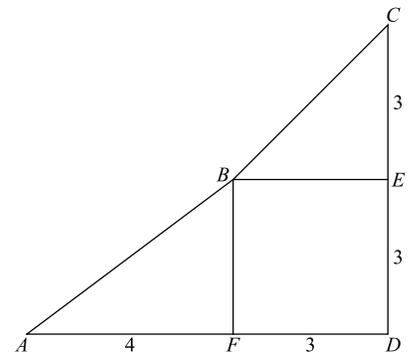

8. A ball is tethered by a 50-foot-long rope to a tree with a cylindrical trunk that is one foot in diameter. The ball swings around the tree, wrapping the rope around the trunk. How many full circles around the tree will the ball make before it hits the tree? Assume that the loops made by the rope are perpendicular to the tree trunk and the thickness of the rope is negligible.

9. Express $\left(\sqrt{500} + \sqrt{125}\right)^2$ as a rational number.

10. Nora has some saltwater with a concentration of 12% salt. How much pure water should she mix with 2 liters of the saltwater so that the resulting solution has a concentration of 3% salt?

11. Let $f(x) = x^2$ and $g(x) = 2x - 9$. Find all x such that $f\big(g(x)\big) = g\big(f(x)\big)$.

12. The operation $\otimes$ is defined by: $a \otimes b = a^2 b + 1$. Find all values of b such that $(2 \otimes 3) \otimes b = 2 \otimes (3 \otimes b)$.

13. Let $f(x) = x^2 + px + q$, where coefficients p and q are integers. If $f\left(1 + \sqrt{3}\right) = -7 + 10\sqrt{3}$, find p and q.

14. A two-digit number with identical digits is multiplied by 99. What is the resulting product, given that the tens digit of the product is 2?

15. Find the positive number x that satisfies the equation $x \cdot \lfloor x \rfloor = 11$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

16. At a certain time one Monday morning, Tilly asked Adira "What time is it?" and Adira replied, "Add one-quarter of the time from midnight until now to half the time from now until midnight. The result is the current time." What time was it? Express your answer in the form *hour:minutes*.

17. Dal walks one mile east, then one mile northeast, then another mile east. Find the square of the distance between Dal's initial and final positions.

18. Find the value of c for which the three lines given by the equations $7x - 12y = 42$, $7x + 20y = 98$, and $21x + 12y = c$ intersect at a single point.

19. Express as a rational number in lowest terms: $\dfrac{2024! + 2025! + 2026!}{2026! - 2025! - 2024!}$

20. How many integers n satisfy the inequality given below?
$$\frac{1}{n^2 - 4} > \frac{1}{100}$$

21. A rectangle 6 units wide and 4 units high is divided into a grid of 1×1 squares. What is the sum of the areas of all squares (of any size) that are contained in the figure?
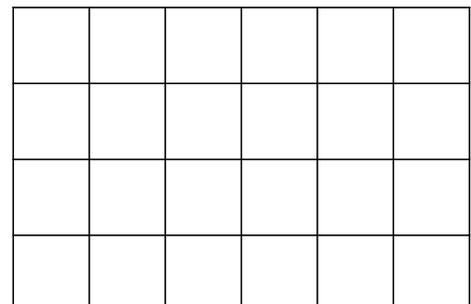
22. How many positive integers divide 2024^3?

23. Isosceles triangle ABC has base $BC = 4$ and a height of 6 units. Four equally spaces lines are drawn parallel to the base, dividing the triangle into regions R_1 through R_5. The odd numbered regions are shaded, as shown in the figure. Find the sum of the areas of the shaded regions. Express your answer as a rational number in lowest terms.
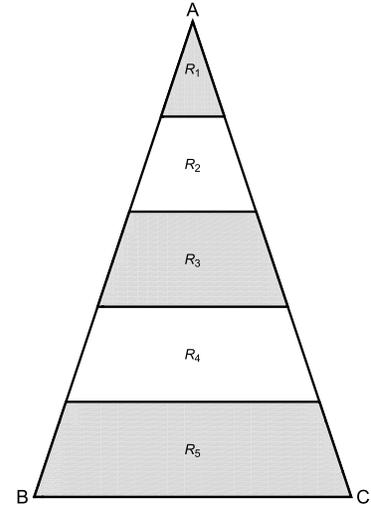
24. Find the value of $\log_3(5) \cdot \log_{25}(27)$. Express your answer as a rational number in lowest terms.

25. What octal (base 8) number added to 5701_8 gives a result of 10000_8?

26. Kite $ABCD$ has side lengths $AB = x + 6$, $BC = 2x - 1$, $CD = y$, and $DA = y - 1$, for some numbers x and y. Find the numerical value of the perimeter of the kite.

27. Brantley eats his dog food three times as fast as Rufus, but takes one and one-quarter the amount of time that Spot takes to eat. If the sum of the eating times of all three dogs is 6 minutes, how many seconds does it take Brantley to eat his dog food?

28. A square with side length 6 units circumscribes a circle. If a point within the square is randomly chosen, what is the probability that it falls within the circle?

29. Lily is choosing what to order at the ice cream shop. She can get a creemee, which has 5 flavors and 2 toppings to choose from, or hard ice cream, which has 12 flavors and 3 toppings to choose from. She can choose one or more (or no) toppings, and she can also add sprinkles to either type of ice cream if she likes. How many possible ice cream orders are there?

30. Find the value of $\cos\left(\arctan\left(-\frac{5}{12}\right)\right)$.

31. If the expression given below is simplified to the form $ax^p y^q$, find $a + p + q$.

$$\left(\frac{1}{81}x^8 y^4\right)^{-3/4} \left(\frac{3}{2}x^{-1/2}y^3\right)^{-2}$$

32. Find the product of the solutions to the equation $(\log_2(x))^2 + \log_2\left(\left(\frac{1}{x}\right)^2\right) = 15$.

33. Find the probability that a random arrangement of the letters in **PARALLEL** contains the consecutive sequence **REAL**. Express your answer as a rational number in lowest terms.

34. In right triangle ABC, point D lies on $\overline{AC}$ and is equidistant from $\overline{AB}$ and $\overline{BC}$. If $AD = 20$ and $DC = 30$, find AB.
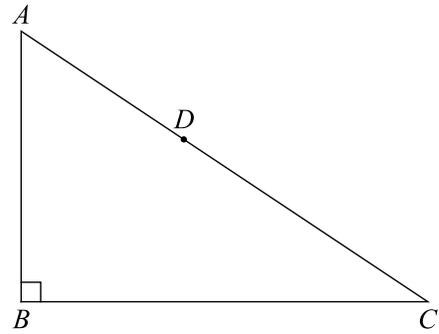

35. For lunch, 100 children had juice, 50 had salad, and 48 had a burger. Each of the children had at least one item, half of them had at least two items, and one-third of them had all three items. How many children were there?

36. Lines L_1 and L_2 are parallel. The equation of L_1 is $x + 2y = 5$ and the y-intercept of L_2 is 1. What is the perpendicular distance between L_1 and L_2?

37. How many ways can 12 identical balls be distributed among three different boxes so that no box is empty, and no box has more than 6 balls?

38. If $\tan(2x) = 5$, find the value of $\cos(4x)$.

39. How many paths are there from point A to point B travelling left to right along horizontal segments, upward along vertical segments, or clockwise along circular arcs in the figure? One such path is shown (thick).
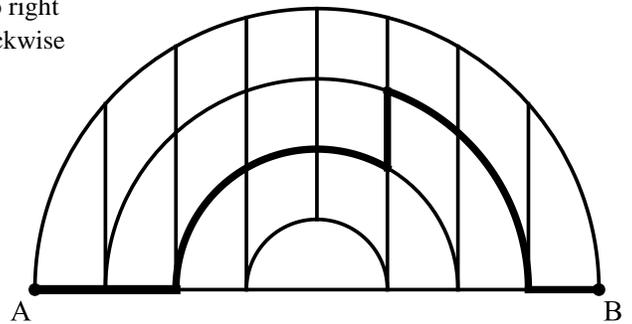
40. In triangle ABC, angle C is twice angle A. If $BA - BC = 2$ and $AC = 5$, find the area of $\triangle ABC$.

41. The roots of $4x^2 + ax + b = 0$ are $\sin(18°)$ and $-\sin(54°)$. Find the ordered pair (a, b).